Vectors and Paths in 2D

James S. Cook
Liberty University
Department of Mathematics

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Abstract

In this article we study vectors in two dimensions and the calculus of paths. We begin by contrasting the different approaches to describing a curve. The implicit viewpoint looks at a curve as the solution set to some equation. The parametric viewpoint looks at a curve as the points which are mapped to by parametric formulas; that is, the parametric viewpoint views a curve as the image of the parametrizing map. The introduction of a new parameter to describe the curve is a bit of a departure from our usual practice of using mainly the Cartesian coordinates x, y. However, a graph $y = f(x)$ is in some sense a middle ground since it is trivial to convert a graph to the equation $y - f(x) = 0$ or to the parametrization $\vec{r}(x) = \langle x, f(x) \rangle$. A graph takes x as its natural parameter. Generally there are infinitely many parametrizations for a given curve. We can think of the parameter as time in many applications. Perhaps most famously, we have position, velocity and acceleration given respectively by the parametrization, its derivative and its second derivative (all vector-valued functions of time). We introduce the calculus of paths in two dimensions. In short, we do calculus componentwise. Vector arithmetic is introduced, we can add vectors either algebraically or geometrically. Unit-vectors and standard angle are critical concepts to understand vectors and the dot-product is extremely useful for the study of angles in the plane. Finally we study various standard calculations of calculus such as slope, concavity, the tangent line, area, surface area and arclength as they can be calculated from the parametric viewpoint. A general guiding principle for application is found in the infinitesimal method; find a rule which is sensible at the level of very small quantities then integrate to find the behavior in the bulk. Finally a word of warning, my lecture has been in a slightly different order than these notes. My apologies, but I thought it best for me to publish these now as I lack time to perform a substantial edit.

1 Curves

In this short section we analyze the concept of a curve from the three major viewpoints prevalent in analytic geometry. For now, we use only Cartesian coordinates[1]. The three views of a curve are:

1. a graph of a function; $graph(f) = \{(x, y) \mid y = f(x), x \in dom(f)\}$
2. a level curve; $C_k = \{(x, y) \mid F(x, y) = k\}$
3. a parametrized curve; $C = \{(x(t), y(t)) \mid t \in J \subseteq \mathbb{R}\}$

These views are not mutually exclusive and each has their advantange and disadvantage. We desire you understand all three in this course. Experiment and question is key, you have to discover these

[1] all of these constructions find parallel versions when other coordinates such as polar, skew-linear or hyperbolics are used to describe $\mathbb{R}^2$.

concepts for yourself. I'll tell you what I think, but don't stop with my comments. Think. Ask your own questions.

2 graphs

Let's begin by reminding ourselves of the definition of a graph:

Definition 2.1. *Graph of a function.*

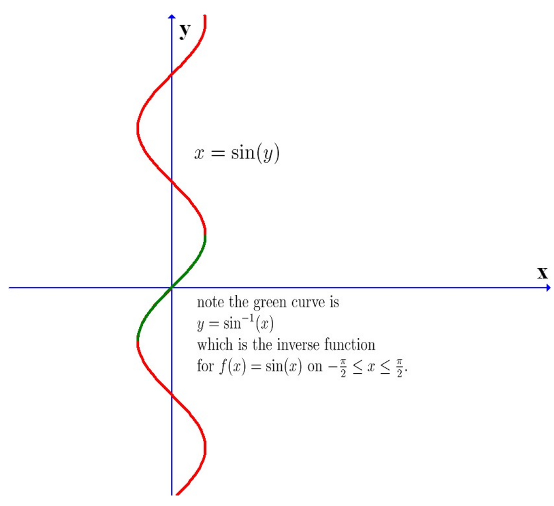
Let $f : dom(f) \to \mathbb{R}$ be a function then

$$graph(f) = \{(x, f(x)) \mid x \in dom(f)\}.$$

We know this is quite restrictive. We must satisfy the vertical line test if we say our curve is the graph of a function.

Example 2.2. *To form a circle centered at the origin of radius R we need to glue together two graphs. In particular we solve the equation $x^2 + y^2 = R^2$ for $y = \sqrt{R^2 - x^2}$ or $y = -\sqrt{R^2 - x^2}$. Let $f(x) = \sqrt{R^2 - x^2}$ and $g(x) = -\sqrt{R^2 - x^2}$ then we find $graph(f) \cup graph(g)$ gives us the whole circle.*

Example 2.3. *On the other hand, if we wish to describe the set of all points such that $\sin(y) = x$ we also face a similar difficulty if we insist on functions having independent variable x. Naturally, if we allow for functions with y as the independent variable then $f(y) = \sin(y)$ has graph $graph(f) = \{(f(y), y) \mid y \in dom(f)\}$. You might wonder, is this correct? I would say a better question is, "is this allowed?". Different books are more or less imaginative about what is permissible as a function. This much we can say, if a shape fails both the vertical and horizontal line tests then it is not the graph of a single function of x or y.*



Example 2.4. *Let $f(x) = mx + b$ for some constants m, b then $y = f(x)$ is the line with slope m and y-intercept b.*

3 level curves

Level curves are amazing. The full calculus of level curves is only partially appreciated even in calculus III, but trust me, this viewpoint has many advantages as you learn more. For now it's simple enough:

Definition 3.1. *Level Curve.*

A level curve is given by a function of two variables $F : dom(F) \subseteq \mathbb{R}^2 \to \mathbb{R}$ and a constant k. In particular, the set of all $(x, y) \in \mathbb{R}^2$ such that $F(x, y) = k$ is called the level-set of F, but more commonly we just say $F(x, y) = k$ is a level curve.

In an algebra class you might have called this the "graph of an equation", but that terminology is dead to us now. For us, it is a level curve. Moreover, for a particular set of points $C \subseteq \mathbb{R}^2$ we can find more than one function F which produces C as a level set. Unlike functions, for a particular curve there is not just one function which returns that curve. This means that it might be important to give both the level-function F and the level k to specify a level curve $F(x, y) = k$.

Example 3.2. *A circle of radius R centered at the origin is a level curve $F(x, y) = R^2$ where $F(x, y) = x^2 + y^2$. We call F the level function (of two variables).*

Example 3.3. *To describe $\sin(y) = x$ as a level curve we simply write $\sin(y) - x = 0$ and identify the level function is $F(x, y) = \sin(y) - x$ and in this case $k = 0$. Notice, we could just as well say it is the level curve $G(x, y) = 1$ where $G(x, y) = x - \sin(y) + 1$.*

Note once more this type of ambiguity is one distinction of the level curve langauge, in constrast, the graph $graph(f)$ of a function $y = f(x)$ and the function f are interchangeable. Some mathematicians insist the rule $x \mapsto f(x)$ defines a function whereas others insist that a function is a set of pairs $(x, f(x))$. I prefer the mapping rule because it's how I think about functions in general whereas the idea of a graph is much less useful in general.

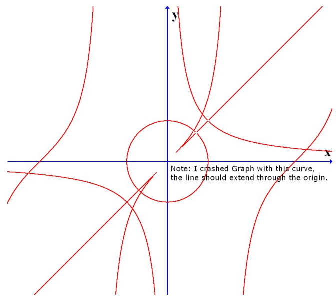
Example 3.4. *A line with slope m and y-intercept b can be described by $F(x, y) = mx + b - y = 0$. Alternatively, a line with x-intercept x_o and y-intercept y_o can be described as the level curve $G(x, y) = \frac{x}{x_o} + \frac{y}{y_o} = 1$.*

Example 3.5. *Level curves need not be simple things. They can be lots of simple things glued together in one grand equation:*

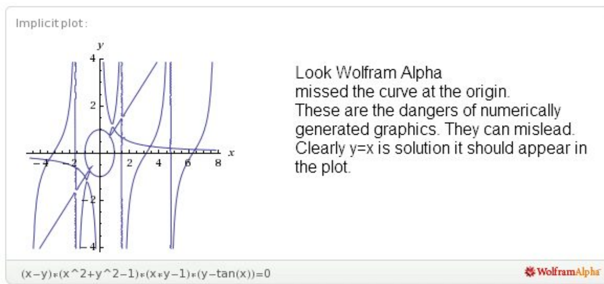
$$F(x, y) = (x - y)(x^2 + y^2 - 1)(xy - 1)(y - \tan(x)) = 0.$$

Solutions to the equation above include the line $y = x$, the unit circle $x^2 + y^2 = 1$, the tilted-hyperbola known more commonly as the reciprocal function $y = \frac{1}{x}$ and finally the graph of the tangent. Some of these intersect, others are disconnected from each other.

It is sometimes helpful to use software to plot equations. However, we must be careful since they are not as reliable as you might suppose. The example above is not too complicated but look what happens with Graph:



Wolfram Alpha shares the same fate:



I hope Mathematica proper fairs better...

Theorem 3.6. *any graph of a function can be wriiten as a level curve.*

If $y = f(x)$ is the graph of a function then we can write $F(x, y) = f(x) - y = 0$ hence the graph $y = f(x)$ is also a level curve.

Not much of a theorem. But, it's true. The converse is not true without a lot of qualification. Its a little harder. Basically, the theorem below gives a criteria for when we can undo a level curve and rewrite is as a single function of x.

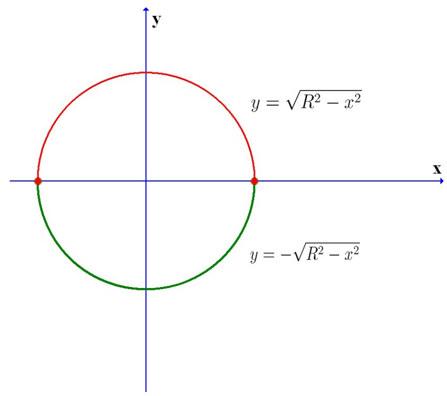
Theorem 3.7. *sometimes a level curve can be locally represented as the graph of a function.*

Suppose (x_o, y_o) is a point on the level curve $F(x, y) = k$ hence $F(x_o, y_o) = k$. We say the level curve $F(x, y) = k$ is **locally represented by a function** $y = f(x)$ at (x_o, y_o) iff $F(x, f(x)) = k$ for all $x \in B_\delta(x_o)$ for some $\delta > 0$. Claim: if

$$\frac{\partial F}{\partial y}(x_o, y_o) = \left(\frac{d}{dy}F(x_o, y)\right)\bigg|_{y=y_o} \neq 0$$

and the $\frac{\partial F}{\partial y}$ is continuous near (x_o, y_o) then $F(x, y) = k$ is locally represented by some function near (x_o, y_o).

The theorem above is called the **implicit function theorem** and its proof is nontrivial. Its proper statement is given in Advanced Calculus (Math 332). I'll just illustrate with the circle: $F(x, y) = x^2 + y^2 = R^2$ has $\frac{\partial F}{\partial y} = 2y$ which is continuous everywhere, however at $y = 0$ we have $\frac{\partial F}{\partial y} = 0$ which means the implicit function theorem might fail. On the circle, $y = 0$ when $x = \pm R$ which are precisely the points where we cannot write $y = f(x)$ for just one function. For any other point we may write either $y = \sqrt{R^2 - x^2}$ or $y = -\sqrt{R^2 - x^2}$ as a local solution of the level curve.



Remark 3.8. *finding the formula for a local solution generally a difficult problem.*

The implicit function theorem is an existence theorem. It merely says there exists a solution given a certain criteria, howeve it does not tell us how to solve the equation to find the formula for the local function. Sometimes we are just content to have an equation which implicitly defines a function of x. For example, most sane creatures do not try to solve $y^5 + y^4 + 3y^2 - y + 1 = x$ for y. Or $\sin(xy) = 3 + y$. When faced with such curves we prefer the level curve description. We've already done some work on this in Calculus I when we did implicit differentiation. The idea of implicit differentiation is only logical when we can write $y = f(x)$, so we must rely on the implicit function theorem as a backdrop to any implicit differentiation problem (see §**??** if you forgot about implicit differentiation).

4 parametrized curves

The idea of a parametrized curve is probably simpler than the definition below appears. In short, we want to take the real number line, or some subset, and paste it into the plane. Think of taking a string and placing it on a table. You can place the string in a great variety of patterns. Imagine the string has little markers placed along it which say $t = 1$, $t = 2$ etc... the value of this label will tell us at which point we are on the string[2]. That label is called the "parameter".

Definition 4.1. *Parametrization of a curve.*

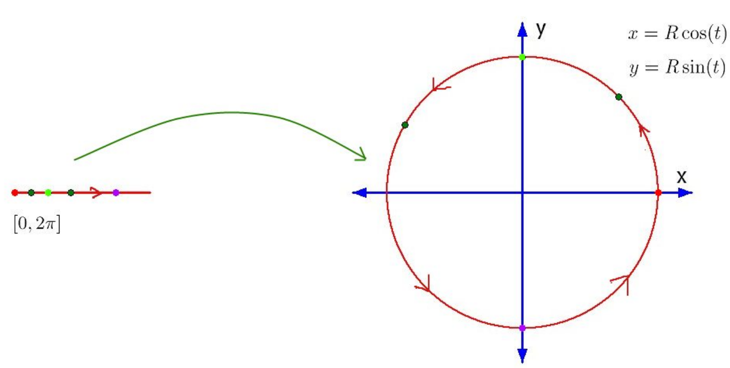
Let C be some curve in the plane. A parametrization of the curve C is a pair of functions $f, g : J \subseteq \mathbb{R} \to \mathbb{R}$ such that $C = \{(f(t), g(t)) \mid t \in J\}$. In other words, a parametric curve is a mapping from $J \subseteq \mathbb{R} \to \mathbb{R}^2$ given by the rule $t \mapsto (f(t), g(t))$ for each $t \in J$. We say that t is the parameter and that the parametric equations for the curve are $x = f(t)$ and $y = g(t)$. In the case $J = [a, b]$ we say that $(f(a), g(a))$ is the initial point and $(f(b), g(b))$ is the terminal point. We often use the notation x for f and y for g when it is convenient. Furthermore, the notation t is just one choice for the label of the parameter, we also may use s or λ or other demarcations.

Finding the parametric equations for a curve does require a certain amount of creativity. However, it's almost always some slight twist on the examples I give in this section.

Example 4.2. *Let $x = R\cos(t)$ and $y = R\sin(t)$ for $t \in [0, 2\pi]$. This is a parametrization of the circle of radius R centered at the origin. We can check this by substituting the equations back into our standard Cartesian equation for the circle:*

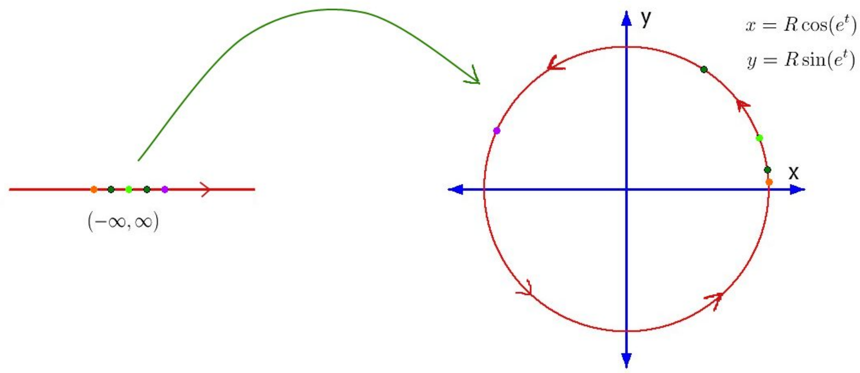
$$x^2 + y^2 = (R\cos(t))^2 + (R\sin(t))^2 = R^2(\cos^2(t) + \sin^2(t))$$

Recall that $\cos^2(t) + \sin^2(t) = 1$ therefore, $x(t)^2 + y(t)^2 = R^2$ for each $t \in [0, 2\pi]$. This shows that the parametric equations do return the set of points which we call a circle of radius R. Moreover, we can identify the parameter in this case as the standard angle from standard polar coordinates.



[2] the parameter can also be thought of as time as the next section discusses at length

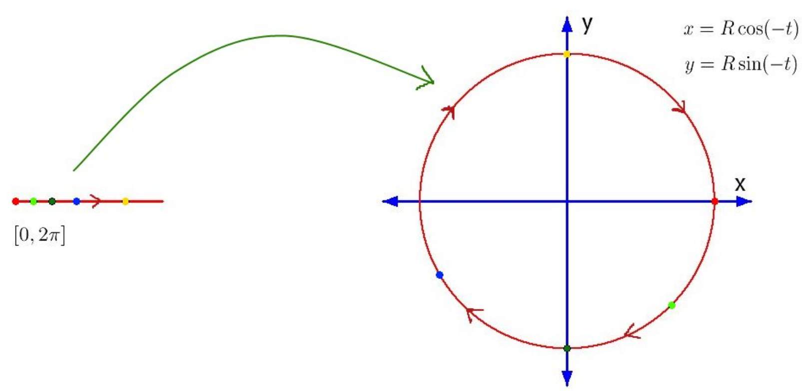
Example 4.3. *Let $x = R\cos(e^t)$ and $y = R\sin(e^t)$ for $t \in \mathbb{R}$. We again cover the circle at t varies since it is still true that $(R\cos(e^t))^2 + (R\sin(e^t))^2 = R^2(\cos^2(e^t) + \sin(e^t)) = R^2$. However, since range$(e^t) = [1, \infty)$ it is clear that we will actually wrap around the circle infinitly many times. The parametrizations from this example and the last do cover the same set, but they are radically different parametrizations: the last example winds around the circle just once whereas this example winds around the circle ∞-ly many times.*



Example 4.4. *Let $x = R\cos(-t)$ and $y = R\sin(-t)$ for $t \in [0, 2\pi]$. This is a parametrization of the circle of radius R centered at the origin. We can check this by substituting the equations back into our standard Cartesian equation for the circle:*

$$x^2 + y^2 = (R\cos(-t))^2 + (R\sin(-t))^2 = R^2(\cos^2(-t) + \sin^2(-t))$$

Recall that $\cos^2(-t) + \sin^2(-t) = 1$ therefore, $x(t)^2 + y(t)^2 = R^2$ for each $t \in [0, 2\pi]$. This shows that the parametric equations do return the set of points which we call a circle of radius R. Moreover, we can identify the parameter an angle measured CW[3] from the positive x-axis. In contrast, the standard polar coordinate angle is measured CCW from the postive x-axis. Note that in this example we cover the circle just once, but the direction of the curve is opposite that of Example 4.2.



The idea of directionality is not at all evident from Cartesian equations for a curve. Given a graph $y = f(x)$ or a level-curve $F(x, y) = k$ there is no intrinsic concept of direction ascribed to the curve. For example, if I ask you whether $x^2 + y^2 = R^2$ goes CW or CCW then you ought not have an answer. I suppose you could ad-hoc pick a direction, but it wouldn't be natural. This means that if we care about giving a direction to a curve we need the concept of the parametrized curve. We can use the ordering of the real line to induce an ordering on the curve.

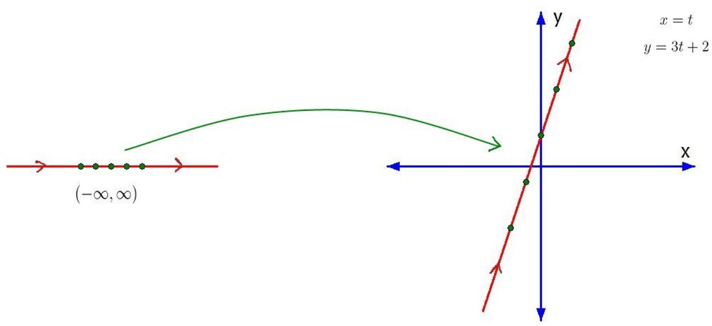
[3] CW is an abbreviation for ClockWise,whereas CCW is an abbreviation for CounterClockWise.

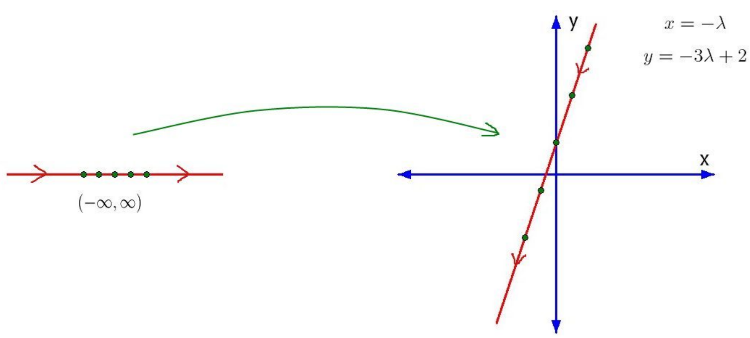
Definition 4.5. *oriented curve.*

> Suppose $f, g : J \subseteq \mathbb{R} \to \mathbb{R}$ are $1-1$ functions. We say the set $\{(f(t), g(t)) \mid t \in J\}$ is an **oriented curve** and say $t \to (f(t), g(t))$ is a consistently oriented **path** which covers C. If $J = [a, b]$ and $(f(a), g(a)) = p$ and $(f(b), g(b)) = q$ then we can say that C is a curve from to p to q.

I often illustrate the orientation of a curve by drawing little arrows along the curve to indicate the direction. Furthermore, in my previous definition of parametrization I did not insist the parametric functions were $1-1$, this means that those parametrizations could reverse direction and go back and forth along a given curve. What is meant by the terms "path", "curve" and "parametric equations" may differ from text to text so you have to keep a bit of an open mind and try to let context be your guide when ambguity occurs. I will try to be uniform in my langauge within this course.

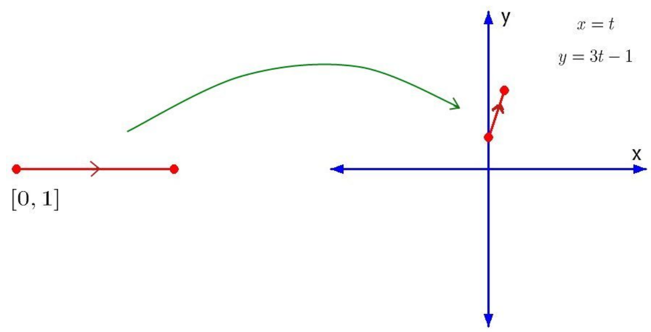
Example 4.6. *The line $y = 3x + 2$ can be parametrized by $x = t$ and $y = 3t + 2$ for $t \in \mathbb{R}$. This induces an orientation which goes from left to right for the line. On the other hand, if we use $x = -\lambda$ and $y = -3\lambda + 2$ then as λ increases we travel from right to left on the curve. So the λ-equations give the line the opposite orientation.*

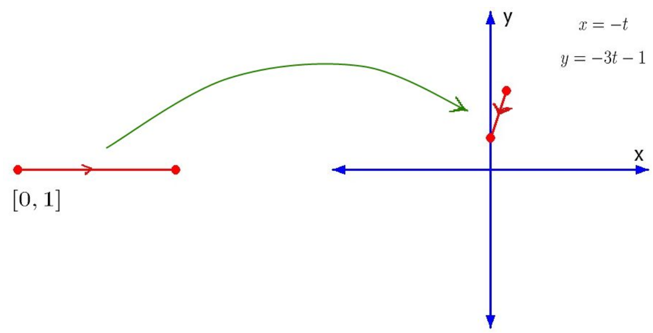




To reverse orientation for $x = f(t), y = g(t)$ for $t \in J = [a, b]$ one may simply replace t by $-t$ in the parametric equations, this gives new equations which will cover the same curve via $x = f(-t), y = g(-t)$ for $t \in [-b, -a]$. This is one of several methods to create a reversed curve. An alternative method is to use $x = f(a + b - t)$ and $y = g(a + b - t)$ for $t \in [a, b]$.

Example 4.7. *The line-segment from $(0, -1)$ to $(1, 2)$ can be parametrized by $x = t$ and $y = 3t - 1$ for $0 \leq t \leq 1$. On the other hand, the line-segment from $(1, 2)$ to $(0, -1)$ is parametrized by $x = -t, y = -3t - 1$ for $-1 \leq t \leq 0$.*





The other method to graph parametric curves is simply to start plugging in values for the parameter and assemble a table of values to plot. I have illustrated that in part by plotting the green dots in the domain of the parameter together with their images on the curve. Those dots are the results of plugging in the parameter to find corresponding values for x, y. I don't find that is a very reliable approach in the same way I find plugging in values to $f(x)$ provides a very good plot of $y = f(x)$. That sort of brute-force approach is more appropriate for a CAS system. There are many excellent tools for plotting parametric curves, hopefully I will have some posted on the course website. In addition, the possibility of animation gives us an even more exciting method for visualization of the time-evolution of a parametric curve. In the next section we connect the parametric viewpoint with physics and such an animation actually represents the physical motion of some object. My focus in the remainder of this section is almost purely algebraic, I could draw pictures to explain, but I wanted the notes to show you that the pictures are not necessary when you understand the algebraic process. That said, the best approach is to do some combination of algebraic manipulation/figuring and graphical reasoning.

Remark 4.8. *Ahhhhhh! what is t ???*

If you are at all like me when I first learned about parametric curves you're probably wondering what is t ? You probably, like me, suppose incorrectly that t should be just like x or y. There is a crucial difference between x and y and t. The notations x and y are actually shorthands for the Cartesian coordinate maps $x : \mathbb{R}^2 \to \mathbb{R}$ and $y : \mathbb{R}^2 \to \mathbb{R}$ where $x(a, b) = a$ and $y(a, b) = b$. When I use the notaion $x = 3$ then you know what I mean, you know that I'm focusing on the vertical line with first coordinate 3. On the other hand, if I say $t = 3$ and ask where is it? Then you should say, you question doesn't make sense. The concept of t is tied to the curve for which it is the parameter. There are infinitely many geometric meanings for t. In other words, if you try to find t in the xy-plane without regard to a curve then you'll never find an answer. It's a meaningless question.

On the other hand if we are given a curve and ask what the meaning of t is for that curve then we ask a meaningful question. There are two popular meanings.

1. the parameter s measures the arclength from some base point on the given curve.

2. the parameter t gives the time along the curve.

In case (1.) for an oriented curve this actually is uniquely specified if we have a starting point. Such a parameterization is called the **arclength parametrization** or **unit-speed** parametrization of a curve. These play a fundamental role in the study of the differential geometry of curves. In case (2.) we have in mind that the curve represents the physical trajectory of some object, as t increases, time goes on and the object moves. I tend to use (2.) as my conceptual backdrop. But, keep in mind that these are just applications of parametric curves. In general, the parameter need not be time or arclength. It might just be what is suggested by algebraic convenience: that is my primary motivator in this section.

5 converting to and from the parametric viewpoint

Let's change gears a bit, we've seen that parametric equations for curves give us a new method to describe particular geometric concepts such as orientability or multiple covering. Without the introduction of the parametric concept these geometric ideas are not so easy to describe. That said, I now turn to the question of how to connect parametric descriptions with Cartesian descriptions of a curve. We'd like to understand how to go both ways if possible:

1. how can we find the Cartesian form for a given parametric curve?

2. how can we find a parametrization of a given Cartesian curve?

In case (2.) we mean to include the ideas of level curves and graphs. It turns out that both questions can be quite challenging for certain examples. However, in other cases, not so much: for example any graph $y = f(x)$ is easily recast as the set of parametric equations $x = t$ and $y = f(t)$ for $t \in dom(f)$. For the standard graph of a function we use x as the parameter.

5.1 how can we find the Cartesian form for a given parametric curve?

Example 5.1. *What curve has parametric equations $x = t$ for $y = t^2$ for $t \in \mathbb{R}$? To find Cartesian equation we eliminate the parameter (when possible)*

$$t^2 = x^2 = y \quad \Rightarrow \quad y = x^2$$

Thus the Cartesian form of the given parametrized curve is simply $y = x^2$.

Example 5.2. *Example 15.2.2: Find parametric equations to describe the graph $y = \sqrt{x+3}$ for $0 \leq x < \infty$. We can use $x = t^2$ and $y = \sqrt{t^2+3}$ for $t \in \mathbb{R}$. Or, we could use $x = \lambda$ and $y = \sqrt{\lambda+3}$ for $\lambda \in [0, \infty)$.*

Example 5.3. *What curve has parametric equations $x = t$ for $y = t^2$ for $t \in [0, 1]$? To find Cartesian equation we eliminate the parameter (when possible)*

$$t^2 = x^2 = y \quad \Rightarrow \quad y = x^2$$

Thus the Cartesian form of the given parametrized curve is simply $y = x^2$, however, given that $0 \leq t \leq 1$ and $x = t$ it follows we do not have the whole parabola, instead just $y = x^2$ for $0 \leq x \leq 1$.

Example 5.4. *Identify what curve has parametric equations $x = \tan^{-1}(t)$ and $y = \tan^{-1}(t)$ for $t \in \mathbb{R}$. Recall that range$(\tan^{-1}(t)) = (-\pi/2, \pi/2)$. It follows that $-\pi/2 < x < \pi/2$. Naturally we just equate inverse tangent to obtain $\tan^{-1}(t) = y = x$. The curve is the open line-segment with equation $y = x$ for $-\pi/2 < x < \pi/2$. This is an interesting parameterization, notice that as $t \to \infty$ we approach the point $(\pi/2, \pi/2)$, but we never quite get there.*

Example 5.5. *Consider $x = \ln(t)$ and $y = e^t - 1$ for $t \geq 1$. We can solve both for t to obtain*

$$t = e^x = \ln(y+1) \quad \Rightarrow \quad y = -1 + exp(exp(x)).$$

The domain for the expression above is revealed by analyzing $x = \ln(t)$ for $t \geq 1$, the image of $[1, \infty)$ under natural log is precisely $[0, \infty)$; $\ln[1, \infty) = [0, \infty)$.

Example 5.6. *Suppose $x = \cosh(t) - 1$ and $y = 2\sinh(t) + 3$ for $t \in \mathbb{R}$. To eliminate t it helps to take an indirect approach. We recall the most important identity for the hyperbolic sine and cosine: $\cosh^2(t) - \sinh^2(t) = 1$. Solve for hyperbolic cosine; $\cosh(t) = x + 1$. Solve for hyperbolic sine; $\sinh(t) = \frac{y-3}{2}$. Now put these together via the identity:*

$$\cosh^2(t) - \sinh^2(t) = 1 \quad \Rightarrow \quad (x+1)^2 - \frac{(y-3)^2}{4} = 1.$$

Note that $\cosh(t) \geq 1$ hence $x + 1 \geq 1$ thus $x \geq 0$ for the curve described above. On the other hand y is free to range over all of $\mathbb{R}$ since hyperbolic sine has range $\mathbb{R}$. You should[4] recognize the equation as a hyperbola centered at $(-1, 3)$.

[4] many students need to review these at this point, we use circles, ellipses and hyperbolas as examples in this course. I'll give examples of each in this section.

5.2 how can we find a parametrization of a given Cartesian curve?

I like this topic more, the preceding bunch of examples, while needed, are boring. The art of parameterizing level curves is much more fun.

Example 5.7. Find parametric equations for the circle centered at (h, k) with radius R.

To begin recall the equation for such a circle is $(x-h)^2 + (y-k)^2 = R^2$. Our inspiration is the identity $\cos^(t) + \sin^2(t) = 1$. Let $x - h = R\cos(t)$ and $y - k = R\sin(t)$ thus

$$\boxed{x = h + R\cos(t)} \qquad \text{and} \qquad \boxed{y = k + R\sin(t).}$$

I invite the reader to verify these do indeed parametrize the circle by explicitly plugging in the equations into the circle equation. Notice, if we want the whole circle then we simply choose any interval for t of length 2π or longer. On the other hand, if you want to select just a part of the circle you need to think about where sine and cosine are positive and negative. For example, if I want to parametrize just the part of the circle for which $x > h$ then I would choose $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

The reason I choose that intuitively is that the parametrization given for the circle above is basically built from polar coordinates[5] centered at (h, k). That said, to be sure about my choice of parameter domain I like to actually plug in some of my proposed domain and make sure it matches the desired criteria. I think about the graphs of sine and cosine to double check my logic. I know that $\cos(-\frac{\pi}{2}, \frac{\pi}{2}) = (0, 1]$ whereas $\sin(-\frac{\pi}{2}, \frac{\pi}{2}) = (-1, 1)$, I see it in my mind. Then I think about the parametric equations in view of those facts,

$$x = h + R\cos(t) \qquad \text{and} \qquad y = k + R\sin(t).$$

I see that x will range over $(h, h+R]$ and y will range over $(k-R, k+R)$. This is exactly what I should expect geometrically for half of the circle. Visualize that $x = h$ is a vertical line which cuts our circle in half. These are the thoughts I think to make certain my creative leaps are correct. I would encourage you to think about these matters. Don't try to just memorize everything, it will not work for you, there are simply too many cases. It's actually way easier to just understand these as a consequence of trigonometry, algebra and analytic geometry.

Example 5.8. Find parametric equations for the level curve $x^2 + 2x + \frac{1}{4}y^2 = 0$ which give the ellipse a CW orientation.

To begin we complete the square to understand the equation:

$$x^2 + 2x + \frac{1}{4}y^2 = 0 \quad \Rightarrow \quad (x+1)^2 + \frac{1}{4}y^2 = 1.$$

We identify this is an ellipse centered at $(-1, 0)$. Again, I use the pythagorean trig. identity as my guide: I want $(x+1)^2 = \cos^2(t)$ and $\frac{1}{4}y^2 = \sin^2(t)$ because that will force the parametric equations to solve the ellipse equation. However, I would like for the equations to describe CW direction so I replace the t with $-t$ and propose:

$$\boxed{x = -1 + \cos(-t)} \qquad \text{and} \qquad \boxed{y = 2\sin(-t)}$$

If we choose $t \in [0, 2\pi)$ then the whole ellipse will be covered. I could simplify $\cos(-t) = \cos(t)$ and $\sin(-t) = -\sin(t)$ but I have left the minus to emphasize the idea about reversing the orientation. In the preceding example we gave the circle a CCW orientation.

[5]we will discuss further in a later section, but this should have been covered in at least your precalculus course.

Example 5.9. Find parametric equations for the part of the level curve $x^2 - y^2 = 1$ which is found in the first quadrant.

We recognize this is a hyperbola which opens horizontally since $x = 0$ gives us $-y^2 = 1$ which has no real solutions. Hyperbolic trig. functions are built for a problem just such as this: recall $\cosh^2(t) - \sinh^2(t) = 1$ *thus we choose $x = \cosh(t)$ and $y = \sinh(t)$. Furthermore, the hyperbolic sine function* $\sinh(t) = \frac{1}{2}\left(e^t - e^{-t}\right)$ *is everywhere increasing since it has derivative* $\cosh(t)$ *which is everywhere positive. Moreover, since* $\sinh(0) = 0$ *we see that* $\sinh(t) \geq 0$ *for $t \geq 0$. Choose non-negative t for the domain of the parametrization:*

$$\boxed{x = \cosh(t), \qquad y = \sinh(t), \qquad t \in [0, \infty).}$$

Example 5.10. Find parametric equations for the part of the level curve $x^2 - y^2 = 1$ which is found in the third quadrant.

Based on our thinking from the last example we just need to modify the solution a bit:

$$\boxed{x = -\cosh(t), \qquad y = \sinh(t), \qquad t \in (-\infty, 0].}$$

Note that if $t \in (-\infty, 0]$ then $-\cosh(t) \leq -1$ and $\sinh(t) \leq 0$, this puts us in the third quadrant. It is also clear that these parametric equations to solve the hyperbola equation since

$$(-\cosh(t))^2 - (\sinh(t))^2 = \cosh^2(t) - \sinh^2(t) = 1.$$

The examples thus far are rather specialized, and in general there is no method to find parametric equations. This is why I said it is an art.

Example 5.11. Find parametric equations for the level curve $x^2y^2 = x - 2$.

This example is actually pretty easy because we can solve for $y^2 = \frac{x-2}{x^2}$ hence $y = \pm\sqrt{\frac{x-2}{x^2}}$. We can choose x as parameter so the parametric equations are just

$$x = t \quad \text{and } y = \sqrt{\frac{t-2}{t^2}}$$

for $t \geq 2$. Or, we could give parametric equations

$$x = t \quad \text{and } y = -\sqrt{\frac{t-2}{t^2}}$$

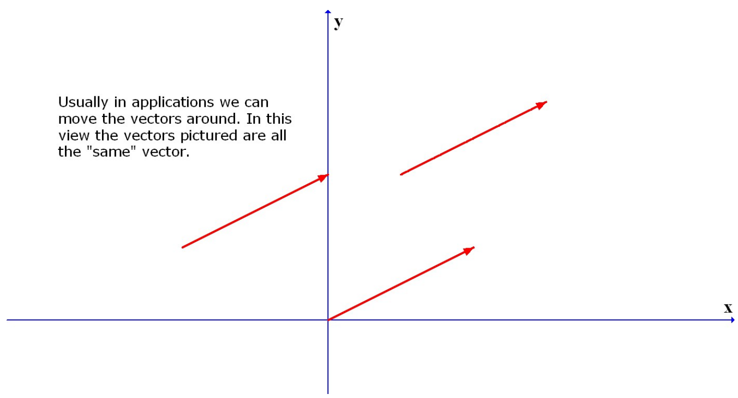
for $t \geq 2$. These parametrizations simply cover different parts of the same level curve.

6 Calculus of Parametrized Curves

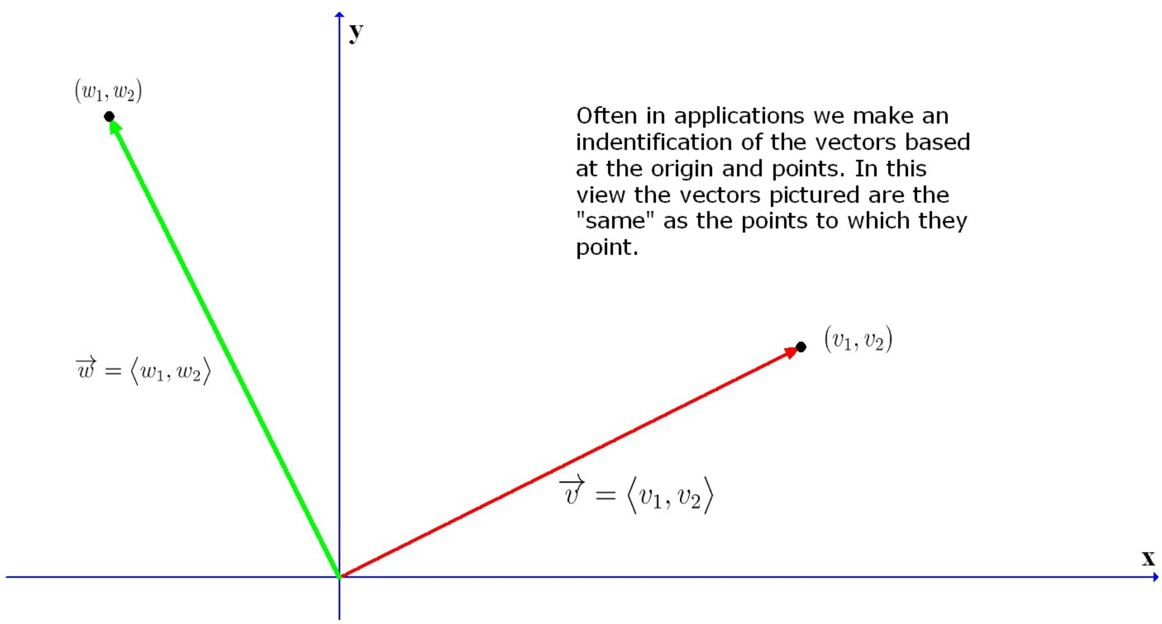
In the preceding section my intention was to introduce you to the basic mathematical structure of parametrized curves and argue for why they are mathematically necessary. In this chapter we take the next step: we learn what we can do with parametrized curves in the plane. In particular, we study the physics of motion from this viewpoint. Parametric equations are the natural choice for physics because in physics we gain a meaning for the parameter. In classical nonrelativistic physics the parameter is time. The nice thing about the parametric viewpoint is that there is no preference between x or y. Motion in the x-direction and motion in the y-direction are both covered by the same calculus. To fully appreciate this formalism it helps to think of the x and y equations together as a single object. This is precisely what the vector formalism does for us. In addition, vectors play an important role in both physics and calculus III. In this course we will work primarily with two-dimensional vectors. The notation I'll use is as follows:

1. (a, b) is a point in $\mathbb{R}^2$
2. $< a, b >$ is a vector in $\mathbb{R}^2$

A vector is a directed line segment whereas a point is a point. Vectors you can move around the plane whereas a point is stuck.



These are our **conventions**[6].



[6]often points and vectors are identified since they are in 1-1 correspondance. Moreover, you could think about vectors which are stuck in place. The conventions I adopt here are simply to avoid a multiplication of notation.

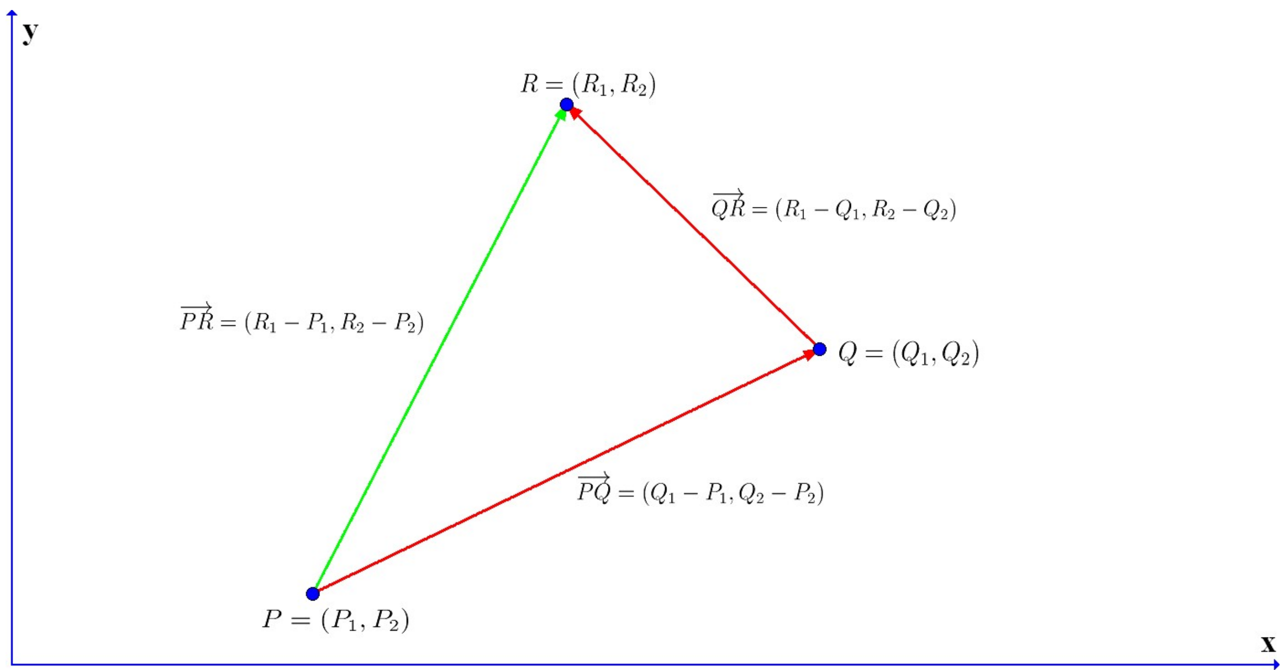
7 vectors in a nutshell

Definition 7.1. *vector terminology.*

Given two points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ we can construct the vector from p to q by

$$\overrightarrow{pq} =< q_1 - p_1, q_2 - p_2 > .$$

We call directed line segments **vectors** and denote the set of all such objects by V, we also write a little arrow over such objects to emphasize that they are vectors. If $\vec{v} \in V$ then we can write the vector in Cartesian form $\vec{v} =< v_1, v_2 >$. We say v_1 is the x-component of $\vec{v}$ and v_2 is the y-component of $\vec{v}$. Two vectors are equal iff both the x and y components are equal.

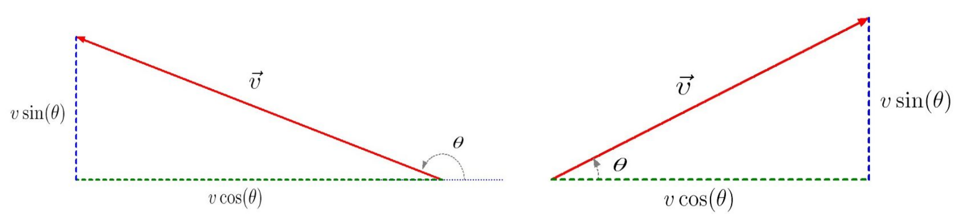


The main thing to remember is that a vector has both a length and a direction. However, it is often more convenient to think of the vector in terms of it Cartesian components. The definition that follows tells us how to work with vectors. They're really quite simple:

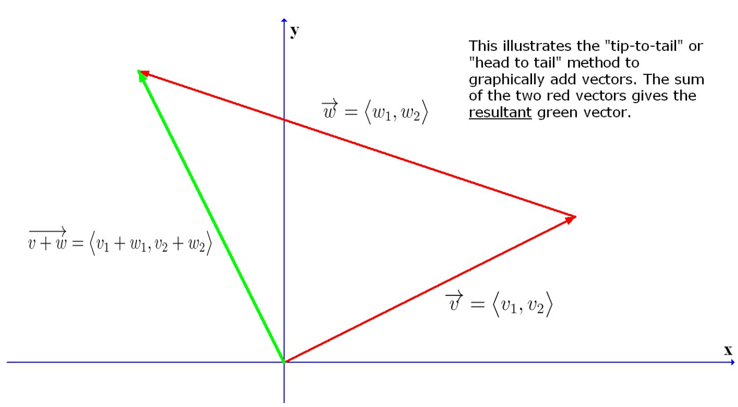
Definition 7.2. *vector addition, subtraction, lengths and standard angles.*

> Suppose $< v_1, v_2 >, < w_1, w_2 > \in V$ and $c \in \mathbb{R}$ we define
>
> 1. addition of vectors is done component-wise;
> $\langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle = \langle v_1 + w_1, v_2 + w_2 \rangle$.
> 2. subtraction of vectors is done component-wise;
> $\langle v_1, v_2 \rangle - \langle w_1, w_2 \rangle = \langle v_1 - w_1, v_2 - w_2 \rangle$.
> 3. scalar multiplication is defined by; $c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle$.
> 4. the length of a vector is defined to by
>
> $$\|\langle v_1, v_2 \rangle\| = \sqrt{(v_1)^2 + (v_2)^2}.$$
>
>   Moreover, if $\vec{v} = \langle v_1, v_2 \rangle$ then we also write $v = \|\vec{v}\|$. If $\vec{v} \neq \langle 0, 0 \rangle$ then the unit vector in the $\vec{v}$-direction is a vector of length one denoted with a hat: $\hat{v} = \frac{1}{v}\vec{v}$. We can always write $\vec{v} = v\hat{v}$ for a nonzero vector. In words, v is the **magnitude** and $\hat{v}$ is the **direction** vector.
> 5. the standard angle of a nonzero vector $\langle v_1, v_2 \rangle$ is an angle θ such that
>
> $$v_1 = v\cos(\theta) \qquad \text{and} \qquad v_2 = v\sin(\theta).$$

I have made a point in (5.) to help you avoid the pitfalls of carelessly solving $\tan(\theta) = \frac{v_2}{v_1}$ for $\theta = \tan^{-1}\left(\frac{v_2}{v_1}\right)$. This formula is only correct for angles that point in quadrants I and IV whereas (5.) needs no qualification.



The definition for vector addition is also very natural if you look at the picture below:



Example 7.3. *Let* $\vec{v} =< 1, 2 >$ *and* $\vec{w} =< 3, 4 >$ *then we can calculate:*

$$\begin{aligned} \vec{v} + \vec{w} &= < 1+3, 2+4 >=< 4, 6 > \\ \vec{v} - \vec{w} &= < 1-3, 2-4 >=< -2, -2 > = \ -2 < 1, 1 > \\ 2\vec{v} - 3\vec{w} &= \ 2 < 1, 2 > -3 < 3, 4 > = < 2, 4 > - < 9, 12 > = < -7, -8 > \end{aligned}$$

Note that $v = \sqrt{1+4} = \sqrt{5}$ *and* $w = \sqrt{9+16} = 5$. *However, the length of* $\vec{v} + \vec{w}$ *is not* $v + w$. *Instead,* $||\vec{v} + \vec{w}|| = || < 4, 6 > || = \sqrt{16+36} = \sqrt{52}$.

Example 7.4. Find a vector of length 6 **in the same direction as** $\vec{v} =< 1, 1 >$**.** *Let's call the vector were trying to find* $\vec{w}$. *If* $\vec{w}$ *points in the same direction as* $\vec{v}$ *then there exists some constant* $k > 0$ *such that* $\vec{w} = kv =< k, k >$. *We also want* $w = 6$ *thus* $6 = \sqrt{k^2 + k^2} = k\sqrt{2}$. *Therefore,* $\vec{w} = \frac{6}{\sqrt{2}} < 1, 1 >$.

Example 7.5. *The standard angle for* $< -1, -1 >$ *is an angle* θ *such that* $-1 = \sqrt{2}\cos(\theta)$ *and* $-1 = \sqrt{2}\sin(\theta)$. *We must solve* $\cos(\theta) = -1/\sqrt{2}$ *and* $\sin(\theta) = -1/\sqrt{2}$. *The solution is* $\boxed{\theta = \frac{3\pi}{4}}$. *(in quadrant III)*

Note $\tan^{-1}(-1/-1) = \frac{\pi}{4}$ is not the correct angle in the preceding example. You have to think about both components when determining standard angle. The inverse tangent will mislead you unless you really understand what you are doing.

Example 7.6. *The standard angle for* $< -\sqrt{3}, 1 >$ *is an angle* θ *such that* $-\sqrt{3} = 2\cos(\theta)$ *and* $1 = 2\sin(\theta)$. *We must solve* $\cos(\theta) = -2/\sqrt{3}$ *and* $\sin(\theta) = 1/2$. *The solution is* $\boxed{\theta = \frac{5\pi}{6}}$. *(in quadrant II)*

Once more, the inverse tangent fails us since $\tan^{-1}(-1/\sqrt{3}) = \frac{-\pi}{6}$. To use inverse tangent correctly you must adjust answers when they ought to be in quadrants II and III.

Definition 7.7. *dot-product*

Let $\vec{A} = \langle A_1, A_2\rangle$ and $\vec{B} = \langle B_1, B_2\rangle$ the $\vec{A} \bullet \vec{B} = A_1B_1 + A_2B_2$.

Notice $\vec{A} \bullet \vec{A} = A_1^2 + A_2^2 = ||\vec{A}||^2 = A^2$ in the notation given in this section. Therefore, we can use the dot-product to create the usual formula for vector-length

$$A = \sqrt{\vec{A} \bullet \vec{A}}$$

I usually prove the following theorem in Calculus III, it is essentially the law of cosines:

Theorem 7.8.

$\vec{A} \bullet \vec{B} = AB\cos\theta$ where θ is the angle between $\vec{A}$ and $\vec{B}$.

We worked examples in class which showed how to use the formula above to find the angle between nonzero vectors.

8 calculus for vector-valued functions

A vector valued function of a real variable is an assignment of of vector for each real number in some domain. It's a mapping $t \mapsto \left\langle f(t), g(t) \right\rangle$ for each $t \in J \subset \mathbb{R}$.

Definition 8.1. *vector-valued functions.*

We say $\vec{v}$ is a vector-valued function of a real variable if $\vec{v} : J \subseteq \mathbb{R} \to V$ is a function. In addition if $\vec{v}(t) = \langle v_1(t), v_2(t) \rangle$ then we say v_1 is the x-component function of $\vec{v}$ and v_2 is the y-component function of $\vec{v}$.

If our vectors are all based from the origin outward then we can write $V = \mathbb{R}^2$. Think about it: each point is naturally identified with the vector which points from the origin out to the point. The identification amounts to saying that $\langle a, b \rangle = (a, b)$ in our current notation. With this identification in mind we can write parametric curves in nice vector notation:

Example 8.2. *Recall the circle centered at the origin of radius R had parametric equations $x = R\cos(t)$ and $y = R\sin(t)$ for $t \in [0, 2\pi]$. We can group these together and simply say that the circle is parametrized by $\vec{r}(t) =< R\cos(t), R\sin(t) >$.*

One is often asked to find the parametrization of a line-segment from a point p to a point q. I recommend the following approach: for $0 \leq t \leq 1$ let

$$\vec{r}(t) = p(1-t) + tq.$$

It's easy to calculate $\vec{r}(0) = p$ and $\vec{r}(1) = q$. This formula can also be written as

$$\vec{r}(t) = p + t(q-p) = p + t\ [\overrightarrow{pq}].$$

Example 8.3. Find the parametrization of a line segment which goes from $(1, 3)$ to $(5, 2)$. *We use the comment preceding this example and construct:*

$$\vec{r}(t) = (1, 3) + t[(5, 2) - (1, 3)] =< 1 + 4t, 3 - t >$$

And now for the calculus. In short, we just do everything component by component.

Definition 8.4. *calculus of vector-valued functions.*

We say $\vec{v}$ is a vector-valued function of a real variable where $\vec{v}(t) = \langle v_1(t), v_2(t) \rangle$ then

1. If v_1 and v_2 are both differentiable functions we define

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}\langle v_1, v_2 \rangle = \left\langle \frac{dv_1}{dt}, \frac{dv_2}{dt} \right\rangle.$$

2. If v_1 and v_2 are both integrable functions on $[a, b]$ then we define

$$\int_a^b \vec{v} dt = \int_a^b \langle v_1, v_2 \rangle dt = \left\langle \int_a^b v_1(t) dt, \int_a^b v_2(t) dt \right\rangle.$$

3. We write $\int \vec{v}(t) dt = \vec{V} + \vec{c}$ iff $\vec{V} = \langle V_1, V_2 \rangle$ and $\vec{c} = \langle c_1, c_2 \rangle$ where $\int v_1(t) dt = V_1 + c_1$ and $\int v_2(t) dt = V_2 + c_2$.

We also use the prime notation for differentiation of vector valued functions if it is convenient; this means $\vec{r}\,'(t) = d\vec{r}/dt = \frac{d\vec{r}}{dt}$.

Example 8.5. *Let $\vec{r}(t) = \langle t, \cos(t) \rangle$.*

$$\frac{d\vec{r}}{dt} = \left\langle \frac{d}{dt}(t), \frac{d}{dt}(\cos(t)) \right\rangle = \langle 1, -\sin(t) \rangle.$$

$$\int \vec{r}(t)dt = \left\langle \int tdt, \int \cos(t)dt \right\rangle = \left\langle \frac{1}{2}t^2 + c_1,\ \sin(t) + c_2 \right\rangle.$$

$$\int_0^1 \vec{r}(t)dt = \left\langle \int_0^1 tdt, \int_0^1 \cos(t)dt \right\rangle = \left\langle \frac{1}{2},\ \sin(1) \right\rangle.$$

Often a vector-valued function will be called a space curve since its image is a curve in space.

Theorem 8.6. *fundamental theorems of calculus for space curves.*

$$(I.)\quad \frac{d}{dt}\int_a^t \vec{r}(\tau)d\tau = \vec{r}(t)$$

$$(II.)\quad \int_a^b \frac{d\vec{r}}{dt}dt = \vec{r}(b) - \vec{r}(a)$$

Proof: Apply the FTC part I and II componentwise to arrive at the corresponding theorems above. $\square$

Theorem 8.7. *rules of calculus for space curves.*

Let $\vec{v}, \vec{w} : J \subseteq \mathbb{R} \to \mathbb{R}^2$ and $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable functions and let $c \in \mathbb{R}$

$$(1.)\quad \frac{d}{dt}\Big[\vec{v} + \vec{w}\Big] = \frac{d\vec{v}}{dt} + \frac{d\vec{w}}{dt}$$

$$(2.)\quad \frac{d}{dt}\Big[f\vec{v}\Big] = \frac{df}{dt}\vec{v} + f\frac{d\vec{v}}{dt}$$

$$(3.)\quad \frac{d}{dt}\Big[c\vec{v}\Big] = c\frac{d\vec{v}}{dt}$$

$$(4.)\quad \int \big[\vec{v} + \vec{w}\big]dt = \int \vec{v}dt + \int \vec{w}dt$$

$$(5.)\quad \int c\vec{v}dt = c\int \vec{v}dt$$

The proof of the theorem above is easily derived by simply expanding what the vector notation means and borrowing the corresponding theorems from calculus I to simplify the component expressions. I might ask for this in homework so I'll not offer details here.

9 position, velocity and acceleration in two dimensions

In this section the notations $\vec{r}$, $\vec{v}$ and $\vec{a}$ is special and set-apart. I don't use these as abstract variables here with no set meaning. Instead, these are connected as is described in the definition that follows:

Definition 9.1. *position, velocity and acceleration.*

The position, velocity and acceleration of an object are vector-valued functions of time and we define them as follows:

1. $\vec{r}(t)$ is the position at time t.
2. $\vec{v}(t) = \frac{d\vec{r}}{dt}$ is the velocity at time t.
3. $\vec{a}(t) = \frac{d\vec{v}}{dt}$ is the acceleration at time t.

If we are given the position vector as a function of time then we need only differentiate to find the velocity and acceleration. On the other hand, if we are given the acceleration then we need to integrate and apply initial conditions to obtain the velocity and position.

Example 9.2. *Suppose R and ω are positive constants and the motion of an object is observed to follow the path $\vec{r}(t) =< R\cos(\omega t), R\sin(\omega t) >= R < \cos(\omega t), \sin(\omega t) >$. We wish to calculate the velocity and acceleration as functions of time.*

Differentiate to obtain the velocity

$$\boxed{\vec{v}(t) = R\omega < -\sin(\omega t), \cos(\omega t) > .}$$

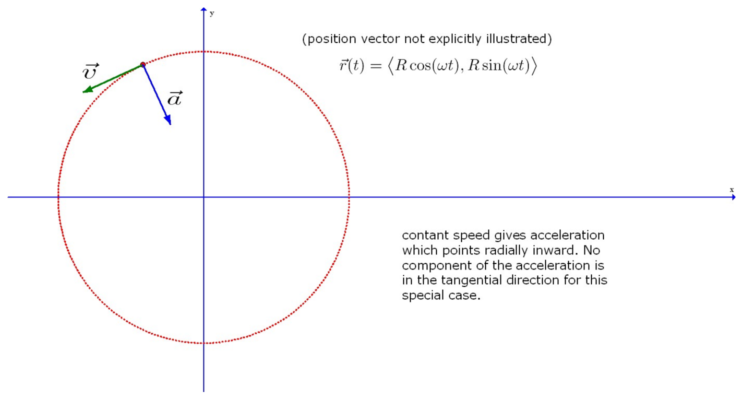
Differentiate once more to obtain the acceleration:

$$\vec{a}(t) = R\omega < -\omega\cos(\omega t), -\omega\sin(\omega t) >= \boxed{-R\omega^2 < \cos(\omega t), \sin(\omega t) > .}$$

Notice we can write that $\vec{a}(t) = -\omega^2\vec{r}(t)$ in this very special example. This means the aacceleration is opposite the direction of the position. Furthermore, we can calculate

$$r = R, \qquad v = R\omega, \qquad a = R\omega^2$$

Thus the magnitudes of the position, velocity and acceleration are all constant. However, their directions are always changing. Perhaps you recognize these equations as the foundational equations describing constant speed circular motion. This acceleration is called the **centripetal** *or center-seeking acceleration since it points towards the center. Here we imagine attaching the acceleration vector to the object which is traveling in the circle.*



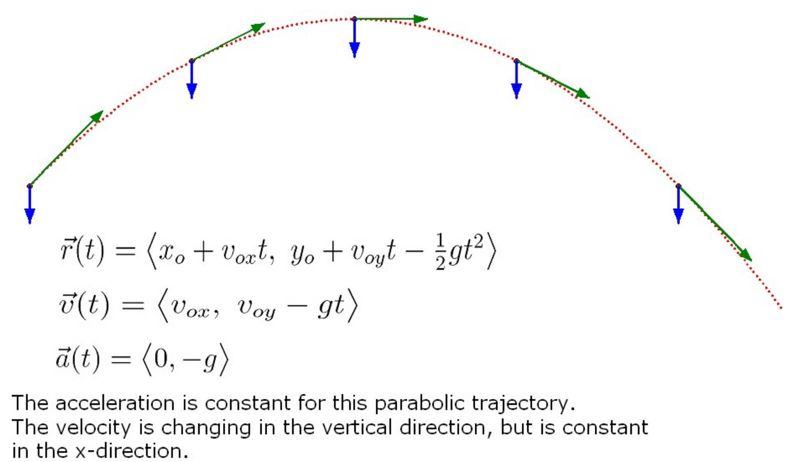
Example 9.3. *Suppose that the acceleration of an object is known to be $\vec{a} =< 0, -g >$ where g is a positive constant. Furthermore, suppose that initially the object is at $\vec{r}_o$ and has velocity $\vec{v}_o$. We wish to calculate the position and velocity as functions of time.*

Integrate the acceleration from 0 *to* t*,*

$$\int_0^t \frac{d\vec{v}}{d\tau} d\tau = \int_0^t a(\tau) d\tau \quad \Rightarrow \quad \vec{v}(t) - \vec{v}(0) = \int_0^t < 0, -g > d\tau \quad \Rightarrow \quad \boxed{\vec{v}(t) = \vec{v}_o + < 0, -gt >}$$

Integrate the velocity from 0 *to* t*,*

$$\int_0^t \frac{d\vec{r}}{d\tau} d\tau = \int_0^t v(\tau) d\tau \quad \Rightarrow \quad \vec{r}(t) - \vec{r}(0) = \int_0^t \big(\vec{v}_o + < 0, -gt >\big) d\tau \quad \Rightarrow \quad \boxed{\vec{r}(t) = \vec{r}_o + t\vec{v}_o + < 0, -\frac{1}{2}gt^2 >}$$



The best understanding of Newtonian Mechanics is given by a combination of both vectors and calculus. We need vectors to phrase the geometry of force addition whereas we need calculus to understand how the position, velocity and acceleration variables change in concert. In calculus II we don't delve too deeply into the geometry of vectors. To do it properly I would also need to discuss dot and cross products and three dimensional geometry. In addition, we will consider a few problems of physics, but the set-up will be fairly predictable and the problems we work on tests will be like those we have already solved. In contrast, I make no such promise in a physics course. In physics we must understand the concepts of physical law. Here our focus is more towards how calculus is used in physics.

9.1 position vs. displacement vs. distance traveled

The position of an object is simply the (x, y) coordinates of the object. Usually it is convenient to think of the position as a vector-valued function of time which we denote $\vec{r}(t)$. The displacement is also a vector, however it compares two possibly distinct positions:

Definition 9.4. *displacement and distance traveled.*

Suppose $\vec{r}(t)$ is the position at time t of some object.

1. The **displacement** from position $\vec{r}_1$ to position $\vec{r}_2$ is the vector $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$.
2. The **distance travelled** during the interval $[t_1, t_2]$ along the curve $t \mapsto \vec{r}(t)$ is given by

$$s_{12} = \int_{t_1}^{t_2} v(t)dt = \int_{t_1}^{t_2} \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt$$

where $v(t) = ||d\vec{r}/dt||$.

Note that the position is the displacement from the origin. Distance travelled is a **scalar** quantity which means it is just a number or if we think of an endpoint as variable it could be a function.

Definition 9.5. *arclength function and speed.*

We define

$$s(t) = \int_{t_1}^{t} v(\tau)d\tau = \int_{t_1}^{t} \sqrt{\frac{dx}{d\tau}^2 + \frac{dy}{d\tau}^2}\, d\tau$$

to be the arclength travelled from time t_1 to t along the parametrized curve $t \mapsto \vec{r}(t)$. Futhermore, we define the **speed** to be the instantaneous rate of change in the arclength; speed is ds/dt.

Notice it is simple to show that the speed is also equal to the magnitude of the velocity; $ds/dt = v$. We'll explore the geometric foundation of the arclength[7] formula in the next section. For now let's examine a few examples:

Example 9.6. *Let $\omega, R > 0$. Suppose $\vec{r}(t) =< R\cos(\omega t), R\sin(\omega t) >$ for $t \geq 0$. We can calculated that*

$$\frac{d\vec{r}}{dt} =< -R\omega\sin(\omega t), R\omega\cos(\omega t) > \quad \Rightarrow \quad v(t) = \sqrt{(-R\omega\sin(\omega t))^2 + (R\omega\cos(\omega t))^2} = \sqrt{R^2\omega^2} = R\omega.$$

Now use this to help calculate the distance travelled during the interval $[0, t]$

$$s(t) = \int_0^t v(\tau)d\tau = \int_{t_1}^t R\omega d\tau = R\omega\tau\Big|_0^t = R\omega t.$$

In other words, $\Delta s = R\omega\Delta t$. On a circle the arclength subtended Δs divided by the radius R is defined to be the radian measure of that arc which we typically denote $\Delta\theta$. We find that $\Delta\theta = \omega\Delta t$ or if you prefer $\omega = \Delta\theta/\Delta t$.

Circular motion which is not at a constant speed can be obtained mathematically by replacing the constant ω with a function of time. Let's examine such an example.

[7] or distance travelled formula if you insist the parameter be time

Example 9.7. *Suppose* $\vec{r}(t) =< R\cos(\theta), R\sin(\theta) >$ *for* $t \geq 0$ *where* $\theta_o, \omega_o, \alpha$ *are constants and* $\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$*. To calculate the distance travelled it helps to first calculate the velocity:*

$$\frac{d\vec{r}}{dt} =< -R(\omega_o + \alpha t)\sin(\theta), R(\omega_o + \alpha t)\cos(\theta) >$$

Next, the speed is the length of the velocity vector,

$$v = \sqrt{[-R(\omega_o + \alpha t)\sin(\theta)]^2 + [R(\omega_o + \alpha t)\cos(\theta)]^2} = R\sqrt{(\omega_o + \alpha t)^2} = R|\omega_o + \alpha t|.$$

Therefore, the distance travelled is given by the integral below:

$$s(t) = \int_0^t R|\omega_o + \alpha\tau| d\tau$$

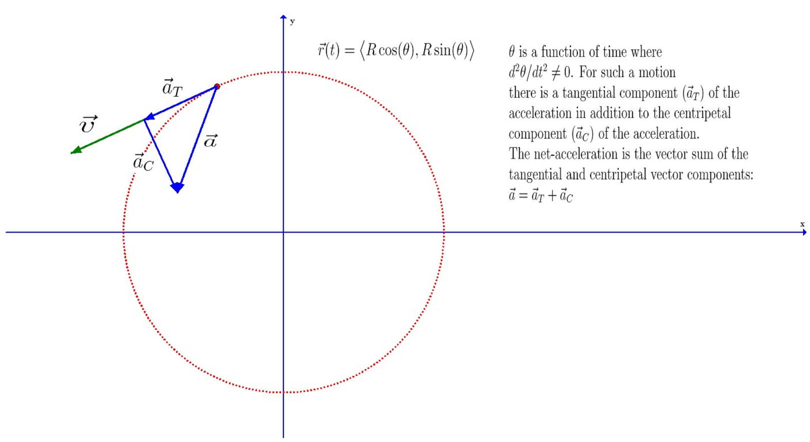
To keep things simple, let's suppose that ω_o, α *are given such that* $\omega_o + \alpha t \geq 0$ *hence* $v = R\omega_o + R\alpha t$*. To suppose otherwise would indicate the motion came to a stopping point and reversed direction, which is interesting, just not to us here.*

$$s(t) = R\int_0^t (\omega_o + \alpha\tau) d\tau = R\omega_o t + \frac{1}{2}R\alpha t^2.$$

Observe that $\theta(t) - \theta_o = (s(t) - s(0))/R$ *thus we find that* $\Delta\theta = \omega_o t + \frac{1}{2}\alpha t^2$ *which is the formula for the angle subtended due to motion at a constant* **angular acceleration** α*. In invite the reader to differentiate the position twice and show that*

$$\vec{a}(t) = \frac{d^2\vec{r}}{dt} = -\underbrace{R\omega^2\Big\langle \cos(\theta(t)), \sin(\theta(t))\Big\rangle}_{centripetal} + \underbrace{R\alpha\Big\langle -\sin(\theta(t)), \cos(\theta(t))\Big\rangle}_{tangential}$$

where $\omega = \omega_o + \alpha t$*.*



Distance travelled is not always something we can calculate in closed form. Sometimes we need to relagate the calculation of the arclength integral to a numerical method. However, the example that follows is still calculable without numerical assistance. It did require some thought.

Example 9.8. *We found that* $\vec{a} =< 0, -g >$ *twice integrated yields a position of* $\vec{r}(t) = \vec{r}_o + t\vec{v}_o + < 0, -\frac{1}{2}gt^2 >$ *for some constant vectors* $\vec{r}_o =< x_o, y_o >$ *and* $\vec{v}_o =< v_{ox}, v_{oy} >$*. Thus,*

$$\vec{r}(t) = \big\langle x_o + v_{ox}t, y_o + v_{oy}t - \frac{1}{2}gt^2\big\rangle$$

From which we can differentiate to derive the velocity,

$$\vec{v}(t) = \langle v_{ox}, v_{oy} - gt \rangle.$$

If you've had any course in physics, or just a proper science education, you should be happy to observe that the zero-acceleration in the x-direction gives rise to constant-velocity motion in the x-direction whereas the gravitational acceleration in the y-direction makes the object fall back down as a consequence of gravity. If you think about $v_{oy} - gt$ *it will be negative for some* $t > 0$ *whatever the initial velocity* v_{oy} *happens to be, this point where* $v_{oy} - gt = 0$ *is the turning point in the flight of the object and it gives the top of the parabolic*[8] *trajectory which is parametrized by* $t \to \vec{r}(t)$*. Suppose* $x_o = y_o = 0$ *and calculate the distance travelled from time* $t = 0$ *to time* $t_1 = v_{oy}/g$*. Additionally, let us assume* $v_{ox}, v_{oy} \geq 0$*.*

$$\begin{aligned} s = \int_0^{t_1} v(t)dt &= \int_0^{t_1} \sqrt{(v_{ox})^2 + (v_{oy} - gt)^2}dt \\ &= \int_{v_{oy}}^{0} \sqrt{(v_{ox})^2 + (u)^2}\left(\frac{du}{-g}\right) \qquad u = v_{oy} - gt \\ &= \frac{1}{g}\int_0^{v_{oy}} \sqrt{(v_{ox})^2 + (u)^2}du \end{aligned}$$

Recall that a nice substitution for an integral such as this is provided by the $\sinh(z)$ *since* $1 + \sinh^2(z) = \cosh^2(z)$ *hence a* $u = v_{ox}\sinh(z)$ *subsitution will give*

$$(v_{ox})^2 + (u)^2 = (v_{ox})^2 + (v_{ox}\sinh(z))^2 = v_{ox}^2\cosh^2(z)$$

and $du = v_{ox}\cosh(z)dz$ *thus,* $\int \sqrt{(v_{ox})^2 + (u)^2}du = \int \sqrt{v_{ox}^2\cosh^2(z)}v_{ox}\cosh(z)dz = \int v_{ox}^2\cosh^2(z)dz$*. Furthermore,* $\cosh^2(z) = \frac{1}{2}(1 + \cosh(2z))$ *hence*

$$\int \sqrt{(v_{ox})^2 + (u)^2}du = \frac{v_{ox}^2}{2}\left[z + \frac{1}{2}\sinh(2z)\right] + c = \frac{v_{ox}^2}{2}\left[z + \sinh(z)\cosh(z)\right] + c$$

Note $u = v_{ox}\sinh(z)$ *and* $v_{ox}cosh(z) = \sqrt{(v_{ox})^2 + (u)^2}$ *hence substituting,*

$$\int \sqrt{(v_{ox})^2 + (u)^2}du = \frac{1}{2}\left[v_{ox}^2\sinh^{-1}\left(\frac{u}{v_{ox}}\right) + u\sqrt{v_{ox}^2 + u^2}\right] + c$$

Well, I didn't think that was actually solvable, but there it is. Returning to the definite integral to calculate s *we can use the antiderivative just calculated together with FTC part II to conclude: (provided* $v_{ox} \neq 0$*)*

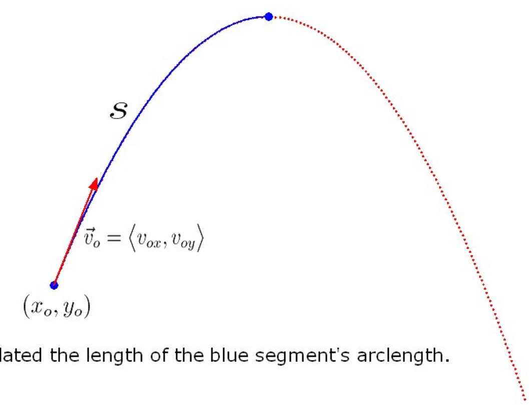
$$\boxed{s = \frac{1}{2g}\left[v_{ox}^2\sinh^{-1}\left(\frac{v_{oy}}{v_{ox}}\right) + v_{oy}\sqrt{v_{ox}^2 + v_{oy}^2}\right]}$$

If $v_{ox} = 0$ *then the problem is much easier since* $v(t) = |v_{oy} - gt| = v_{oy} - gt$ *for* $0 \leq t \leq t_1 = v_{oy}/g$ *hence*

$$s = \int_0^{t_1} v(t)dt = \int_0^{t_1} (v_{oy} - gt)dt = \left[v_{oy}t - \frac{1}{2}gt^2\right]\Bigg|_0^{v_{oy}/g} = \boxed{\frac{v_{oy}^2}{2g}}$$

Interestingly, this is the formula for the height of the parabola even if $v_{ox} \neq 0$*. The initial x-velocity simply determines the horizontal displacement as the object is accelerated vertically by gravity.*

[8] no, we have not shown this is a parabola, I invite the reader to verify this claim. That is find A, B, C such that the graph $y = Ax^2 + Bx + C$ is the same set of points as $\vec{r}(\mathbb{R})$.



We calculated the length of the blue segment's arclength.

Remark 9.9. *physics, grrr...*

I have made a few physical comments in this section, however, you should not confuse these comments with your responsibilities. I intend for you to understand the mathematics. This means that if you are given $\vec{r}$ or $\vec{v}$ or $\vec{a}$ then you can calculate whatever you are not given provided sufficient initial data. Also, if the integration isn't crazy then you can calculate the distance travelled. Notice that while these calculations do use some physical terminology, like position, velocity, speed or acceleration, they are hardly physics problems. We have made no effort to understand what a force is or how it is applied. In short, the calculations we explore here are easy physics problems. I say they are easy because we know exactly what to calculate once the problem is stated. It's just a matter of applying the rules of calculus which we already know and love.

10 geometric analysis in parametric setting

In this section we seek to understand tangent lines and arclength. We begin with the problem of tangent lines. Since we have taken a little time to introduce vectors this problem is simple.

Definition 10.1. *tangent vector to a path $t \mapsto \vec{r}(t)$.*

The tangent vector at $\vec{r}(t_o)$ is the vector $\vec{v}$ defined below:

$$\vec{v}_o = \frac{d\vec{r}}{dt}\bigg|_{t=t_o}.$$

If $\vec{v}_o \neq 0$ then the tangent line to $t \mapsto \vec{r}(t)$ at $\vec{r}(t_o) = \vec{r}_o$ is given parametrically by

$$\lambda \mapsto \vec{L}(\lambda) = \vec{r}_o + \lambda \vec{v}_o$$

where we call the **direction vector** for the line $\vec{L}$.

Example 10.2. Find the tangent line to the curve $\vec{r}(t) =< 1+t, 2+t^2 >$ at the point $(3,6)$.

To begin we find if this point is actually on the path; do the equations $3 = 1+t$ and $6 = 2+t^2$ have a simultaneous solution? Answer: yes, $t = 2$ is the solution which solves both equations ($t = -2$ is also a solution for $6 = 2+t^2$ however it does not solve $3 = 1+t$). Calculate:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \big\langle 1, 2t \big\rangle \quad \Rightarrow \quad \textit{hence,} \;\; \vec{v}(2) =< 1,4 >$$

and we find the tangent line $\vec{L}(\lambda) = (3,6) + \lambda < 1,4 >$.

We could use t as the parameter for the tangent line, however I have not done this because I don't want you to think the parameter for the curve and its tangent have to be the same. In the preceding example $\lambda = 0$ matches up with $t = 2$; $\vec{r}(2) = \vec{L}(0)$. Finding the direction of the tangent line is very similar to finding the slope of the tangent line. When we found the slope we first calculated the derivative function which gives us the slope at many points beside the point in question. We had to evaluate $\frac{dy}{dx}\big|_{x=x_o}$ in order to calculate the slope to the tangent line to $y = f(x)$ at $x = x_o$. In the same way, we calculate the *velocity vector field* $\frac{d\vec{r}}{dt}$ which once evaluated $\frac{d\vec{r}}{dt}\big|_{t=t_o}$ gives us a direction vector for the tangent line at $\vec{r}(t_o)$. Again, we find the tangent vector for many points beside the point in question, but evaluation singles out the direction vector we desire. It is called a *vector field* because we envision the velocity vectors glued to the points of tangency. The tangent vector $\frac{d\vec{r}}{dt}$ is most naturally based at the point $\vec{r}(t)$. [9]

Example 10.3. *Let a be a constant and suppose $x = e^{-t}\cos(t)$ and $y = e^{-t}\sin(t)$ then $\vec{r}(t) = e^{-t}\big\langle \cos(t), \sin(t)\big\rangle$ and we can calculate from the product rule for space curves,*

$$\begin{aligned} \frac{d\vec{r}}{dt} &= -e^{-t}\big\langle \cos(t), \sin(t)\big\rangle + e^{-t}\big\langle -\sin(t), \cos(t)\big\rangle \\ &= e^{-t}\big\langle -\cos(t) - \sin(t), -\sin(t) + \cos(t)\big\rangle \end{aligned}$$

The mapping $t \mapsto \frac{d\vec{r}}{dt}$ gives us the tangent vector field to $\vec{r}(t) = e^{-t}\big\langle \cos(t), \sin(t)\big\rangle$.

We can ask a question: **is there a point which has a tangent vector pointing in the $< 0,1 >$ direction ?**

Solution: *We search for solutions to $\frac{d\vec{r}}{dt} = k < 0,1 >$ for some $k \neq 0$. This means we need to solve:*

$$e^{-t}(-\cos(t) - \sin(t)) = 0 \qquad \textit{and} \qquad e^{-t}(-\sin(t) + \cos(t)) = k.$$

Since $e^{-t} \neq 0$ we must have $-\cos(t) - \sin(t) = 0$ hence $\tan(t) = -1$ which has solutions $t = -\frac{\pi}{4} + n\pi$ for $n \in \mathbb{Z}$. Which is just a fancy way of saying that $t = \ldots, -\frac{9\pi}{4}, -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots$. When $t = -\frac{\pi}{4} + 2m\pi$ then

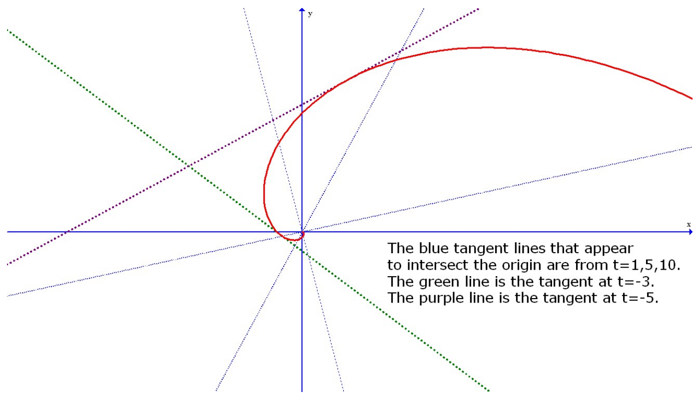
$$-\sin(t) + \cos(t) = -\left(\frac{-\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} = \sqrt{2}$$

[9] In calculus III when we study space curves in $\mathbb{R}^3$ it is natural to study unit tangent vector field($\vec{T}$) along with the unit normal($\vec{N}$) and unit binormal vector fields($\vec{B}$) of a space curve. For a regular smooth curve the $\{\vec{T}, \vec{N}, \vec{B}\}$ vectors together form a moving coordinate system which is called a moving frame. The classical differential geometry tied to the moving frame is beautifully encapsulated by the Frenet-Serret equations which explain how $\{\vec{T}, \vec{N}, \vec{B}\}$ change along the curve.

and we find that $k = \sqrt{2}e^{-\frac{\pi}{4}+2m\pi}$ for any $m \in \mathbb{Z}$ yields the desired direction. On the other hand when $t = -\frac{\pi}{4} + (2m+1)\pi$ we find $k = -\sqrt{2}e^{-\frac{\pi}{4}+(2m+1)\pi}$ for any $m \in \mathbb{Z}$ which means the tangent vector points opposite the desired direction. The tangent lines at $\vec{r}(-\frac{\pi}{4} + 2m\pi)$ are

$$\vec{L}_m(t) = e^{-\frac{\pi}{4}+2m\pi}\langle\sqrt{2}/2, -\sqrt{2}/2\rangle + t\sqrt{2}e^{-\frac{\pi}{4}+2m\pi}\langle 1, 0\rangle = \frac{e^{-\frac{\pi}{4}+2m\pi}}{\sqrt{2}}\Big\langle 1 + 2t, -1\Big\rangle.$$

The curve in the last example winds very tightly around the origin as $t \to \infty$. It's hard to picture the curve directly but the tangent lines pictured give us some idea.



The family of tangent lines found in the last example were all vertical tangent lines. A vertical tangent line is not covered as the linearization of a function of x. This is one of the geometric deficiencies of using graphs of the form $y = f(x)$. Perhaps you recall that the slope of a line is defined in every case except it is a vertical line. We face no such exception for the parametric case. However, if the velocity vector is zero at a point the analysis does require some thought. There might not be a well-defined tangent line at such a point thus many discussions of the geometry of curves rule out such a case by insisting the curve is *smooth*.

Definition 10.4. *smooth curve $t \mapsto \vec{r}(t)$.*

If $\frac{d\vec{r}}{dt}$ exists and $\frac{d\vec{r}}{dt} \neq \langle 0, 0\rangle$ for all $t \in dom(\vec{r})$ then we say $t \mapsto \vec{r}(t)$ is **smooth** or **non-stop**. A point for which the tangent vector either is zero or fails to exist is called a **critical point** for the path.

When the curve is smooth we can sometimes locally express the parametric curve as the graph of a function. We test if $y = f(x)$ is the same curve as $t \mapsto \vec{r}(t) = \big\langle x(t), y(t)\big\rangle$ by substituting the parametric formulas for $x(t)$ into f and checking if $y(t)$ is the result. We insist that $y(t) = f(x(t))$ for all t in at least some subset $J \subseteq dom(\vec{r})$. Differentiate implicitly to find:

$$\frac{dy}{dt} = \frac{df}{dx}\bigg|_{x=x(t)} \frac{dx}{dt}$$

Thus, if $y = f(x)$ and $\vec{r}(t_o) = \big\langle x(t_o), y(t_o)\big\rangle = \big\langle x_o, f(x_o)\big\rangle$ then

$$\boxed{\frac{dy}{dx}\bigg|_{x_o} = \frac{\frac{dy}{dt}\big|_{t_o}}{\frac{dx}{dt}\big|_{t_o}}}$$

Example 10.5. *Suppose* $x = \cos(t)$ *and* $y = \cos(t)$ *then we have* $\vec{r}(t) = \big\langle \cos(t), \cos(t)\big\rangle$ *hence* $\frac{d\vec{r}}{dt} = \big\langle -\sin(t), -\sin(t)\big\rangle$. *We find many critical points: if* $t = n\pi$ *for* $n \in \mathbb{Z}$ *then* $\frac{d\vec{r}}{dt} = 0$. *However, for* $t \neq n\pi$ *for* $n \in \mathbb{Z}$ *we find the curve is smooth. Let* t_o *be some point where this curve is smooth, we calculate*

$$\frac{dy}{dx}\bigg|_{x=\cos(t_o)} = \frac{\sin(t_o)}{\sin(t_o)} = 1.$$

This is no surprise since we can easily eliminate t *by equating* $\cos(t) = y = x$. *Note that the parametric equations, while giving the same set of points as* $y = x$ *for* $-1 \leq x \leq 1$, *is quite different as it covers this line segment infinitely many times as it oscillated between the points* $(-1, 1)$ *and* $(1, 1)$.

Example 10.6. *Suppose* $x = \sinh(t) + t$ *and* $y = e^{2t}$ *we can calculate the slope of the corresponding Cartesian curve* $y = f(x)$ *at* $(0, 1)$. *Note* $t = 0$ *yields* $x = 1$ *and* $y = 1$ *thus calculate*

$$\frac{dy}{dx}\bigg|_{x=1} = \frac{2e^{2t}}{\cosh(t) + 1}\bigg|_{t=0} = \frac{2}{1+1} = 1.$$

This would be less easy to arrive at through implicit differentiation of the Cartesian equation defining this curve. Just for fun, note $t = \ln(\sqrt{y})$ *hence* $x = \sinh(\ln(\sqrt{y})) + \ln(\sqrt{y})$. *Thus,*

$$1 = [\cosh((\ln(\sqrt{y}))) + 1]\frac{1}{2y}\frac{dy}{dx}$$

Plug in $y = 1$ *to obtain* $1 = [cosh(0) + 1]\frac{1}{2}\frac{dy}{dx} = \frac{dy}{dx}$. *Moral of story? Igore parametric techniques at your own peril.*

If $\frac{d\vec{r}}{dt}$ does not exist at a particular point it could signify the path has abruptly changed direction so the past-pointing and future-pointing tangents differ[10]. However, if the velocity vector $\frac{d\vec{r}}{dt}\big|_{t=t_o} = \big\langle 0, 0\big\rangle$ then it may just be a point like that on the top of a hill where the velocity might tend to zero just as the object barely makes it over the hill. There might still be a well-defined direction for the curve which can be detected by examining how the curve approaches the point of zero-tangent.

Example 10.7. *Let* $\vec{r}(t) = \big\langle t^3, t^3\big\rangle$. *Calculate* $\frac{d\vec{r}}{dt} = \big\langle 3t^2, 3t^2\big\rangle$ *thus observe* $t = 0$ *yields the critical point* $\vec{r}(0) = \big\langle 0, 0\big\rangle$. *In this case notice:*

$$\lim_{t \to 0} \frac{dy/dt}{dx/dt} = \lim_{t \to 0} \frac{3t^2}{3t^2} = 1$$

Thus we identify that $dy/dx = 1$ *at* $(0, 0)$.

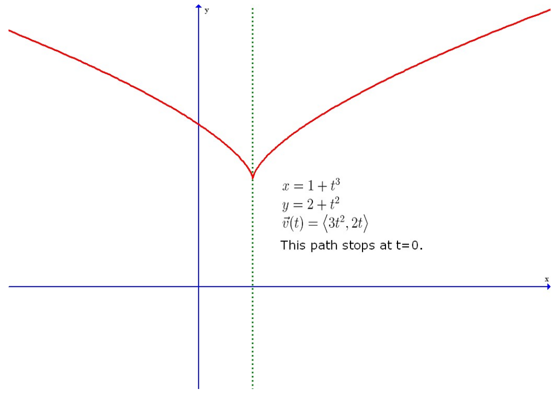
However, critical points are much like limits of indeterminant form. There are many possible types of behaviour.

Example 10.8. *Let* $\vec{r}(t) = \big\langle 1 + t^3, 2 + t^2\big\rangle$ *hence* $\frac{d\vec{r}}{dt} = \big\langle 3t^2, 2t\big\rangle$ *thus observe* $t = 0$ *yields the critical point* $\vec{r}(0) = \big\langle 1, 2\big\rangle$. *Notice:*

$$\lim_{t \to 0^+} \frac{dy/dt}{dx/dt} = \lim_{t \to 0^+} \frac{2t}{3t^2} = \lim_{t \to 0^+} \frac{2}{3t} = \infty \quad \textit{and} \quad \lim_{t \to 0^-} \frac{dy/dt}{dx/dt} = \lim_{t \to 0^-} \frac{2t}{3t^2} = \lim_{t \to 0^-} \frac{2}{3t} = -\infty$$

Thus we identify that dy/dx *is not defined at* $(0, 0)$. *This is a vertical tangent.*

[10] this happened in Example 10.5 at points $(1, 1)$ and $(-1, -1)$



In any event, we are most interested in smooth curves in this course so don't worry about finding other possible cases. For now you should mainly be interested in be able to find the parametric equations for the tangent line for a given smooth curve. In addition, you should work on developing a picture in your mind for our calculations. Draw pictures. Test your hunches. Of course Mathematica or another CAS will produce nice plots of parametric curves and their tangents but you must test your understanding on simple examples as to make sure you understand what the CAS is telling you if nothing else. I wouldn't ask a question which required the full assistance of a CAS on the test. Obviously the problem I give will be algebraically accessible and mainly test your comptence in understanding the definitions we have explored in this section. Some of your homework is probably deliberately messy so the CAS is the only way to go. That is for breadth not for the test.

Remark 10.9. *higher derivatives for parametric curves*

To calculate $y''(x)$ in terms of $x(t)$ and $y(t)$ we can just differentiate (omitting explicit notation for the t-dependence, let it be understood all of the objects below depend appropriately on t)

$$\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

implicitly a second time to obtain by the product and chain rules:

$$\frac{d^2y}{dt^2} = \frac{d^2f}{dx^2}\frac{dx}{dt}^2 + \frac{df}{dx}\frac{d^2x}{dt^2}$$

Hence substitute $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ for $\frac{df}{dx}$ to find

$$\frac{d^2y}{dt^2} = \frac{d^2f}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\frac{d^2x}{dt^2}$$

Therefore, if $y = f(x(t))$ then

$$\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2} - \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

Example 10.10. *Let $x = t+1$ and $y = t^3$ calculate $\frac{dx}{dt} = 1$ and $\frac{d^2x}{dt^2} = 0$ whereas $\frac{dy}{dt} = 3t^2$ and $\frac{d^2y}{dt^2} = 6t$ it follows that*

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3} = 6t$$

Hence for $t < 0$ the curve is concave down whereas for $t > 0$ the curve is concave up. From $x = t+1$ we find CD for $x < 1$ and CU for $x > 1$.

If you eliminate t to find $y = (x-1)^3$ and it's easy to calculate that the result we just found should be agreeable from your previous work on concavity.

10.1 motivating the arclength formula

To begin we assume the curve $t \mapsto \vec{r}(t) = \langle x(t), y(t) \rangle$ is smooth and $t \in [a,b]$. Let us begin by partitioning the closed interval as we did in the integration theory:

$$t_o = a, t_1 = a + \Delta t, t_2 = a + 2\Delta t, \dots, t_{n-1} = a + (n-1)\Delta t, t_n = b$$

where $\Delta t = (b-a)/n$ and $n \in \mathbb{N}$ is the number of subintervals in the partition. Let us define $\vec{r}_k = \vec{r}(t_k) = \langle x_k, y_k \rangle$ for $k = 0, 1, 2, \dots n$. For each subinterval $[t_{k-1}, t_k]$ we can construct the tangent line from $\vec{r}_{k-1}$ to $\vec{r}_k$ by the usual construction for a line segment: however, we set-up the parameter of each line segment as to make the segment's parameter match the parameter of the given curve:

$$\vec{L}_k(t) = \vec{r}_{k-1} + \left(\frac{t - t_{k-1}}{\Delta t}\right)(\vec{r}_k - \vec{r}_{k-1}) \qquad \text{for } t \in [t_{k-1}, t_k].$$

As a check on the formula above, note $\vec{L}_k(t_{k-1}) = \vec{r}_{k-1}$ whereas $\vec{L}_k(t_k) = \vec{r}_k$. We calculate the length of the line segment $t \mapsto \vec{L}_k(t)$ for $t \in [t_{k-1}, t_k]$ by calculating the length of the displacement from $\vec{r}_{k-1}$ to $\vec{r}_k$:

$$\Delta s_k = ||\vec{r}_k - \vec{r}_{k-1}|| = \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$$

The total length of this piecewise-polygonal approximation to the smmoth path is

$$\Delta s_n = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}.$$

If we allow $n \to \infty$ then the approximation should converge to the arclength.

$$\begin{aligned}
\Delta s &= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\left(\frac{x_k - x_{k-1}}{\Delta t}\right)^2 + \left(\frac{y_k - y_{k-1}}{\Delta t}\right)^2} \Delta t.
\end{aligned}$$

By construction, $t_k = t_{k-1} + \Delta t$ so we may identify that as $n \to \infty$ and $\Delta t \to 0$ since $\Delta t = (b-a)/n$ the difference quotients tend to derivatives:

$$\frac{x_k - x_{k-1}}{\Delta t} = \frac{x(t_{k-1} + \Delta t) - x(t_{k-1})}{\Delta t} \quad \longrightarrow \quad \frac{dx}{dt}(t_{k-1}).$$

Likewise,

$$\frac{y_k - y_{k-1}}{\Delta t} = \frac{y(t_{k-1} + \Delta t) - y(t_{k-1})}{\Delta t} \quad \longrightarrow \quad \frac{dy}{dt}(t_{k-1})$$

Hence,

$$\Delta s = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,\Delta t$$
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

where in the last step we identified the Riemann sum as the arclength integral we sought from the outset. Notice that in the case of a graph $y = f(x)$ we can take $x = t$ and we derive $\Delta s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.\square$

Remark 10.11. *heuristic metric calculation*

The infinitesimal argument should be understood as a shorthand for the technical argument we covered before this remark:

1. $ds^2 = dx^2 + dy^2$ since a tiny triangle with sides dx and dy has a hypoteneuse which is really close to the arclength.
2. Thus $ds = \sqrt{dx^2 + dy^2}$
3. Factor out a dt to obtain $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}dt$.
4. add up all the little ds's by integrating.

This infinitesimal argument should be thought of as a nice notation to summarize a more technically correct summation argument(which is what follows this remark). I usually emphasize the infinitesimal approach in calculus because it gets to the heart of the matter quickly and provides a very general framework to solve a myriad of applied problems. The infinitesimal ds^2 is called the **metric**. In general relativity the metric is sometimes used as the central object of interest in the theory. You can calculate the influence of gravity in curved space through the mathematics of the metric. The calculus for curved space involves **tensors** and **manifolds**. The calculations in tensor calculus are very very fun.

In both notations there is a question we have neglected. In particular, how do we know our calculation is independent of parametrization? If the arclength calculated depends on the choice of parameter then it is not what we would want to call arclength. Geometrically, it is clear that the arclength is an intrinsic geometric property of a given curve. I prove independence of the arclength formula from choice of parameter in calculus III if time permits.

11 arclength examples

It might be helpful to note Stewart discusses arclength for graphs and parametric curves in separate sections: see §9.1 and page 669-670 for his arclength examples. The distinction between the examples in this section and those in the previous section on distance travelled is that we make no assumption that the parameter is time. The calculations here are merely geometrical.

Example 11.1. .

Consider the following parametric curve where $0 \le t \le 3$, find arclength.

$$x = e^t + e^{-t} \qquad \dot{x} = e^t - e^{-t} \qquad \dot{x}^2 + \dot{y}^2 = (e^t - e^{-t})^2 + 4$$
$$y = 5 - 2t \qquad \dot{y} = -2$$

$$S = \int_0^3 \sqrt{e^{2t} - 2 + e^{-2t} + 4}\,dt = \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}}\,dt = \int_0^3 (e^t + e^{-t})\,dt$$
$$= (e^t - e^{-t})\Big|_0^3$$
$$= \boxed{e^3 - 1/e^3}$$

Example 11.2. .

Let $\alpha(t) = (r\cos t, r\sin t) \qquad 0 \le t \le 2\pi$

$$x = r\cos t \qquad \frac{dx}{dt} = -r\sin t \qquad \therefore \left(\frac{dx}{dt}\right)^2 = r^2\sin^2 t$$
$$y = r\sin t \qquad \frac{dy}{dt} = r\cos t \qquad \therefore \left(\frac{dy}{dt}\right)^2 = r^2\cos^2 t$$

$$S = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt = \int_0^{2\pi} r\,dt = \boxed{2\pi r}$$

Which is good since $x^2 + y^2 = r^2$, is a circle of radius r.

Example 11.3. .

Let $\alpha(x) = (x, \ln(\cos(x))) \qquad 0 \le x \le \pi/4 \qquad \begin{matrix} x = x \\ y = \ln(\cos(x)) \end{matrix}$

$y = \ln(\cos(x))$ for $0 \le x \le \pi/4$

$$S = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \int_0^{\pi/4} \sqrt{1 + \left(\frac{-\sin(x)}{\cos(x)}\right)^2}\,dx$$
$$= \int_0^{\pi/4} \sqrt{1 + \tan^2(x)}\,dx \qquad ; \qquad \tan^2(x) + 1 = \sec^2(x)$$
$$= \int_0^{\pi/4} \sec(x)\,dx$$
$$= \int_1^{1+\sqrt{2}} \frac{du}{u}$$

$$u = \sec(x) + \tan(x)$$
$$\frac{du}{u} = \sec(x)\,dx$$
$$u(0) = \sec(0) + \tan(0) = 1$$
$$u(\pi/4) = \sec(\pi/4) + \tan(\pi/4) = \sqrt{2} + 1$$

$$= \left(\ln|u|\right)\Big|_1^{1+\sqrt{2}}$$
$$= \ln(1+\sqrt{2}) - \ln(1)$$
$$= \boxed{\ln(1+\sqrt{2})}$$

Example 11.4. .

Find circumference of an ellipse: $x = a\cos t$, $y = b\sin t$; $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \sin^2 t + \cos^2 t = 1$

$\frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = b\cos t$ where we are given $0 \le t \le 2\pi$.

$$
\begin{aligned}
s &= \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt \\
&= \int_0^{2\pi} a\sqrt{\sin^2 t + \left(\frac{b\cos t}{a}\right)^2}\, dt \\
&= \int_0^{2\pi} a\sqrt{1 - \cos^2 t + \left(\frac{b\cos t}{a}\right)^2}\, dt \\
&= \int_0^{2\pi} a\sqrt{1 - \left(1 - (b/a)^2\right)\cos^2 t}\, dt \quad : \quad \underline{\text{define}}\ \beta = \sqrt{|1-(b/a)^2|} \\
&= \int_0^{2\pi} a\sqrt{1 - \beta^2\cos^2 t}\, dt
\end{aligned}
$$

This integral is not elementary. We'll need a specific a and b in order to proceed. Note that when $a = b$ then $\beta = 0$ so $s = 2\pi a$ happily. Consider $a = 1$, $b = \sqrt{2}$ then $\beta = 1$

$$
\int_0^{2\pi} \sqrt{1 - \cos^2 t}\, dt = \int_0^{2\pi} |\sin t|\, dt = \int_0^{\pi} \sin t\, dt - \int_{\pi}^{2\pi} \sin t\, dt = 4
$$

For most choices of a & b we would need to do this integral numerically

Example 11.5. .

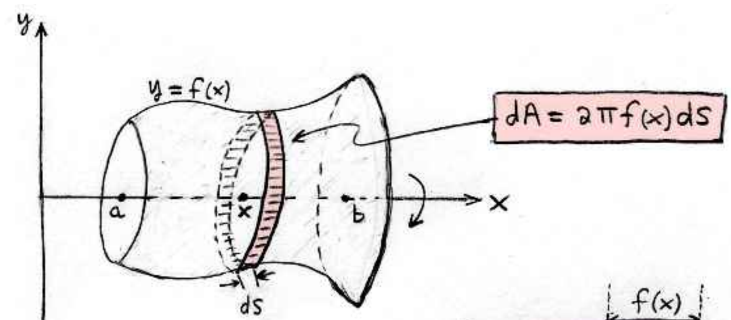
$y = \frac{x^3}{6} + \frac{1}{2x}$ with $1/2 \le x \le 1$ find arclength.

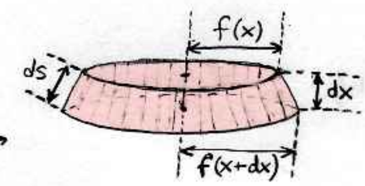
$\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2} \quad \therefore \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)$

$$
\begin{aligned}
s &= \int_{1/2}^{1} \sqrt{1 + \tfrac{1}{4}\left(x^4 - 2 + \tfrac{1}{x^4}\right)}\, dx = \int_{1/2}^{1} \sqrt{\tfrac{1}{4}\left(x^4 + 2 + \tfrac{1}{x^4}\right)}\, dx = \int_{1/2}^{1} \sqrt{\tfrac{1}{4}\left(x^2 + \tfrac{1}{x^2}\right)^2}\, dx \\
&= \int_{1/2}^{1} \tfrac{1}{2}\left(x^2 + \tfrac{1}{x^2}\right) dx = \tfrac{1}{2}\left[\tfrac{x^3}{3} - \tfrac{1}{x}\right]\Big|_{1/2}^{1} = \frac{31}{48} = \boxed{0.6458}
\end{aligned}
$$

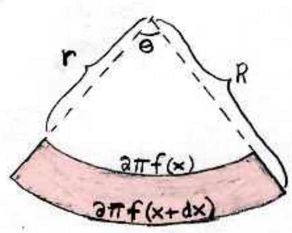
12 surface area

Imagine we take a graph $y = f(x)$ for $a \leq x \leq b$ and rotate it around the x-axis. We suppose $f(x) \geq 0$ for the purposes of this discussion. This creates a surface of revolution. You may recall from calculus I that we calculated the volume contained inside such surfaces for a variety of cases. For now we will just focus on the case of a surface revolved around the x-axis. Let us focus on just a small portion of the surface. In particular the bit from x to $x + dx$. Suppose ds is the arclength of the graph from $(x, f(x))$ to $(x + dx, f(x + dx))$. The straight-line distance from $(x, f(x))$ to $(x + dx, f(x + dx))$ is identical to ds in this infinitesimal limit. Notice then we can calculate the area of this conical ribbon which has radii $f(x)$ and $f(x + dx)$ at its edges and a length of ds along the edge as described below[11]:





The strip at x can be flipped and illustrated as shown
Or, we can lay the strip flat as below:



$$\theta = \frac{2\pi f(x)}{r}$$

$$\theta = \frac{2\pi f(x+dx)}{R}$$

The area of this partial annulus is given by

$$\begin{aligned} dA &= \pi R^2 \left(\frac{\theta}{2\pi}\right) - \pi r^2 \left(\frac{\theta}{2\pi}\right) \\ &= \frac{1}{2}\theta\,(R^2 - r^2) \\ &= \frac{1}{2}\theta\,(R+r)(R-r) \quad : \left[\begin{array}{l}\text{observe } \theta \approx \pi \text{ since} \\ f(x+dx) \approx f(x) \text{ as} \\ dx \approx 0.\end{array}\right] \\ &= \frac{1}{2}(\pi)(2f(x))\,ds \\ &= \underline{2\pi f(x)\,ds}. \end{aligned}$$

Or, in terms of dx, $dA = 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ where $y = f(x)$.

Furthermore, while we assumed an increasing function for the ease of visualization this formula holds for the case that f is decreasing. Note that ds is positive and we assumed from the outset

[11]thanks to Ginny for this idea

that $f(x) \geq 0$. Recall $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}dx$ hence the total surface area is thus found from the following integration:

$$\boxed{A = \int_a^b 2\pi f(x)\sqrt{1 + \frac{df}{dx}^2}\, dx}$$

All of this said, we can state a more general formula for parametric curves around an arbitrary axis in the plane. Suppose that $t \mapsto \langle x(t), y(t) \rangle$ is a parametric curve and $\mathcal{L}$ is a line in the plane. Suppose this parametric curve does not cross the axis and any perpendicular bisector of the axis crosses the curve in at most one point. Let $r(t)$ be the distance from the curve to the axis then we can by the same argument as given for $y = f(x)$ derive that the area of the surface or revolution formed by rotation the parameterized curve for $a \leq t \leq b$ is simply:

$$\boxed{A = \int_a^b 2\pi r(t)\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt}$$

In the case of the graph we had $t = x$ and $r(t) = f(x)$. For an abitrary example the real problem is geometrically determining the formula for $r(t)$.

Example 12.1. Problem: *Let $R > 0$ and $x = R\cos(t)$ and $y = R\sin(t)$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Find the surface area of the surface formed by revolving the given curve around the y-axis.*

Solution: *the cruve we consider is a half-circle centered at the origin with radius R. Since the axis $\mathcal{L}$ is the y-axis the distance[12] to a point $(x(t), y(t))$ on the curve is clearly $x(t)$. Recall $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}dt$ hence we calculate $ds = \sqrt{R^2\sin^2(t) + R^2\cos^2(t)}dt = Rdt$. The area of a typical infinitesimal ribbon is $dA = 2\pi r(t)ds = 2\pi(R\cos(t))(Rdt)$ for each $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Add together all the little dA's by integration to find the total surface area:*

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi R^2\cos(t)dt = 2\pi R^2\sin(t)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \boxed{4\pi R^2.}$$

Therefore, the surface area of a sphere of radius R is $4\pi R^2$. It is interesting to note that the integral of this formula with respect to R yields the volume of a sphere; $V = \frac{4}{3}\pi R^3$. Likewise, the circumference of the disk of radius R is $2\pi R$ which once integrated yields the area πR^2. This pattern does not hold for all solids. For example, if you think about a cube of side length x then the $V = x^3$ whereas the surface area is $A = 6x^2$. The symmetry of the sphere or circle is very special and the pattern continues for higher dimensional spheres[13]

[12] distance from a point to a set is by definition the distance from the point to the closest point in the set

[13] $x^2 + y^2 = 1$ gives unit circle or S_1, $x^2 + y^2 + z^2 = 1$ gives unit spherical shell or S_2, $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$ gives the unit n-sphere S_n. You can find a derivation of the hypervolume of the higher dimensional spheres in Apostol if you're interested.

Example 12.2. Problem: *Find the surface area of the surface of revolution formed by rotating $y = \sqrt{x}$ around the x-axis for $0 \le x \le 1$.*

Solution: *We have*

$$\begin{aligned} dA &= 2\pi\sqrt{x}\sqrt{1 + \left[\frac{1}{2\sqrt{x}}\right]^2}\ dx \\ &= 2\pi\sqrt{x\left[1 + \frac{1}{4x}\right]}\ dx \\ &= 2\pi\sqrt{x + \frac{1}{4}}\ dx \\ &= \pi\sqrt{4x+1}\ dx \end{aligned}$$

We can integrate the expression above with the help of a $u = 4x + 1$ subsitution. Note

$$\int \sqrt{4x+1}\ dx = \int \sqrt{u}\frac{du}{4} = \frac{2u^{\frac{3}{2}}}{12} + c = \frac{1}{6}(4x+1)^{\frac{3}{2}} + c.$$

Thus,

$$A = \int_0^1 \pi\sqrt{4x+1}\ dx = \frac{\pi}{6}(4x+1)^{\frac{3}{2}}\Big|_0^1 = \boxed{\frac{\pi}{6}(\sqrt{125} - 1).}$$

Example 12.3. Problem: *Find the surface area of an open right-circular cone of height h and radius R.*

Solution: *we can view this as a surface of revolution. Take the line $y = Rx/h$ for $0 \le x \le h$ and rotate it around the x-axis. Observe that*

$$dA = 2\pi\left(\frac{Rx}{h}\right)\sqrt{1 + \frac{R^2}{h^2}}\ dx = \frac{2\pi Rx}{h^2}\sqrt{h^2 + R^2}\ dx.$$

Now integrate over $0 \le x \le h$ to find the total area:

$$A = \int_0^h \frac{2\pi Rx}{h^2}\sqrt{h^2+R^2}\ dx = \frac{2\pi Rx^2}{2h^2}\sqrt{h^2+R^2}\Big|_0^h = \boxed{\pi R\sqrt{h^2+R^2}.}$$

Notice that the formula above checks nicely in the limits $h \to 0$ and $R \to 0$ where we find $A \to \pi R^2$ and $A \to 0$ respective. Can you see why this makes sense?

13 physics

In this section we examine a few variable force work problems, variable pressure hydrostatic force problems and finally the center of mass problem for a homogeneous laminate in the plane.

13.1 work and force with calculus

The basic physical concepts used here are as follows:

1. work W due to a force F over a displacement Δx is defined to be $W = F\Delta x$ provided the force is exerted in the direction of the displacement and is constant.

2. the force F exerted over an area A by a pressure P is defined to be $F = PA$ provided the pressure is constant over the area A.

In the examples we consider in this section we cannot simply multiply as described above because the requisite idealizations are not met in our examples in the finite case. In other words, the forces are variable and the pressures are not constant. However, if we instead consider an infinitesimal displacement dx or an infinitesimal area dA we can in fact realize the idealized physical laws. It is true that $dW = Fdx$ because the F does not change over the tiny displacement dx. For the problem of the dam, we can say $dF = PdA$ if our dA is a horizontal strip since the pressure is constant over a certain depth. I'll leave the rest of the details for the examples. Mainly we need the following physical equations to complete the examples:

1. $F = mg$, near the surface of the earth this is the force of gravity on a mass m.

2. $P = \rho g d$, is the pressure due to water at a depth d where $\rho \approxeq 1000kg/m^3$ is the density of water.

Example 13.1. . *This is a variable work due to variable mass problem.*

Find magnitude of work required to lift a 10m cable with a uniformly distributed mass of 10 kg.

Lets define l to be the length of the cable being lifted, this will vary from $l = 10\,m \longrightarrow l = 0m$ as the cable is lifted.

We should find the work dW done as we lift length l a distance dl,

$$
\begin{aligned}
dW &= F\,dl \\
&= -mg\,dl \\
&= -\lambda l g\,dl
\end{aligned}
$$

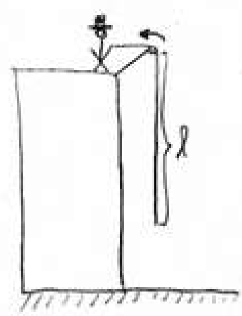
$\lambda = \frac{\text{mass}}{\text{length}} = \frac{10kg}{10m} = 1\ kg/m$

"λ is linear mass density"

$m = \lambda l$.

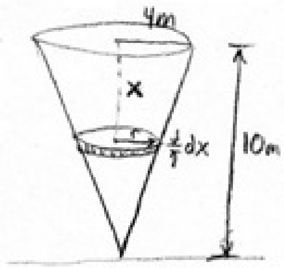
Now add up the work

$$
\begin{aligned}
W &= \int_{10}^{0} -\lambda g\, l\,dl \\
&= -\lambda g \left(\tfrac{1}{2} l^2 \right|_{10}^{0} \\
&= \lambda g\, \tfrac{1}{2}(10)^2 \\
&= 1 \cdot 9.8 \cdot 50 \\
&= \boxed{490\ J}
\end{aligned}
$$

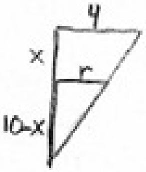


Example 13.2. . *This is a variable work due to variable mass problem.*

Find minimum work to pump water out of a full cone.



$$dm = \rho\, dV = \rho \pi r^2 dx$$



$$\frac{4}{10} = \frac{r}{10-x}$$

$$r = \frac{2}{5}(10-x)$$

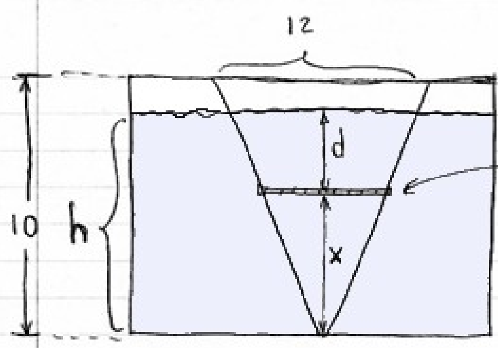
$$\begin{aligned} dW &= (dm)\, g \cdot x \\ &= (\rho \pi r^2 dx)\, g\, x \\ &= \rho g \pi \, \frac{4(10-x)^2}{25}\, x\, dx \\ &= \frac{4\pi\rho g}{25}(100x - 20x^2 + x^3)\, dx \end{aligned}$$

$$\begin{aligned} W &= \int_0^{10} \frac{4\pi\rho g}{25}(100x - 20x^2 + x^3)\, dx \\ &= \frac{4\pi\rho g}{25}\left(50 \times 10^2 - \frac{20}{3} \times 10^3 + \frac{10^4}{4}\right) \\ &= \frac{4\pi(1000)(9.8)}{25}\left(50 \times 10^2 - \frac{20}{3} \times 10^3 + \frac{10^4}{4}\right) \\ &= \boxed{4.11 \times 10^6\ \text{J}} \end{aligned}$$

$\boxed{\text{Remark: this is a magic cone, we cannot just tip it over or drill a hole in the bottom. We are forced to pump out the top :)}}$

Example 13.3. . *The triangular dam problem.*

Find the hydrostatic force on the triangular region pictured below. Assume the water of density ρ is upto level h



width is w which clearly depends linearly on x.

$w = mx + b$

$$w(0) = m(0) + b = 0 \quad \therefore \; \underline{b=0}$$
$$w(10) = m(10) + 0 = 12 \quad \therefore \; m = \frac{12}{10} = \frac{6}{5} \quad \therefore \; \boxed{w = \frac{6}{5}x}$$

This formula for w checks because $W(10) = \frac{6}{5}(10) = 12$ as it should.
Then the area of strip is $dA = w\,dx = \frac{6}{5}x\,dx$.
Now set up the pressure $P = \rho g d$,

$$x + d = h \Rightarrow d = h - x \Rightarrow P = \rho g(h - x)$$

Again the connection between force & pressure is $P = \frac{dF}{dA}$
so $dF = P\,dA$, thus

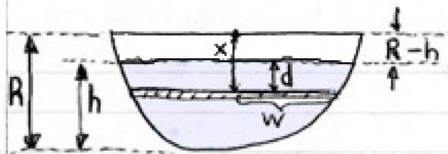
$$dF = \rho g(h-x)\frac{6}{5}x\,dx$$

Now sum the forces for the strips at x in $0 \leq x \leq h$,

$$\begin{aligned} F &= \int_0^h \frac{6\rho g}{5}(hx - x^2)\,dx \\ &= \frac{6\rho g}{5}\left(\frac{1}{2}hx^2 - \frac{1}{3}x^3\right|_0^h \\ &= \frac{6\rho g}{5}\left(\frac{1}{2}h^3 - \frac{1}{3}h^3\right) \\ &= \boxed{\frac{1}{5}\rho g h^3} \end{aligned}$$

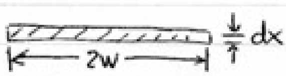
Example 13.4. . *The hemispherical dam problem.*

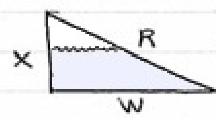
Find hydrostatic force on half-barrel pictured below, well just the end piece. The radius of barrel is R and the water of density ρ is filled to height h. Lets find force dF on a strip of area dA at position x and depth d below the surface.



$$\Rightarrow \quad d = x-(R-h)$$

From picture above we find $P=\rho g d = \rho g(x-R+h)$.
(I set up the pressure wrong in notes, it is the depth that should determine the pressure, specifically $P=\rho g d$.) Next;





$$w=\sqrt{R^2-x^2}$$

$$dA = 2w\,dx = 2\sqrt{R^2-x^2}\,dx$$

Ok so our choice of x makes dA relatively pretty, (you can try defining x differently but it'll make the square root nasty...) Ok, we know $P=\frac{dF}{dA}$ so $dF = P\,dA$

$$dF = \rho g\,(x-R+h)\,2\sqrt{R^2-x^2}\,dx$$

Now we just need to add-up the forces, $R-h \le x \le R$

$$\begin{aligned}
F &= \int_{R-h}^{R} 2\rho g\,(x-R+h)\sqrt{R^2-x^2}\,dx \\
&= \int_{R-h}^{R} 2\rho g\,x\sqrt{R^2-x^2}\,dx + \int_{R-h}^{R} 2\rho g\,(h-R)\sqrt{R^2-x^2}\,dx \\
&= 2\rho g \underbrace{\int_{R-h}^{R} x\sqrt{R^2-x^2}\,dx}_{u-\text{substitute}} + 2\rho g\,(h-R)\underbrace{\int_{R-h}^{R} \sqrt{R^2-x^2}\,dx}_{\text{trig}-\text{substitute.}}
\end{aligned}$$

The hemispherical dam problem continued:

$$\int x\sqrt{R^2-x^2}\,dx = \int \sqrt{w}\,\frac{dw}{-2} \qquad w = R^2 - x^2, \quad dw = -2x\,dx, \quad x\,dx = \frac{-dw}{2}$$

$$= -\frac{1}{2}\int w^{\frac{1}{2}}\,dw$$

$$= -\frac{1}{2}\,\frac{2}{3}\,w^{\frac{3}{2}} + C$$

$$= -\frac{1}{3}\left(R^2 - x^2\right)^{3/2} + C$$

$$\int \sqrt{R^2-x^2}\,dx = \int (R\cos\theta)(R\cos\theta\,d\theta) \qquad x = R\sin\theta, \quad dx = R\cos\theta\,d\theta, \quad R^2 - x^2 = R^2\cos^2\theta, \quad \sqrt{R^2-x^2} = R\cos\theta$$

$$= \int R^2\cos^2\theta\,d\theta$$

$$= \int \frac{R^2}{2}\left(1 + \cos(2\theta)\right)d\theta$$

$$= \frac{R^2}{2}\int d\theta + \frac{R^2}{2}\int \cos(2\theta)\,d\theta$$

$$= \frac{R^2}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$$

$$F = 2\rho g\int_{R-h}^{R} x\sqrt{R^2-x^2}\,dx + 2\rho g(h-R)\int_{R-h}^{R}\sqrt{R^2-x^2}\,dx$$

$$= \frac{-2\rho g}{3}\left(R^2-x^2\right)^{3/2}\Big|_{R-h}^{R} + \rho g(h-R)R^2\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{x=R-h}^{x=R}$$

$$= +\frac{2}{3}\rho g\left(R^2-(R-h)^2\right)^{3/2} + \rho g(h-R)R^2\left(\sin^{-1}\left(\frac{x}{R}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{x}{R}\right)\right)\right)\Big|_{R-h}^{R}$$

$$= \frac{2}{3}\rho g\left(R^2-(R-h)^2\right)^{3/2} + \rho g(h-R)R^2\left(\sin^{-1}\left(\frac{R}{R}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{R}{R}\right)\right)\right)$$

$$- \rho g(h-R)R^2\left(\sin^{-1}\left(\frac{R-h}{R}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{R-h}{R}\right)\right)\right)$$

$$= \boxed{\frac{2}{3}\rho g\left(R^2-(R-h)^2\right)^{3/2} + \rho g(h-R)R^2\left[\frac{\pi}{2} + \frac{1}{2}\sin\left(2\,\frac{\pi}{2}\right) - \sin^{-1}\left(\frac{R-h}{R}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{R-h}{R}\right)\right)\right]}$$

13.2 center of mass

The concept of center of mass is a ubiquituous topic in mechanics. In a nutshell it allows us to idealize shapes with finite size as if they were just a point mass. This is a tremendous simplification as it allows us to think of just one particle at a time rather than the infinity of atoms that make up a solid. I'll prove this idealization is reasonable in physics, but for here we just want to see how the center of mass is calculate via calculus.

Moments & Centers of Mass (com)

In the discrete case we define $\vec{r}_{cm} = \dfrac{\sum m_i \vec{r}_i}{\sum m_i}$ the sum is over all the particles.
For 2 dimensional case we have

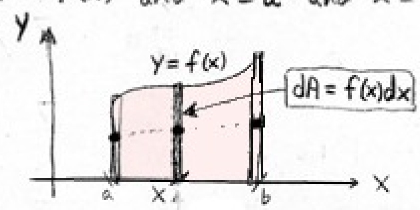
$$\vec{r}_{cm} = (X_{cm}, Y_{cm})$$

$$X_{cm} = \left(\sum m_i x_i\right) / \left(\sum m_i\right) = M_x / M$$

$$Y_{cm} = \left(\sum m_i y_i\right) / \left(\sum m_i\right) = M_y / M$$

M_x = moment of intertia w.r.t x-axis.
M = total mass.

What then is the generalization of this to a continous region. Lets see how to find the c.o.m. of the region bounded by $y=0$, $y=f(x)$ and $x=a$ and $x=b$. Assume uniform density.



Each strip has its c.o.m. at $(x, \frac{1}{2}f(x))$. We can treat it like a bunch of particles with dm each

$$\rho = \frac{dm}{dA} \rightarrow dm = \rho\, dA$$

So we find the c.o.m. of this system in the natural way,

$$\bar{x} = \frac{\int_a^b \rho x f(x)dx}{\int_a^b \rho f(x)dx} \quad \text{AND} \quad \bar{y} = \frac{\int_a^b \rho \frac{1}{2} f(x) f(x)dx}{\int \rho f(x)\, dx}$$

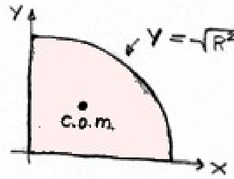
Or in terms of moments & mass we have

$M_y \equiv \rho \int x f(x) dx$	$\bar{x} = \dfrac{M_y}{M}$
$M_x \equiv \rho \int \frac{1}{2}[f(x)]^2 dx$	$\bar{y} = \dfrac{M_x}{M}$
$M = \rho \int f(x)dx = \rho\,(\text{AREA})$	

We assume ρ to be a constant.

Example 13.5. . *Center of mass problem.*

Find the moments of intertia and c.o.m for a quarter-circle of uniform density ρ with radius R



$y = \sqrt{R^2 - x^2} = f(x)$

$$M = \int_0^R \rho f(x)\,dx = \rho \int_0^R f(x)\,dx = \boxed{\frac{\pi}{4}\rho R^2 = M}$$

mass is simply area times density.

$$M_y = \rho \int_0^R x\sqrt{R^2 - x^2}\,dx = \rho \int_{R^2}^{0} \frac{-1}{2}\sqrt{u}\,du$$

$u = R^2 - x^2$ $\quad u(R) = 0$

$du = -2x\,dx$ $\quad u(0) = R^2$

$$= \frac{\rho}{2}\,\frac{2}{3}\,u^{3/2}\Big|_0^{R^2} = \frac{\rho}{3}(R^2)^{3/2} = \boxed{\frac{1}{3}\rho R^3 = M_x}$$

$$M_x = \rho \int_0^R \frac{1}{2}(R^2 - x^2)\,dx = \frac{\rho}{2}\left(R^2 x - \frac{1}{3}x^3\right|_0^R = \boxed{\frac{1}{3}\rho R^3 = M_y}$$

Now we can find the center of mass,

$$X_{cm} = \frac{M_y}{M} = \frac{\frac{1}{3}\rho R^3}{\frac{\pi}{4}\rho R^2} = \frac{4}{3\pi}R \cong 0.4244R$$

$$Y_{cm} = \frac{M_x}{M} = \frac{\frac{1}{3}\rho R^3}{\frac{\pi}{4}\rho R^2} = \frac{4}{3\pi}R \cong 0.4244R$$

From the symmetry of the region we could have anticipated $X_{cm} = Y_{cm}$.

Notice we could work problems where the density depended on x without much trouble, however if the density depended on both x and y at once then it would not be easy given our current tools. In calculus III we can treat problems which allow both x and y to vary so we relegate that more interesting class of problems to that course.