Chapter 7

local extrema for multivariate functions

In this chapter I show how the multivariate Taylor series and the theory of quadratic forms give a general form of the second derivative test. In particular we recover the second derivative tests of calculus I and III as special cases. There are technical concerns about remainders and convergence that I set aside for this chapter. The techniques developed here are not entirely general, there are exceptional cases but that is not surprising, we had the same trouble in calculus I. If you read the fine print you'll find we really only have nice theorems for continuously differentiable functions. When functions have holes or finite jump discontinuities we have to treat those separately.

7.1 Taylor series for functions of two variables

Our goal here is to find an analogue for Taylor's Theorem for function from $\mathbb{R}^n$ to $\mathbb{R}$. Recall that if $g : U \subseteq \mathbb{R} \to \mathbb{R}$ is *smooth* at $a \in \mathbb{R}$ then we can compute as many derivatives as we wish, moreover we can generate the Taylor's series for g centered at a:

$$\boxed{g(a+h) = g(a) + g'(a)h + \frac{1}{2}g''(a)h^2 + \frac{1}{3!}g''(a)h^3 + \cdots = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}h^n}$$

The equation above assumes that g is analytic at a. In other words, the function actually matches it's Taylor series near a. This concept can be made rigorous by discussing the remainder. If one can show the remainder goes to zero then that proves the function is analytic. (*read p117-127 of Edwards for more on these concepts, I did cover some of that in class this semester, Theorem 6.3 is particularly interesting*).

7.1.1 deriving the two-dimensional Taylor formula

The idea is fairly simple: create a function on $\mathbb{R}$ with which we can apply the ordinary Taylor series result. Much like our discussion of directional derivatives we compose a function of two variables

with linear path in the domain. Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ be smooth with smooth partial derivatives of all orders. Furthermore, let $(a, b) \in U$ and construct a line through (a, b) with direction vector (h_1, h_2) as usual:

$$\phi(t) = (a, b) + t(h_1, h_2) = (a + th_1, b + th_2)$$

for $t \in \mathbb{R}$. Note $\phi(0) = (a, b)$ and $\phi'(t) = (h_1, h_2) = \phi'(0)$. Construct $g = f \circ \phi : \mathbb{R} \to \mathbb{R}$ and differentiate, note we use the chain rule for functions of several variables in what follows:

$$\begin{aligned} g'(t) = (f \circ \phi)'(t) &= f'(\phi(t))\phi'(t) \\ &= \nabla f(\phi(t)) \cdot (h_1, h_2) \\ &= h_1 f_x(a + th_1, b + th_2) + h_2 f_y(a + th_1, b + th_2) \end{aligned}$$

Note $g'(0) = h_1 f_x(a, b) + h_2 f_y(a, b)$. Differentiate again (I omit $(\phi(t))$ dependence in the last steps),

$$\begin{aligned} g''(t) &= h_1 f'_x(a + th_1, b + th_2) + h_2 f'_y(a + th_1, b + th_2) \\ &= h_1 \nabla f_x(\phi(t)) \cdot (h_1, h_2) + h_2 \nabla f_y(\phi(t)) \cdot (h_1, h_2) \\ &= h_1^2 f_{xx} + h_1 h_2 f_{yx} + h_2 h_1 f_{xy} + h_2^2 f_{yy} \\ &= h_1^2 f_{xx} + 2h_1 h_2 f_{xy} + h_2^2 f_{yy} \end{aligned}$$

Thus, making explicit the point dependence, $g''(0) = h_1^2 f_{xx}(a, b) + 2h_1 h_2 f_{xy}(a, b) + h_2^2 f_{yy}(a, b)$. We may construct the Taylor series for g up to quadratic terms:

$$\begin{aligned} g(0 + t) &= g(0) + tg'(0) + \frac{1}{2} g''(0) + \cdots \\ &= f(a, b) + t[h_1 f_x(a, b) + h_2 f_y(a, b)] + \frac{t^2}{2}\left[h_1^2 f_{xx}(a, b) + 2h_1 h_2 f_{xy}(a, b) + h_2^2 f_{yy}(a, b)\right] + \cdots \end{aligned}$$

Note that $g(t) = f(a + th_1, b + th_2)$ hence $g(1) = f(a + h_1, b + h_2)$ and consequently,

$$\begin{aligned} f(a + h_1, b + h_2) = f(a, b) &+ h_1 f_x(a, b) + h_2 f_y(a, b) + \\ &+ \frac{1}{2}\left[h_1^2 f_{xx}(a, b) + 2h_1 h_2 f_{xy}(a, b) + h_2^2 f_{yy}(a, b)\right] + \cdots \end{aligned}$$

Omitting point dependence on the 2nd derivatives,

$$\boxed{f(a + h_1, b + h_2) = f(a, b) + h_1 f_x(a, b) + h_2 f_y(a, b) + \tfrac{1}{2}\left[h_1^2 f_{xx} + 2h_1 h_2 f_{xy} + h_2^2 f_{yy}\right] + \cdots}$$

Sometimes we'd rather have an expansion about (x, y). To obtain that formula simply substitute $x - a = h_1$ and $y - b = h_2$. Note that the point (a, b) is fixed in this discussion so the derivatives are not modified in this substitution,

$$\begin{aligned} f(x, y) = f(a, b) &+ (x - a) f_x(a, b) + (y - b) f_y(a, b) + \\ &+ \frac{1}{2}\left[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)\right] + \cdots \end{aligned}$$

At this point we ought to recognize the first three terms give the tangent plane to $z = f(z, y)$ at $(a, b, f(a, b))$. The higher order terms are nonlinear corrections to the linearization, these quadratic terms form a *quadratic form*. If we computed third, fourth or higher order terms we'd find that, using $a = a_1$ and $b = a_2$ as well as $x = x_1$ and $y = x_2$,

$$\boxed{f(x,y) = \sum_{n=0}^{\infty}\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\cdots\sum_{i_n=0}^{n}\frac{1}{n!}\frac{\partial^{(n)}f(a_1,a_2)}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_n}}(x_{i_1}-a_{i_1})(x_{i_2}-a_{i_2})\cdots(x_{i_n}-a_{i_n})}$$

Let me expand the third order case just for fun:

$$\sum_{i_1,i_2,i_3=0}^{3} \xrightarrow[\text{three terms}]{\text{order}} \sum_{\substack{i_1+i_2+i_3=3\\ i_1,i_2,i_3=0}}^{3} \frac{1}{3!}\left(\frac{\partial^{(3)} f(a,b)}{\partial x_{i_1}\partial x_{i_2}\partial x_{i_3}}\right)(x_{i_1}-a_{i_1})(x_{i_2}-a_{i_2})(x-a_{i_3}) =$$

$$\begin{aligned}
= \frac{\partial^3 f}{\partial x\partial x\partial x}(a,b)\,(x-a)^3 + \frac{\partial^3 f}{\partial x\partial x\partial y}(x-a)^2(y-b)^2 &+ f_{yyx}(x-a)(y-b)^2\\
+ f_{xyx}\,(x-a)^2(y-b)^2 &+ f_{yxy}\,(x-a)(y-b)^2\\
+ f_{yxx}\,(x-a)^2(y-b)^2 &+ f_{xyy}\,(x-a)(y-b)^2\\
&+ f_{yyy}\,(y-b)^3
\end{aligned}$$

Thus,

$$\begin{aligned}
f(x,y) = {} & f(a,b) + f_x(x-a) + f_y(y-b) + \\
& + \frac{1}{2}\left(f_{xx}(x-a)^2 + 2f_{xy}(x-a)(y-b) + f_{yy}(y-b)^2\right)\\
& + \frac{1}{3!}\left(f_{xxx}(x-a)^3 + 3f_{xxy}(x-a)^2(y-b) + 3f_{xyy}(x-a)(y-b)^2 + f_{yyy}(y-b)^3\right)\\
& + \cdots
\end{aligned}$$

Fortunately we're only really interested in the $n = 0, 1, 2$ order terms. Conceptually, $n = 0$ tells us where to base the tangent plane, $n = 1$ tell us how to build the tangent plane. We will soon discuss how $n = 2$ show us if the tangent plane is at the top or bottom of a hill if we're at a critical point. We pause to play with multivariate series:

Example 7.1.1.

<u>Example</u>: $f(x,y) = \sin(x)\cos(y)$

$$= \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right)\left(1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \frac{1}{6!}y^6 + \cdots\right)$$

$$= \underbrace{x - \frac{1}{2}xy^2 + \frac{1}{4!}xy^4 - \frac{1}{3!}x^3 + \frac{1}{3!2!}x^3y^2 + \frac{1}{5!}x^5}_{\text{terms upto } 5^{\text{th}} \text{ order}} + \cdots$$

$$= x + \frac{1}{3!}\left(\frac{-3!}{2}xy^2 - x^3\right) + \frac{1}{5!}\left(x^5 + \frac{5!}{4!}xy^4 + \frac{5!}{3!2!}x^3y^2\right)$$

$$= f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}\left[f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y\right.$$

$$+ \frac{1}{3!}\left(f_{xxx}(0,0)x^3 + 3f_{xxy}(0,0)x^2y + 3f_{xyy}(0,0)xy^2 + f_{yyy}(0,0)y^3\right) + \cdots$$

Neat way to calculate partial derivatives of high orders w/o actually calculating them.

<u>Note</u>:

$$f_y(0,0) = 0$$
$$f_{yyy}(0,0) = 0$$
$$f_{xxx}(0,0) = -1$$
$$f_{xyy}(0,0) = -1 \quad \left(\text{since } \frac{3!}{2} = \frac{3\cdot 2}{2} = 3\right)$$
$$f_{xxxyy}(0,0) = \frac{5!}{3!2!5} = \frac{\cancel{5}\cdot 4\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{\cancel{3}\cdot 2\cdot \cancel{2}\cdot \cancel{5}} = 2.$$

- You can use the Cauchy-Product for series to calculate much higher orders w/o doing the whole series multiplication.

Of course many functions of two variables cannot be separated into a product of a function of x and a function of y. In those cases we'd have to calculate the Taylor series directly.

Example 7.1.2.

Example: choose your path carefully.

$$f(x,y) = \sin(x+y)$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x+y)^{2n+1}}{(2n+1)!}$$
$$= x+y - \frac{1}{3!}(x+y)^3 + \frac{1}{5!}(x+y)^5 + \dots$$

Versus

$$f(x,y) = \sin(x+y)$$
$$= \sin x \cos y + \sin y \cos x$$
$$= \left(x - \frac{1}{3!}x^3 + \dots\right)\left(1 - \frac{1}{2}y^2 + \dots\right) + \left(y - \frac{1}{3!}y^3 + \dots\right)\left(1 - \frac{1}{2}x^2 + \dots\right)$$

which is better? are they the same?

(YES! thank you absolute convergence $\Rightarrow$ rearrangement ok)

generally you can't just shift terms w/o changing I.O.C. ... there be dragons.

Example 7.1.3.

<u>Example</u>: center $f(x,y) = xy$ about $(1,1)$.

$f(1,1) = 1$

$f_x(1,1) = y|_{(1,1)} = 1$

$f_y(1,1) = x|_{(1,1)} = 1$

$f_{xx}(1,1) = 0$

$f_{yy}(1,1) = 0$

$f_{xy}(1,1) = 1$

Higher derivatives are all dead.

<u>Idea</u>: use Taylor's Th$^{\underline{m}}$ to do sneaky algebra.

$$\boxed{f(x,y) = 1 + 1(x-1) + 1(y-1) + \frac{1}{2}(x-1)(y-1)}$$

centered at $(1,1)$.

7.2 Taylor series for functions of many variables

Consider $f : V \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ and denote $x \longmapsto f(x)$ where $x = (x_1, x_2, \ldots, x_n)$ then consider point $p = (a_1, a_2, \ldots, a_n)$ and construct path $\varphi : \mathbb{R} \to V$ via the rule:

$$\varphi(t) = (a_1 + th_1,\ a_2 + th_2, \ldots, a_n + th_n)$$

Note $\varphi(0) = (a_1, \ldots, a_n) = p$ and $\varphi'(0) = (h_1, \ldots, h_n) = h$. Form the composite $g = f \circ \varphi : \mathbb{R} \to V \longrightarrow \mathbb{R}$ and <u>assume</u> f has continuous partial derivatives of <u>all</u> orders. The Taylor series for g centered at zero has form $g(0+\lambda) = g(0) + \lambda g'(0) + \frac{1}{2}\lambda^2 g''(0) + \cdots$ we only intend to calculate upto this order. Focus on $g'(0)$ since $g(0) = f(\varphi(0)) = f(p)$ is easy.

$$\begin{aligned} g'(t) &= \frac{d}{dt}\left[f(x_1, x_2, \ldots, x_n)\right] \qquad \text{where } x_j = a_j + th_j \text{ for } j = 1, 2, \ldots, n. \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a + th)\, \frac{d}{dt}(a_j + th_j) \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a + th) \cdot h_j \end{aligned}$$

$$\boxed{g'(0) = \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a)}$$

Likewise,

$$\begin{aligned} g''(t) &= \frac{d}{dt}\left(\sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a + th)\right) \\ &= \sum_{j=1}^{n}\left(h_j \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k \partial x_j}(a + th)\, \frac{d}{dt}(a_k + th_k)\right) \\ &= \underline{\sum_{j,k=1}^{n} h_j h_k \frac{\partial^2 f}{\partial x_k \partial x_j}(a + th)} \end{aligned}$$

using continuous diff to commute partials!

Continuing:

$$g''(0) = \sum_{j,k=1}^{n} h_j h_k \frac{\partial^2 f}{\partial x_k \partial x_j}(a)$$

Put it all together,

$$g(0+\lambda) = g(0) + \lambda g'(0) + \tfrac{1}{2}\lambda^2 g''(0) + \cdots$$
$$= f(a) + \lambda \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a) + \tfrac{1}{2}\lambda^2 \sum_{j,k=1}^{n} h_j h_k \frac{\partial^2 f}{\partial x_k \partial x_j}(a) + \cdots$$

Finally, $g(0+1) = f(\varphi(1)) = f(a+h) = f(a_1+h_1, a_2+h_2, \ldots, a_n+h_n)$

$$\boxed{f(a+h) = f(a) + \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a) + \sum_{j,k=1}^{n} \frac{h_j h_k}{2!} \frac{\partial^2 f}{\partial x_k \partial x_j}(a) + \cdots}$$

We can substitute $x = a+h \longrightarrow h = x-a$
which means $h_1 = x_1 - a_1, \ldots, h_n = x_n - a_n$ and then

$$\boxed{f(x) = f(a) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a)(x_j - a_j) + \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial x_k \partial x_j} \frac{1}{2!}(x_k - a_k)(x_j - a_j) + \cdots}$$

Summary: We can approximate $f(x)$ locally by a polynomial in x, y with terms upto order two by stringing together partial derivatives of orders 1 and 2 appropriately. Generally a 2nd order polynomial in $\mathbb{R}[x_1, x_2, \ldots, x_n]$ is called a <u>quadratic form</u> on $\mathbb{R}^n$.

7.3 quadratic forms, conic sections and quadric surfaces

Conic sections and quadratic surfaces are common examples in calculus. For example:

$$x^2 + y^2 = 4 \qquad \text{level curve; generally has form } f(x, y) = k$$

$$x^2 + 4y^2 + z^2 = 1 \qquad \text{level surface; generally has form } F(x, y, z) = k$$

Our goal is to see what linear algebra and multivariate calculus have to say about conic sections and quadric surfaces. (these notes borrowed from my linear algebra notes)

7.3.1 quadratic forms and their matrix

We are primarily interested in the application of this discussion to $\mathbb{R}^2$ and $\mathbb{R}^3$, however, these concepts equally well apply to arbitrarily high finite dimensional problems where the geometry is not easily pictured.

Definition 7.3.1.

Generally, a **quadratic form** Q is a function $Q : \mathbb{R}^n \to \mathbb{R}$ whose formula can be written $Q(\vec{x}) = \vec{x}^T A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ where $A \in \mathbb{R}^{n\times n}$ such that $A^T = A$. In particular, if $\vec{x} = [x, y]^T$ and $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ then

$$\vec{x}^T A\vec{x} = ax^2 + bxy + byx + cy^2 = ax^2 + 2bxy + y^2.$$

The $n = 3$ case is similar,denote $A = [A_{ij}]$ and $\vec{x} = [x, y, z]^T$ so that

$$\vec{x}^T A\vec{x} = A_{11}x^2 + 2A_{12}xy + 2A_{13}xz + A_{22}y^2 + 2A_{23}yz + A_{33}z^2.$$

Generally, if $[A_{ij}] \in \mathbb{R}^{n\times n}$ and $\vec{x} = [x_i]^T$ then the quadratic form

$$\vec{x}^T A\vec{x} = \sum_{i,j} A_{ij}x_ix_j = \sum_{i=1}^{n} A_{ii}x_i^2 + \sum_{i<j} 2A_{ij}x_ix_j.$$

In case you wondering, yes you could write a given quadratic form with a different matrix which is not symmetric, but we will find it convenient to insist that our matrix is symmetric since that choice is always possible for a given quadratic form.

You should notice can write a given quadratic form in terms of a dot-product:

$$\vec{x}^T A\vec{x} = \vec{x} \cdot (A\vec{x}) = (A\vec{x}) \cdot \vec{x} = \vec{x}^T A^T \vec{x}$$

Some texts actually use the middle equality above to define a symmetric matrix.

Example 7.3.2.

$$2x^2 + 2xy + 2y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 7.3.3.

$$2x^2 + 2xy + 3xz - 2y^2 - z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 1 & 3/2 \\ 1 & -2 & 0 \\ 3/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Proposition 7.3.4.

The values of a quadratic form on $\mathbb{R}^n - \{0\}$ is completely determined by it's values on the $(n-1)$-sphere $S_{n-1} = \{\vec{x} \in \mathbb{R}^n \mid ||\vec{x}|| = 1\}$. In particular, $Q(\vec{x}) = ||\vec{x}||^2 Q(\hat{x})$ where $\hat{x} = \frac{1}{||\vec{x}||}\vec{x}$.

Proof: Let $Q(\vec{x}) = \vec{x}^T A\vec{x}$. Notice that we can write any nonzero vector as the product of its magnitude $||x||$ and its direction $\hat{x} = \frac{1}{||\vec{x}||}\vec{x}$,

$$Q(\vec{x}) = Q(||\vec{x}||\hat{x}) = (||\vec{x}||\hat{x})^T A||\vec{x}||\hat{x} = ||\vec{x}||^2 \hat{x}^T A\hat{x} = ||x||^2 Q(\hat{x}).$$

Therefore $Q(\vec{x})$ is simply proportional to $Q(\hat{x})$ with proportionality constant $||\vec{x}||^2$. $\square$

The proposition above is very interesting. It says that if we know how Q works on unit-vectors then we can extrapolate its action on the remainder of $\mathbb{R}^n$. If $f : S \to \mathbb{R}$ then we could say $f(S) > 0$ iff $f(s) > 0$ for all $s \in S$. Likewise, $f(S) < 0$ iff $f(s) < 0$ for all $s \in S$. The proposition below follows from the proposition above since $||\vec{x}||^2$ ranges over all nonzero positive real numbers in the equations above.

Proposition 7.3.5.

If Q is a quadratic form on $\mathbb{R}^n$ and we denote $\mathbb{R}^n_* = \mathbb{R}^n - \{0\}$

1.(negative definite) $Q(\mathbb{R}^n_*) < 0$ iff $Q(S_{n-1}) < 0$
2.(positive definite) $Q(\mathbb{R}^n_*) > 0$ iff $Q(S_{n-1}) > 0$
3.(non-definite) $Q(\mathbb{R}^n_*) = \mathbb{R} - \{0\}$ iff $Q(S_{n-1})$ has both positive and negative values.

7.3.2 almost an introduction to eigenvectors

Eigenvectors and eigenvalues play an important role in theory and application. In particular, eigenvalues and eigenvectors allow us to (if possible) *diagonalize* a matrix. This essentially is the problem of choosing coordinates for a particular system which most clearly reveals the true nature of the system. For example, the fact that $2xy = 1$ is a hyperbola is clearly seen once we change to coordinates whose axes point along the eigenvectors for the quadratic form $Q(x, y) = 2xy$.

Likewise, in the study of rotating rigid bodies the eigenvectors of the inertia tensor give the so-called principle axes of inertia. When a body is set to spin about such an axes through its center of mass the motion is natural, smooth and does not wobble. The inertia tensor gives a quadratic form in the angular velocity which represents the rotational kinetic energy. I've probably assigned a homework problem so you can understand this paragraph. In any event, there are many motivations for studying eigenvalues and vectors. I explain much more theory for e-vectors in the linear course.

Definition 7.3.6.

Let $A \in \mathbb{R}^{n\times n}$. If $v \in \mathbb{R}^{n\times 1}$ is **nonzero** and $Av = \lambda v$ for some $\lambda \in \mathbb{C}$ then we say v is an **eigenvector** with **eigenvalue** λ of the matrix A.

Proposition 7.3.7.

Let $A \in \mathbb{R}^{n\times n}$ then λ is an eigenvalue of A iff $det(A-\lambda I) = 0$. We say $P(\lambda) = det(A-\lambda I)$ the **characteristic polynomial** and $det(A-\lambda I) = 0$ is the **characteristic equation**.

Proof: Suppose λ is an eigenvalue of A then there exists a nonzero vector v such that $Av = \lambda v$ which is equivalent to $Av - \lambda v = 0$ which is precisely $(A-\lambda I)v = 0$. Notice that $(A-\lambda I)0 = 0$ thus the matrix $(A-\lambda I)$ is singular as the equation $(A-\lambda I)x = 0$ has more than one solution. Consequently $det(A-\lambda I) = 0$.

Conversely, suppose $det(A-\lambda I) = 0$. It follows that $(A-\lambda I)$ is singular. Clearly the system $(A-\lambda I)x = 0$ is consistent as $x = 0$ is a solution hence we know there are infinitely many solutions. In particular there exists at least one vector $v \neq 0$ such that $(A-\lambda I)v = 0$ which means the vector v satisfies $Av = \lambda v$. Thus v is an eigenvector with eigenvalue λ for A. $\square$

Example 7.3.8. *Let* $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ *find the e-values and e-vectors of* A.

$$det(A-\lambda I) = det\begin{bmatrix} 3-\lambda & 1 \\ 3 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda) - 3 = \lambda^2 - 4\lambda = \lambda(\lambda-4) = 0$$

We find $\lambda_1 = 0$ *and* $\lambda_2 = 4$. *Now find the e-vector with e-value* $\lambda_1 = 0$, *let* $u_1 = [u, v]^T$ *denote the e-vector we wish to find. Calculate,*

$$(A-0I)u_1 = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3u+v \\ 3u+v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Obviously the equations above are redundant and we have infinitely many solutions of the form $3u + v = 0$ *which means* $v = -3u$ *so we can write,* $u_1 = \begin{bmatrix} u \\ -3u \end{bmatrix} = u\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. *In applications we often make a choice to select a particular e-vector. Most modern graphing calculators can calculate e-vectors. It is customary for the e-vectors to be chosen to have length one. That is a useful choice for certain applications as we will later discuss. If you use a calculator it would likely give*

$u_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ *although the* $\sqrt{10}$ *would likely be approximated unless your calculator is smart.*

Continuing we wish to find eigenvectors $u_2 = [u, v]^T$ *such that* $(A - 4I)u_2 = 0$. *Notice that* u, v *are disposable variables in this context, I do not mean to connect the formulas from the* $\lambda = 0$ *case with the case considered now.*

$$(A - 4I)u_1 = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -u + v \\ 3u - 3v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Again the equations are redundant and we have infinitely many solutions of the form $v = u$. *Hence,* $u_2 = \begin{bmatrix} u \\ u \end{bmatrix} = u \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *is an eigenvector for any* $u \in \mathbb{R}$ *such that* $u \neq 0$.

Theorem 7.3.9.

A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric iff there exists an orthonormal eigenbasis for A.

There is a geometric proof of this theorem in Edwards[1] (see Theorem 8.6 pgs 146-147) . I prove half of this theorem in my linear algebra notes by a non-geometric argument (full proof is in Appendix C of Insel,Spence and Friedberg). It might be very interesting to understand the connection between the geometric verse algebraic arguments. We'll content ourselves with an example here:

Example 7.3.10. *Let* $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. *Observe that* $det(A - \lambda I) = -\lambda(\lambda + 1)(\lambda - 3)$ *thus* $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 3$. *We can calculate orthonormal e-vectors of* $v_1 = [1, 0, 0]^T$, $v_2 = \frac{1}{\sqrt{2}}[0, 1, -1]^T$ *and* $v_3 = \frac{1}{\sqrt{2}}[0, 1, 1]^T$. *I invite the reader to check the validity of the following equation:*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Its really neat that to find the inverse of a matrix of orthonormal e-vectors we need only take the transpose; note $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

7.3.3 quadratic form examples

Example 7.3.11. *Consider the quadric form* $Q(x, y) = x^2 + y^2$. *You can check for yourself that* $z = Q(x, y)$ *is a cone and* Q *has positive outputs for all inputs except* $(0, 0)$. *Notice that* $Q(v) = ||v||^2$

[1] think about it, there is a 1-1 correspondance between symmetric matrices and quadratic forms

so it is clear that $Q(S_1) = 1$. We find agreement with the preceding proposition.

Next, think about the application of $Q(x, y)$ to level curves; $x^2 + y^2 = k$ is simply a circle of radius $\sqrt{k}$ or just the origin.

Finally, let's take a moment to write $Q(x, y) = [x, y] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = \lambda_2 = 1$.

Example 7.3.12. *Consider the quadric form $Q(x, y) = x^2 - 2y^2$. You can check for yourself that $z = Q(x, y)$ is a hyperboloid and Q has non-definite outputs since sometimes the x^2 term dominates whereas other points have $-2y^2$ as the dominent term. Notice that $Q(1, 0) = 1$ whereas $Q(0, 1) = -2$ hence we find $Q(S_1)$ contains both positive and negative values and consequently we find agreement with the preceding proposition.*

Next, think about the application of $Q(x, y)$ to level curves; $x^2 - 2y^2 = k$ yields either hyperbolas which open vertically ($k > 0$) or horizontally ($k < 0$) or a pair of lines $y = \pm\frac{x}{2}$ in the $k = 0$ case.

Finally, let's take a moment to write $Q(x, y) = [x, y] \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 1$ and $\lambda_2 = -2$.

Example 7.3.13. *Consider the quadric form $Q(x, y) = 3x^2$. You can check for yourself that $z = Q(x, y)$ is parabola-shaped trough along the y-axis. In this case Q has positive outputs for all inputs except $(0, y)$, we would call this form* **positive semi-definite**. *A short calculation reveals that $Q(S_1) = [0, 3]$ thus we again find agreement with the preceding proposition (case 3).*
Next, think about the application of $Q(x, y)$ to level curves; $3x^2 = k$ is a pair of vertical lines: $x = \pm\sqrt{k/3}$ or just the y-axis.
Finally, let's take a moment to write $Q(x, y) = [x, y] \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 3$ and $\lambda_2 = 0$.

Example 7.3.14. *Consider the quadric form $Q(x, y, z) = x^2 + 2y^2 + 3z^2$. Think about the application of $Q(x, y, z)$ to level surfaces; $x^2 + 2y^2 + 3z^2 = k$ is an ellipsoid.*
Finally, let's take a moment to write $Q(x, y, z) = [x, y, z] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 1$ and $\lambda_2 = 2$ and $\lambda_3 = 3$.

The examples given thus far are the simplest cases. We don't really need linear algebra to understand them. In contrast, e-vectors and e-values will prove a useful tool to unravel the later examples.

Proposition 7.3.15.

If Q is a quadratic form on $\mathbb{R}^n$ with matrix A and e-values $\lambda_1, \lambda_2, \ldots, \lambda_n$ with orthonormal e-vectors $v_1, v_2, \ldots, v_n$ then

$$Q(v_i) = {\lambda_i}^2$$

for $i = 1, 2, \ldots, n$. Moreover, if $P = [v_1|v_2|\cdots|v_n]$ then

$$Q(\vec{x}) = (P^T\vec{x})^T P^T A P P^T \vec{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where we defined $\vec{y} = P^T\vec{x}$.

Let me restate the proposition above in simple terms: we can transform a given quadratic form to a diagonal form by finding orthonormalized e-vectors and performing the appropriate coordinate transformation. Since P is formed from orthonormal e-vectors we know that P will be either a rotation or reflection. This proposition says we can remove "cross-terms" by transforming the quadratic forms with an appropriate rotation.

Example 7.3.16. *Consider the quadric form $Q(x, y) = 2x^2 + 2xy + 2y^2$. It's not immediately obvious (to me) what the level curves $Q(x, y) = k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x, y) = [x, y]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:*

$$det(A - \lambda I) = det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (\lambda - 2)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$$

Therefore, the e-values are $\lambda_1 = 1$ and $\lambda_2 = 3$.

$$(A - I)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u + v = 0$ to give $v = -u$ choose $u = 1$ then normalize to get the vector above. Next,

$$(A - 3I)\vec{u}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

I just solved $u - v = 0$ to give $v = u$ choose $u = 1$ then normalize to get the vector above. Let $P = [\vec{u}_1|\vec{u}_2]$ and introduce new coordinates $\vec{y} = [\bar{x}, \bar{y}]^T$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x &= \tfrac{1}{2}(\bar{x} + \bar{y}) \\ y &= \tfrac{1}{2}(-\bar{x} + \bar{y}) \end{aligned} \quad \text{or} \quad \begin{aligned} \bar{x} &= \tfrac{1}{2}(x - y) \\ \bar{y} &= \tfrac{1}{2}(x + y) \end{aligned}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield[2]*:*

$$\tilde{Q}(\bar{x}, \bar{y}) = \bar{x}^2 + 3\bar{y}^2$$

[2] technically $\tilde{Q}(\bar{x}, \bar{y})$ is $Q(x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y}))$

It is clear that in the barred coordinate system the level curve $Q(x,y) = k$ is an ellipse. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of $Q(x,y) = 2x^2 + 2xy + 2y^2 = k$ is an ellipse rotated by 45 degrees.

Example 7.3.17. *Consider the quadric form $Q(x,y) = x^2+2xy+y^2$. It's not immediately obvious (to me) what the level curves $Q(x,y) = k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x,y) = [x,y]\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:*

$$det(A - \lambda I) = det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (\lambda - 1)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$$

Therefore, the e-values are $\lambda_1 = 0$ and $\lambda_2 = 2$.

$$(A - 0)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u + v = 0$ to give $v = -u$ choose $u = 1$ then normalize to get the vector above. Next,

$$(A - 2I)\vec{u}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

I just solved $u - v = 0$ to give $v = u$ choose $u = 1$ then normalize to get the vector above. Let $P = [\vec{u}_1|\vec{u}_2]$ and introduce new coordinates $\vec{y} = [\bar{x}, \bar{y}]^T$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x &= \tfrac{1}{2}(\bar{x} + \bar{y}) \\ y &= \tfrac{1}{2}(-\bar{x} + \bar{y}) \end{aligned} \quad or \quad \begin{aligned} \bar{x} &= \tfrac{1}{2}(x - y) \\ \bar{y} &= \tfrac{1}{2}(x + y) \end{aligned}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x}, \bar{y}) = 2\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x,y) = k$ is a pair of paralell lines. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of $Q(x,y) = x^2 + 2xy + y^2 = k$ is a line with slope -1. Indeed, with a little algebraic insight we could have anticipated this result since $Q(x,y) = (x+y)^2$ so $Q(x,y) = k$ implies $x + y = \sqrt{k}$ thus $y = \sqrt{k} - x$.

Example 7.3.18. *Consider the quadric form $Q(x,y) = 4xy$. It's not immediately obvious (to me) what the level curves $Q(x,y) = k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x,y) = [x,y]\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:*

$$det(A - \lambda I) = det\begin{bmatrix} -\lambda & 2 \\ 2 & -\lambda \end{bmatrix} = \lambda^2 - 4 = (\lambda + 2)(\lambda - 2) = 0$$

Therefore, the e-values are $\lambda_1 = -2$ *and* $\lambda_2 = 2$.

$$(A+2I)\vec{u}_1 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u + v = 0$ *to give* $v = -u$ *choose* $u = 1$ *then normalize to get the vector above. Next,*

$$(A-2I)\vec{u}_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

I just solved $u - v = 0$ *to give* $v = u$ *choose* $u = 1$ *then normalize to get the vector above. Let* $P = [\vec{u}_1|\vec{u}_2]$ *and introduce new coordinates* $\vec{y} = [\bar{x}, \bar{y}]^T$ *defined by* $\vec{y} = P^T\vec{x}$. *Note these can be inverted by multiplication by* P *to give* $\vec{x} = P\vec{y}$. *Observe that*

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x &= \tfrac{1}{2}(\bar{x} + \bar{y}) \\ y &= \tfrac{1}{2}(-\bar{x} + \bar{y}) \end{aligned} \quad \text{or} \quad \begin{aligned} \bar{x} &= \tfrac{1}{2}(x - y) \\ \bar{y} &= \tfrac{1}{2}(x + y) \end{aligned}$$

The proposition preceding this example shows that substitution of the formulas above into Q *yield:*

$$\tilde{Q}(\bar{x}, \bar{y}) = -2\bar{x}^2 + 2\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x, y) = k$ *is a hyperbola. If we draw the barred coordinate system superposed over the* xy*-coordinate system then you'll see that the graph of* $Q(x, y) = 4xy = k$ *is a hyperbola rotated by* 45 *degrees.*

Remark 7.3.19.

I made the preceding triple of examples all involved the same rotation. This is purely for my lecturing convenience. In practice the rotation could be by all sorts of angles. In addition, you might notice that a different ordering of the e-values would result in a redefinition of the barred coordinates. [3]

We ought to do at least one 3-dimensional example.

Example 7.3.20. *Consider the quadric form defined below:*

$$Q(x, y, z) = [x, y, z] \begin{bmatrix} 6 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Denote the matrix of the form by A *and calculate the e-values/vectors:*

$$\begin{aligned} det(A - \lambda I) &= det \begin{bmatrix} 6-\lambda & -2 & 0 \\ -2 & 6-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} \\ &= [(\lambda - 6)^2 - 4](5 - \lambda) \\ &= (5 - \lambda)[\lambda^2 - 12\lambda + 32](5 - \lambda) \\ &= (\lambda - 4)(\lambda - 8)(5 - \lambda) \end{aligned}$$

Therefore, the e-values are $\lambda_1 = 4$, $\lambda_2 = 8$ and $\lambda_3 = 5$. After some calculation we find the following orthonormal e-vectors for A:

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $P = [\vec{u}_1|\vec{u}_2|\vec{u}_3]$ and introduce new coordinates $\vec{y} = [\bar{x}, \bar{y}, \bar{z}]^T$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rcl} x &=& \frac{1}{2}(\bar{x}+\bar{y}) \\ y &=& \frac{1}{2}(-\bar{x}+\bar{y}) \\ z &=& \bar{z} \end{array} \quad \text{or} \quad \begin{array}{rcl} \bar{x} &=& \frac{1}{2}(x-y) \\ \bar{y} &=& \frac{1}{2}(x+y) \\ \bar{z} &=& z \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x}, \bar{y}, \bar{z}) = 4\bar{x}^2 + 8\bar{y}^2 + 5\bar{z}^2$$

It is clear that in the barred coordinate system the level surface $Q(x, y, z) = k$ is an ellipsoid. If we draw the barred coordinate system superposed over the xyz-coordinate system then you'll see that the graph of $Q(x, y, z) = k$ is an ellipsoid rotated by 45 degrees around the $z-axis$.

Remark 7.3.21.

There is a connection between the shape of level curves $Q(x_1, x_2, \ldots, x_n) = k$ and the graph $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ of f. I'll discuss $n = 2$ but these comments equally well apply to $w = f(x, y, z)$ or higher dimensional examples. Consider a critical point (a, b) for $f(x, y)$ then the Taylor expansion about (a, b) has the form

$$f(a + h, b + k) = f(a, b) + Q(h, k)$$

where $Q(h, k) = \frac{1}{2}h^2 f_{xx}(a, b) + hkf_{xy}(a, b) + \frac{1}{2}h^2 f_{yy}(a, b) = [h, k][Q](h, k)$. Since $[Q]^T = [Q]$ we can find orthonormal e-vectors $\vec{u}_1, \vec{u}_2$ for $[Q]$ with e-values λ_1 and λ_2 respective. Using $U = [\vec{u}_1|\vec{u}_2]$ we can introduce rotated coordinates $(\bar{h}, \bar{k}) = U(h, k)$. These will give

$$Q(\bar{h}, \bar{k}) = \lambda_1\bar{h}^2 + \lambda_2\bar{k}^2$$

Clearly if $\lambda_1 > 0$ and $\lambda_2 > 0$ then $f(a, b)$ yields the local minimum whereas if $\lambda_1 < 0$ and $\lambda_2 < 0$ then $f(a, b)$ yields the local maximum. Edwards discusses these matters on pgs. 148-153. In short, supposing $f \approx f(p) + Q$, if all the e-values of Q are positive then f has a local minimum of $f(p)$ at p whereas if all the e-values of Q are negative then f reaches a local maximum of $f(p)$ at p. Otherwise Q has both positive and negative e-values and we say Q is non-definite and the function has a saddle point. If all the e-values of Q are positive then Q is said to be **positive-definite** whereas if all the e-values of Q are negative then Q is said to be **negative-definite**. Edwards gives a few nice tests for ascertaining if a matrix is positive definite without explicit computation of e-values.

7.4 local extrema from eigenvalues and quadratic forms

We have all the tools we need, let's put them to use now.

Example 7.4.1.

$$f(x,y) = x^2 - 2xy + y^2$$

$$\nabla f = \langle 2x - 2y,\ 2y - 2x \rangle \quad \therefore \quad \nabla f = 0 \Rightarrow \underline{y = x}.$$

Infinitely many critical points, have the form (a,a).

$$f_{xx}(a,a) = 2, \quad f_{xy}(a,a) = -2, \quad f_{yy}(a,a) = 2$$

Hence, expanding about (a,a),

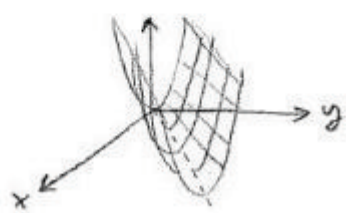
$$\begin{aligned} f(x,y) &= f(a,a) + Q_{(a,a)}(x,y) \\ &= a^2 - 2a^2 + a^2 + \tfrac{1}{2}\left(2(x-a)^2 - 4(x-a)(y-a) + 2(y-a)^2\right) \\ &= \underline{(x-a)^2 - 2(x-a)(y-a) + (y-a)^2}. \end{aligned}$$

Note that

$$[Q_{(a,a)}] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{aligned} \det\left[[Q_{(a,a)}] - \lambda I\right] &= \det\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} \\ &= (\lambda - 1)^2 - 1 \\ &= \lambda^2 - 2\lambda \\ &= \lambda(\lambda - 2) \quad \therefore \quad \underline{\lambda_1 = 0,\ \lambda_2 = 2}. \end{aligned}$$

This is a semi-definite form. Each point along $y = x$ gives local min. Note that $f(x,y) = (x-y)^2$ thus $f(x,y) \geq 0$ hence $f(a,a) = 0$ is the global min. for f as well.



(Zero e-values for Q indicate this type of behaviour.)

Example 7.4.2.

$$f(x,y) = \exp(-x^2-y^2)$$
$$f_x(x,y) = -2x\exp(-x^2-y^2)$$
$$f_y(x,y) = -2y\exp(-x^2-y^2)$$
$$f_{xx}(x,y) = -2\exp(-x^2-y^2) + 4x^2\exp(-x^2-y^2)$$
$$f_{xy}(x,y) = 4xy\exp(-x^2-y^2)$$
$$f_{yy}(x,y) = (4y^2-2)\exp(-x^2-y^2)$$

Note $\nabla f(x,y) = 0 = \langle -2x, -2y\rangle e^{-x^2-y^2} \Rightarrow \underline{x=y=0}$

Only critical pt. is $(0,0)$. We find

$$Q(x,y) = \tfrac{1}{2} f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \tfrac{1}{2} f_{yy}(0,0)y^2$$
$$= \tfrac{1}{2}(-2)x^2 + 0 + \tfrac{1}{2}(-2)y^2$$
$$= -x^2-y^2$$

Thus, we find $(0,0)$ gives local max since Q is negative definite.

$$\boxed{f(x,y) = f(0,0) + Q(x,y) = 1 - x^2 - y^2 + \cdots}$$

Note,

$$[Q] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{thus } \lambda_1 = \lambda_2 = -1 < 0$$

$\therefore$ <u>negative definite</u>

$\therefore$ <u>local maximum</u>.

<u>Remark</u>: this is actually a <u>global</u> max. since $-x^2-y^2 \leq 0$ and $\exp((-\infty, 0]) = (0, 1]$.

Example 7.4.3.

Example: Let $f(x,y) = 2x^2 - xy - 3y^2 - 3x + 7y$ find all critical points and analyze those pts. Find local extrema.

$$\nabla f = \langle 4x - y - 3, -x - 6y + 7 \rangle = \langle 0, 0 \rangle$$

$$\begin{aligned} 4x - y - 3 &= 0 \longrightarrow y = 4x - 3 \\ -x - 6y + 7 &= 0 \longrightarrow -x - 6(4x-3) + 7 = 0 \\ & \qquad -25x + 18 + 7 = 0 \\ & \qquad \Rightarrow -25x = -25 \\ & \qquad \Rightarrow \underline{x = 1} \Rightarrow \underline{y = 4-3 = 1}. \end{aligned}$$

We find a single critical point; $(1,1)$.

$f_x(x,y) = 4x - y - 3$: note $f_x(1,1) = 4 - 1 - 3 = 0$

$f_y(x,y) = -x - 6y + 7$: and $\underbrace{f_y(1,1) = -1 - 6 + 7 = 0}_{\text{good check on earlier calculation.}}$

$f_{xx}(x,y) = 4$

$f_{xy}(x,y) = -1$

$f_{yy}(x,y) = -6$

We find $Q(x,y) = 2(x-1)^2 - 2(x-1)(y-1) - 3(y-1)^2$ hence,

$f(x,y) = f(1,1) + Q(x,y)$, $f(1,1) = 2 - 1 - 3 - 3 + 7 = 2$

$$\therefore \boxed{f(x,y) = 2 + 2(x-1)^2 - 2(x-1)(y-1) - 3(y-1)^2}$$

Note this is not an approximation since higher terms all vanish. It seems likely this is a saddle point, but we need not guess. Note the matrix of Q is

$$[Q] = \begin{bmatrix} 2 & -1 \\ -1 & -3 \end{bmatrix}$$

$$\begin{aligned} 0 = \det([Q] - \lambda I) &= \det\begin{pmatrix} 2-\lambda & -1 \\ -1 & -3-\lambda \end{pmatrix} \\ &= (\lambda+3)(\lambda-2) - 1 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda+3)(\lambda-2) \quad \therefore \underline{\lambda_1 = -3 \ \& \ \lambda_2 = 2}. \end{aligned}$$

Thus $f(1,1)$ is neither max nor min near $(1,1)$. It is at the saddle point $(1,1,2)$.

Example 7.4.4.

<u>Example</u>: $f(x,y) = \sin x \cosh y$

$$\nabla f = \langle \cos x \cosh y, \sin x \sinh y \rangle$$

Thus critical pts must have

① $\cos(x)\cosh y = 0$
② $\sin x \sinh y = 0$

Note $\cosh y = \frac{1}{2}(e^y + e^{-y}) \neq 0$ for all $y \in \mathbb{R}$ hence we <u>need</u> $\cos(x) = 0$. But then $\cos(x) = 0 \Rightarrow \sin x \neq 0$ hence we also <u>need</u> $\sinh y = \frac{1}{2}(e^y - e^{-y}) = 0$. Note

$$\begin{aligned} \sinh y = \tfrac{1}{2}(e^y - e^{-y}) = 0 &\Rightarrow e^y = e^{-y} \\ &\Rightarrow y = -y \\ &\Rightarrow \underline{y = 0}. \end{aligned}$$

We find a whole family of critical points. Namely

$$(x,y) \in \mathbb{R}^2 \text{ such that } \cos(x) = 0 \ \underline{\text{AND}}\ y = 0$$

That gives critical pts. $= \{(n\pi + \frac{\pi}{2}, 0) \mid n \in \mathbb{Z}\}$.

$$\begin{aligned} f_{xx}(x,y) &= -\sin x \cosh y \\ f_{xy}(x,y) &= \cos x \sinh y \\ f_{yy}(x,y) &= \sin x \cosh y \end{aligned}$$

Thus $f_{xx}(x,0) = -\sin x$, $f_{xy}(x,0) = 0$ and $f_{yy}(x,0) = \sin x$.

If $x = 2k\pi + \frac{\pi}{2}$ then $\sin(x) = 1$ whereas if $x = (2k+1)\pi + \frac{\pi}{2}$ then $\sin(x) = -1$ for all $k \in \mathbb{Z}$.

To summarize, $f_{xx}(n\pi + \frac{\pi}{2}, 0) = (-1)^n$ and $f_{yy}(n\pi + \frac{\pi}{2}, 0) = (-1)^{n+1}$.

We can approximate $f(x,y) = \sin x \cosh y$ by

$$\boxed{f(x,y) \cong (-1)^n + \tfrac{1}{2}(-1)^n\left(x - n\pi - \tfrac{\pi}{2}\right)^2 - \tfrac{1}{2}(-1)^n y^2}$$

This is again a saddle-type for each $n \in \mathbb{Z}$.

$$[Q] = \begin{bmatrix} \frac{1}{2}(-1)^n & 0 \\ 0 & -\frac{1}{2}(-1)^n \end{bmatrix} \quad \therefore \ \underline{\lambda_1 = \pm\tfrac{1}{2} \quad \lambda_2 = \mp\tfrac{1}{2}}$$

Chapter 8

on manifolds and multipliers

In this chapter we show the application of the most difficult results in this course, namely the implicit and inverse mapping theorems. Our first application is in the construction of manifolds as graphs or level sets. Then once we have a convenient concept of a manifold we discuss the idea of Lagrange multipliers. The heart of the method combines orthogonal complements from linear algebra along side the construction of tangent spaces in this course. Hopefully this chapter will help you understand why the implicit and inverse mapping theorems are so useful and also why we need manifolds to make sense of our problems. The patching definition for a manifold is not of much use in this chapter although we will mention how it connects to the other two formulations of a manifold in $\mathbb{R}^m$ in the context of a special case.

8.1 surfaces in $\mathbb{R}^3$

Manifolds or surfaces play a role similar to functions in this course. Our goal is not the study of manifolds alone but it's hard to give a complete account of differentiation unless we have some idea of what is a tangent plane. This subsection does break from the larger pattern of thought in this chapter. I include it here to try to remind how surfaces and tangent planes are described in $\mathbb{R}^3$. We need some amount of generalization beyond this section because the solution of max/min problems with constraints will take us into higher dimensional surfaces even for problems that only involve two or three spatial dimensions. We treat those questions in the next chapter.

There are three main methods to describe surfaces:

1. As a **graph**: $S = \{(x, y, z) \mid z = f(x, y) \text{ where } (x, y) \in dom(f)\}$.
2. As a **level surface**: $S = \{(x, y, z) \mid F(x, y, z) = 0\}$
3. As a **parametrized surface**: $S = \{X(u, v) \mid (u, v) \in dom(X)\}$

Let me remind you we found the tangent plane at $(x_o, y_o, z_o) \in S$ for each of these formalisms as follows (continuing to use the same notation as above):

1. For the **graph**: $z = z_o + f(x_o, y_o) + f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o)$.
2. For the **level surface**: plane through (x_o, y_o, z_o) with normal $(\nabla F)(x_o, y_o, z_o)$
3. For the **parametrized surface**: find (u_o, v_o) with $X(u_o, v_o) = (x_o, y_o, z_o)$, the tangent plane goes through $X(u_o, v_o)$ and has normal $N(u_o, v_o) = X_u(u_o, v_o) \times X_v(u_o, v_o)$.

Perhaps you recall that the normal vector field to the surface S was important in the formulation of surface integrals to calculate the flux of vector fields.

Example 8.1.1. *The plane through the point $\vec{r}_o$ with normal $\vec{n} =< a, b, c >$ can be described as:*

1. *all $\vec{r} \in \mathbb{R}^3$ such that $(\vec{r} - \vec{r}_o) \cdot \vec{n} = 0$.*
2. *all $(x, y, z) \in \mathbb{R}^3$ such that $a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$*
3. *if $c \neq 0$, the graph $z = f_3(x, y)$ where $f_3(x, y) = z_o + \frac{a}{c}(x - x_o) + \frac{b}{c}(y - y_o)$*
4. *if $b \neq 0$, the graph $y = f_3(x, z)$ where $f_2(x, z) = y_o + \frac{a}{b}(x - x_o) + \frac{c}{b}(z - z_o)$*
5. *if $a \neq 0$, the graph $x = f_1(y, z)$ where $f_1(y, z) = x_o + \frac{b}{a}(y - y_o) + \frac{c}{a}(z - z_o)$*
6. *given any two linearly independent vectors $\vec{a}, \vec{b}$ in the plane, the plane is the image of the mapping $X : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $X(u, v) = \vec{r}_o + u\vec{a} + v\vec{b}$*

Example 8.1.2. *The sphere of radius R centered about the origin can be described as:*

1. *all $(x, y, z) \in \mathbb{R}^3$ such that $F(x, y, z) = x^2 + y^2 + z^2 = R^2$*
2. *the graphs of $z = f_\pm(x, y)$ where $f_\pm(x, y) = \pm\sqrt{R^2 - x^2 - y^2}$*
3. *for $(u, v) \in [0, 2\pi] \times [0, \pi]$, $X(u, v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)$*

You may recall that the level surface concept allowed by far the easiest computation of the normal of the tangent plane for a particular point. For example, $\nabla F =< 2x, 2y, 2z >$ in the preceding example. Contrast that to calculation of $X_u \times X_v$ where the $\times$ denotes the dreaded cross-product. Of course each formalism has its place in calculus III.

Remark 8.1.3.

In this warm-up section we have hopefully observed this much about surfaces in $\mathbb{R}^3$:

1. the tangent plane is always 2-dimensional, it is really a plane in the traditional sense of the term.
2. the normal to the tangent plane is always 1-dimensional, the normal through a particular point on the surface is just a line which is orthogonal to all possible tangents through the point.
3. the dimension of the tangent plane and normal give the total dimension of the ambient space; $2 + 1 = 3$.

8.2 manifolds as level sets

We will focus almost exclusively on the level surface formulation of a manifold in the remainder of this chapter. We say $M \subseteq \mathbb{R}^n$ is a **manifold** of dimension $p \leq n$ if M has a p-dimensional tangent plane for each point on M. In other words, M is a p-dimensional manifold if it can be locally approximated by $\mathbb{R}^p$ at each point on M. Moreover, the set of all vectors normal to the tangent space will be $n - p$ dimensional.

These are general concepts which encompasses lines, planes volumes and much much more. Let me illustrate by example:

Example 8.2.1. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x, y) = y - x - 1$ note that $g(x, y) = 0$ gives the line $y - x - 1 = 0$ commonly written as $y = x + 1$; note that the line has direction vector $< -1, 1 >$. Furthermore, $\nabla g =< 1, -1 >$ which is orthogonal to $< -1, 1 >$.*

Example 8.2.2. *Let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by $g(x, y, z) = y - x - 1$ note that $g(x, y, z) = 0$ gives the plane $y - x - 1 = 0$. Furthermore, $\nabla g =< 1, -1, 0 >$ which gives the normal to the plane $g = 0$.*

Example 8.2.3. *Let $g : \mathbb{R}^4 \to \mathbb{R}$ be defined by $g(x, y, z, t) = y - x - 1$ note that $g(x, y, z, t) = 0$ gives the hyperplane $y - x - 1 = 0$. Furthermore, $\nabla g =< 1, -1, 0, 0 >$ which gives the normal to the hyperplane $g = 0$. What does that mean? It means that if I take any vector in the hyperplane it is orthogonal to $< 1, -1, 0, 0 >$. Let $\vec{r}_1, \vec{r}_2$ be points in the solution set of $g(x, y, z, t) = 0$. Denote $\vec{r}_1 = (x_1, y_1, z_1, t_1)$ and $\vec{r}_1 = (x_2, y_2, z_2, t_2)$, we have $y_1 = x_1 + 1$ and $y_2 = x_2 + 1$. The vector in the hyperplane is found from the difference of these points:*

$$\vec{v} = \vec{r}_2 - \vec{r}_1 = (x_2, x_2 + 1, z_2, t_2) - (x_1, x_1 + 1, z_1, t_1) = (x_2 - x_1, x_2 - x_1, z_2 - z_1, t_2 - t_1).$$

It's easy to see that $\vec{v} \cdot \nabla g = 0$ hence ∇g is perpendicular to an arbitrary vector in the hyperplane

If you've begun to develop an intuition for the story we're telling this last example ought to bug you a bit. Why is the difference of points a tangent vector? What happened to the set of all tangent vectors pasted together or the differential or the column space of the derivative? All those concepts still apply but since we were looking at a linear space the space itself matched the tangent hyperplane. The point of the triple of examples above is just to constrast the nature of the equation $g = 0$ in various contexts. We find the dimension of the ambient space changes the dimension of the level set. Basically, we have one equation $g = 0$ and n-unknowns then the inverse image of zero gives us a $(n-1)$-dimensional manifold. If we wanted to obtain a $n-2$ dimensional manifold then we would need two equations which were independent. Before we get to that perhaps I should give a curvy example.

Example 8.2.4. *Let $g : \mathbb{R}^4 \to \mathbb{R}$ be defined by $g(x, y, z, t) = t+x^2+y^2-2z^2$ note that $g(x, y, z, t) = 0$ gives a three dimensional subset of $\mathbb{R}^4$, let's call it M. Notice $\nabla g =< 2x, 2y, -4z, 1 >$ is nonzero everywhere. Let's focus on the point $(2, 2, 1, 0)$ note that $g(2, 2, 1, 0) = 0$ thus the point is on M. The tangent plane at $(2, 2, 1, 0)$ is formed from the union of all tangent vectors to $g = 0$ at the point $(2, 2, 1, 0)$. To find the equation of the tangent plane we suppose $\gamma : \mathbb{R} \to M$ is a curve with $\gamma' \neq 0$ and $\gamma(0) = (2, 2, 1, 0)$. By assumption $g(\gamma(s)) = 0$ since $\gamma(s) \in M$ for all $s \in \mathbb{R}$. Define $\gamma'(0) =< a, b, c, d >$, we find a condition from the chain-rule applied to $g \circ \gamma = 0$ at $s = 0$,*

$$\begin{aligned} \frac{d}{ds}\big(g \circ \gamma(s) \big) = \big(\nabla g\big)(\gamma(s)) \cdot \gamma'(s) = 0 \qquad &\Rightarrow \qquad \nabla g(2, 2, 1, 0) \cdot < a, b, c, d >= 0 \\ &\Rightarrow \qquad < 4, 4, -4, 1 > \cdot < a, b, c, d >= 0 \\ &\Rightarrow \qquad 4a + 4b - 4c + d = 0 \end{aligned}$$

Thus the equation of the tangent plane is $4(x - 2) + 4(y - 2) - 4(z - 1) + t = 0$. In invite the reader to find a vector in the tangent plane and check it is orthogonal to $\nabla g(2, 2, 1, 0)$. However, this should not be surprising, the condition the chain rule just gave us is just the statement that $< a, b, c, d >\in Null(\nabla g(2, 2, 1, 0)^T)$ and that is precisely the set of vector orthogonal to $\nabla g(2, 2, 1, 0)$.

One more example before we dive into the theory of Lagrange multipliers. (which is little more than this section applied to word problems plus the powerful orthogonal complement theorem from linear algebra)

Example 8.2.5. *Let $G : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by $G(x, y, z, t) = (z + x^2 + y^2 - 2, z + y^2 + t^2 - 2)$. In this case $G(x, y, z, t) = (0, 0)$ gives a two-dimensional manifold in $\mathbb{R}^4$ let's call it M. Notice that $G_1 = 0$ gives $z + x^2 + y^2 = 2$ and $G_2 = 0$ gives $z + y^2 + t^2 = 2$ thus $G = 0$ gives the intersection of both of these three dimensional manifolds in $\mathbb{R}^4$ (no I can't "see" it either). Note,*

$$\nabla G_1 =< 2x, 2y, 1, 0 > \qquad \nabla G_2 =< 0, 2y, 1, 2t >$$

It turns out that the inverse mapping theorem says $G = 0$ describes a manifold of dimension 2 if the gradient vectors above form a linearly independent set of vectors. For the example considered here the gradient vectors are linearly dependent at the origin since $\nabla G_1(0) = \nabla G_2(0) = (0, 0, 1, 0)$.

In fact, these gradient vectors are colinear along along the plane $x = t = 0$ *since* $\nabla G_1(0, y, z, 0) = \nabla G_2(0, y, z, 0) =< 0, 2y, 1, 0 >$. *We again seek to contrast the tangent plane and its normal at some particular point. Choose* $(1, 1, 0, 1)$ *which is in* M *since* $G(1, 1, 0, 1) = (0 + 1 + 1 - 2, 0 + 1 + 1 - 2) = (0, 0)$. *Suppose that* $\gamma : \mathbb{R} \to M$ *is a path in* M *which has* $\gamma(0) = (1, 1, 0, 1)$ *whereas* $\gamma'(0) =< a, b, c, d >$. *Note that* $\nabla G_1(1, 1, 0, 1) =< 2, 2, 1, 0 >$ *and* $\nabla G_2(1, 1, 0, 1) =< 0, 2, 1, 1 >$. *Applying the chain rule to both* G_1 *and* G_2 *yields:*

$$(G_1 \circ \gamma)'(0) = \nabla G_1(\gamma(0)) \cdot < a, b, c, d >= 0 \qquad \Rightarrow \qquad < 2, 2, 1, 0 > \cdot < a, b, c, d >= 0$$
$$(G_2 \circ \gamma)'(0) = \nabla G_2(\gamma(0)) \cdot < a, b, c, d >= 0 \qquad \Rightarrow \qquad < 0, 2, 1, 1 > \cdot < a, b, c, d >= 0$$

This is two equations and four unknowns, we can solve it and write the vector in terms of two free variables correspondant to the fact the tangent space is two-dimensional. Perhaps it's easier to use matrix techiques to organize the calculation:

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We calculate, $rref \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \end{bmatrix}$. *It's natural to chose* c, d *as free variables then we can read that* $a = d/2$ *and* $b = -c/2 - d/2$ *hence*

$$< a, b, c, d >=< d/2, -c/2 - d/2, c, d >= \tfrac{c}{2} < 0, -1, 2, 0 > + \tfrac{d}{2} < 1, -1, 0, 2 >$$

We can see a basis for the tangent space. In fact, I can give parametric equations for the tangent space as follows:

$$X(u, v) = (1, 1, 0, 1) + u < 0, -1, 2, 0 > +v < 1, -1, 0, 2 >$$

Not surprisingly the basis vectors of the tangent space are perpendicular to the gradient vectors $\nabla G_1(1, 1, 0, 1) =< 2, 2, 1, 0 >$ *and* $\nabla G_2(1, 1, 0, 1) =< 0, 2, 1, 1 >$ *which span the* **normal plane** N_p *to the tangent plane* T_p *at* $p = (1, 1, 0, 1)$. *We find that* T_p *is orthogonal to* N_p. *In summary* $T_p^{\perp} = N_p$ *and* $T_p \oplus N_p = \mathbb{R}^4$. *This is just a fancy way of saying that the normal and the tangent plane only intersect at zero and they together span the entire ambient space.*

Remark 8.2.6.

The reason I am bothering with these seemingly bizarre examples is that the method of Lagrange multipliers comes down to the observation that both the constraint and objective function's gradient vectors should be normal to the tangent plane of the constraint surface. This means they must both reside in the normal to the tangent plane and hence they will either be colinear or for several constraints they will be linearly dependent. The geometry we consider here justifies the method. Linear algebra supplies the harder part which is that if two vectors are both orthogonal to the tangent plane then they must both be in the orthogonal complement to the tangent plane. The heart of the method of Lagrange multipliers is the orthogonal complement theory from linear algebra. Of course, you can be heartless and still sucessfully apply the method of Lagrange.

8.3 Lagrange multiplier method for one constraint

8.4 Lagrange multiplier method for several constraints