EXAMPLES OF ABSOLUTE CONVERGENCE, RATIO & ROOT TESTS

Defn/ the series $\sum a_n$ is <u>absolutely convergent</u> if the series $\sum |a_n|$ is convergent.

Defn/ If $\sum |a_n|$ diverges and $\sum a_n$ converges then $\sum a_n$ is said to be <u>conditionally convergent</u>.

Thm/ absolute convergence $\Rightarrow$ convergence.

1.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ has $a_n = \dfrac{(-1)^{n+1}}{n}$ and $|a_n| = \dfrac{1}{n}$

Note $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by $p=1$ series test.

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is <u>conditionally convergent</u>.

2.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ has $a_n = \dfrac{(-1)^{n+1}}{n^2}$ thus $|a_n| = \dfrac{1}{n^2}$

notice $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by $p=2$ series test.

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is <u>absolutely convergent</u>.

<u>Remark</u>: see Riemann's Rearrangement Lemma for why absolute convergence is so nice. In short, conditionally convergent series can be rearranged to converge to <u>ANY</u> real value.

3.) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ let's study

$$a_n = \frac{\cos(n)}{n^2} \Rightarrow |a_n| = \frac{|\cos(n)|}{n^2} \leq \frac{1}{n^2}$$

Note $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2}$ converges by Direct Comparison to the $p = 2$ series $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is absolutely convergent

Hence $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is convergent.

Th$^{\underline{m}}$ (Ratio Test): Let $\sum a_n$ be series with $a_n \neq 0$. then

Let $L = \displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ then

(1.) If $L < 1$ then $\sum a_n$ is absolutely convergent

(2.) If $L > 1$ or $L = \infty$ then $\sum a_n$ diverges

(3.) If $L = 1$ then no conclusion given here.

4.) $S = \displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$ then $a_n = \dfrac{3^n}{n!}$

thus $$\frac{a_{n+1}}{a_n} = \left(\frac{3^{n+1}}{(n+1)!}\right)\left(\frac{n!}{3^n}\right) = \frac{3 \cdot 3^n \cdot n!}{(n+1)n!3^n} = \frac{3}{n+1}$$

Observe $L = \displaystyle\lim_{n\to\infty} \left(\frac{3}{n+1}\right) = 0 < 1$ thus

S converges absolutely by the Ratio Test.

5.) $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$ has $a_n = \frac{(-1)^n n^n}{n!}$ thus $|a_n| = \frac{n^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left[\frac{(n+1)^{n+1}}{(n+1)!}\right]\left[\frac{n!}{n^n}\right]$$

$$= \frac{(n+1)^{n+1}\, n!}{(n+1)\, n!\, n^n}$$

$$= \frac{(n+1)^n}{n^n} \frac{n!}{n!}$$

$$= \left(\frac{n+1}{n}\right)^n$$

$$= \left(1 + \frac{1}{n}\right)^n$$

Recall,

$$\left(1 + \frac{1}{n}\right)^n = \exp\left(n \ln\left(1 + \frac{1}{n}\right)\right)$$

$$= \exp\left(\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}\right)$$

extending n cont.
L'Hop.

$$= \exp\left(\frac{\left(\frac{1}{1+\frac{1}{n}}\right)\frac{-1}{n^2}}{\frac{-1}{n^2}}\right)$$

$$= \exp\left(\frac{1}{1 + 1/n}\right)$$

Thus $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \exp\left(\frac{1}{1+0}\right) = \exp(1) = e$

Therefore, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = e > 1 \therefore \sum (-1)^n \frac{n^n}{n!}$ diverges by Ratio Test.

$Th^{\underline{m}}$ (Root Test)

Let $S = \sum a_n$ be series and $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ then

(1.) If $L < 1$ then S is absolutely convergent

(2.) If $L > 1$ or $L = \infty$ then S diverges

(3.) If $L = 1$ then this says nothing.

Remark: logically, case (3.) is not needed, but textbooks tend to include it to remove possible confusion.

6.) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{\tan^{-1}(n)}{2}\right]^n$ has $a_n = \left[\frac{\tan^{-1}(n)}{2}\right]^n$ thus

$$\sqrt[n]{|a_n|} = \left(\left(\frac{\tan^{-1}(n)}{2}\right)^n\right)^{1/n} = \frac{\tan^{-1}(n)}{2}$$

Hence $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left[\frac{\tan^{-1}(n)}{2}\right] = \frac{\pi}{4} < 1$

Therefore, $\displaystyle\sum_{n=1}^{\infty}\left[\frac{\tan^{-1}(n)}{2}\right]^n$ converges abs. by Root Test.

7.) $\displaystyle S = \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(\ln(n))^n}$ has $|a_n| = \dfrac{1}{(\ln(n))^n}$ thus

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left[\frac{1}{\ln(n)}\right] = 0 < 1 \therefore S \text{ conv. abs.}$$

by Root Test.