(This is the coversheet for the homework. The problems refer to Anton and Rorres 10th ed. of *Elementary Linear Algebra: applications version.* See Problem Sets 1, 2 or 3 for further formatting.)

Problem 116 § 6.5 # 2 (least squares fit)

Problem 117 § 6.5 # 4 (least squares fit)

Problem 118 § 6.5 # 8 (least squares fit)

Problem 119 § 6.6 # 2 (Fourier series)

Problem 120 § 6.6 # 4 (Fourier series)

Problem 121 § 7.1 #2 (verify orthogonal matrix induces length-preserving map)

Problem 122 § 7.2 #2 (symmetric matrix, find orthonormal e-basis to diagonalize)

Problem 123 § 7.2 #6 (symmetric matrix, find orthonormal e-basis to diagonalize)

Problem 124 § 7.2 #16 (find the spectral decompositions)

Problem 125 § 7.2 #20 (notice, all you have to do is show the given formula is symmetric and guess the eigenvectors which have those values... it's not that hard, just think)

Problem 126 § 7.3 # 6 (change coordinates which remove the cross-terms)

Problem 127 § 7.3 # 14 (identify conic by proper choice of coordinates)

Problem 128 § 7.3 # 16 (identify conic by proper choice of coordinates)

Problem 129 § 7.3 #28 (choose k to make the quadratic form positive definite)

Problem 130 § 7.4 #2 and 4 (find min/max of a quadratic form on the unit-circle)

Problem 131 § 7.4 #6 (find min/max of a quadratic form on the unit-sphere)

Note: there is much more to say about § 7.4 of Anton, I cut short the homework here and leave the rest for the Advanced Calculus course where we have partial derivatives and the general theory of differentiation at our disposal. In short, at a critical point we can approximate a mapping by a quadratic form. The eigenvalues reveal the local behaviour of the function. This is how the Hessian Theorem is derived. You might note that Anton takes the 2nd derivative test as a given and derives algebraic consequences. I prefer the view that algebra gives us the 2nd-derivative theorem modulo a few analytical details. I derive the 2nd derivative theorem from the Method of Lagrange Multipliers paired with the multivariate Taylor Theorem in my Math 231 notes. In any event, I hope you enjoyed this semester. Now, go finish the Matlab you've put off all these many weeks!

$(0,1), (2,0), (3,1), (3,2)$

PROBLEM 116 (§6.5#2) find least squares fit line for

Let $y = mx + b$ then

$$\left.\begin{aligned} (0,1) &: \ 1 = b \\ (2,0) &: \ 0 = 2x + b \\ (3,1) &: \ 1 = 3x + b \\ (3,2) &: \ 2 = 3x + b \end{aligned}\right\} \longrightarrow \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 3 & 1 \\ 3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}}_{B}$$

Approx. Solve $Ay = B$ by solving $A^TAy = A^TB$

$$\begin{bmatrix} 0 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 3 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 22 & 8 \\ 8 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 22 & 8 \\ 8 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \frac{1}{88-64} \begin{bmatrix} 4 & -8 \\ -8 & 22 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 36-32 \\ -72+88 \end{bmatrix}$$

Thus $\vec{y} = \frac{1}{24}\begin{bmatrix} 4 \\ 16 \end{bmatrix} \Rightarrow m = \frac{4}{24} = \frac{1}{6}$ & $b = \frac{16}{24} = \frac{2}{3}$

Hence $\boxed{y = \frac{1}{6}x + \frac{2}{3}}$ is best fit line for given data.

PROBLEM 117 (§6.5#4) fit best fit by linear least squares cubic to $(-1,-14), (0,-5), (1,-4), (2,1), (3,22)$

Let $y = ax^3 + bx^2 + cx + d$ and plug in data

$$\underbrace{\begin{bmatrix} -14 \\ -5 \\ -4 \\ 1 \\ 22 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{V}$$

need to solve $A^TB = A^TAV$ for V.

We calculate that $A^TB = (584, 184, 78, 0)$ and

$$A^TA = \begin{bmatrix} 795 & 277 & 97 & 37 \\ 277 & 99 & 35 & 15 \\ 97 & 35 & 15 & 5 \\ 37 & 15 & 5 & 5 \end{bmatrix}$$

thus calculate $\text{rref}\left[A^TA \,|\, A^TB\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right]$

Therefore, $\boxed{y = 2x^3 - 4x^2 + 3x - 5}$ is best-fit cubic to the given data.

<u>Remark</u>: if you order the parameters a, b, c, d with differing meaning your normal eq^{n}'s will look different. But, the answer should match.

PROBLEM 118 (§6.5#8) A business has $S(x) =$ sales in thousands for the x^{th} month and we're given $S(1) = 4.0$, $S(2) = 4.4$, $S(3) = 5.2$, $S(4) = 6.4$, $S(5) = 8.0$. Assume $S(x) = ax^2 + bx + c$ and find a, b, c to fit data best. Then use model to project sales in the twelth month

$$4.0 = a + b + c$$
$$4.4 = 4a + 2b + c$$
$$5.2 = 9a + 3b + c$$
$$6.4 = 16a + 4b + C$$
$$8.0 = 25a + 5a + C$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{q}} = \underbrace{\begin{bmatrix} 4.0 \\ 4.4 \\ 5.2 \\ 6.4 \\ 8.0 \end{bmatrix}}_{\vec{b}}$$

Solve normal eq$^{\underline{s}}$'s $A^TA\vec{q} = A^T\vec{b}$ to find best-fit choice for the parameters $(a, b, c) = \vec{q}$.

$$A^TA = \begin{bmatrix} 1 & 4 & 9 & 16 & 25 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 979 & 225 & 55 \\ 225 & 55 & 15 \\ 55 & 15 & 5 \end{bmatrix}$$

We calculate, $A^T\vec{b} = (370.8, 94.0, 28.0)$

$$\text{rref}\left[A^TA \mid A^T\vec{b}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.2 \\ 0 & 1 & 0 & -0.2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Therefore, $\boxed{S(x) = 0.2x^2 - 0.2x + 4}$ is best-fit quadratic model

$$S(12) = 0.2(12)^2 - 0.2(12) + 4 = \boxed{30.4}$$ ← estimate of sales for month 12 given our model.

PROBLEM 119 (§6.6 #2) Find least-squares approx of $f(x)=x^2$ on $[0,2\pi]$ by (a.) trig polynomial of order 3 or less (b.) order n or less.

We solve (b.) since (a.) is just special case. We seek Fourier coefficients a_k and b_k such that $f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$ we've derived (or see text, my notes etc...) the formulas

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)dx \quad \& \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\, dx$$

I'll be lazy and use complex notation $e^{ikx} = \cos kx + i\sin kx$

$$\int x^2 e^{ikx}dx = (Ax^2+Bx+C)e^{ikx}$$

$$x^2 e^{ikx} = \left(2Ax+B + ik\left(Ax^2+Bx+C\right)\right)e^{ikx}$$

Hence equating coefficients we deduce,

$$1 = ikA, \quad 0 = 2A + ikB, \quad 0 = B + ikC$$

$$A = \frac{-i}{k}, \quad B = \frac{-2A}{ik} = \frac{-2(-i/k)}{ik} = \frac{2}{k^2}, \quad C = \frac{-B}{ik} = \frac{-2}{ik^3} = \frac{2i}{k^3}$$

Thus $\displaystyle\int x^2 e^{ikx}dx = \left(\frac{-i}{k}x^2 + \frac{2}{k^2}x + \frac{2i}{k^3}\right)\left(\cos kx + i\sin kx\right) + C$

$$\int\Big(\underbrace{x^2\cos kx}_{\text{I}} + i\underbrace{x^2 \sin kx}_{\text{II}}\Big)dx = \underbrace{\frac{2x}{k^2}\cos kx - \left(\frac{2}{k^3} - \frac{x^2}{k}\right)\sin kx}_{\text{I}} + i\Big[\underbrace{\left(\frac{2}{k^3} - \frac{x^2}{k}\right)\cos kx + \frac{2}{k^2}\sin kx}_{\text{II}}\Big] + C$$

This shows that

$$\int_0^{2\pi} x^2 \cos kx\, dx = \left(\frac{2x}{k^2}\cos kx - \left(\frac{2}{k^3} - \frac{x^2}{k}\right)\underbrace{\sin kx}_{0}\right)\Bigg|_0^{2\pi}$$

$$= \frac{4\pi}{k^2}(1) - 0 \Rightarrow a_k = \frac{1}{\pi}\cdot\frac{4\pi}{k^2} = \underline{\frac{4}{k^2}}.$$

$$\int_0^{2\pi} x^2 \sin kx\, dx = \left[\left(\frac{2}{k^3} - \frac{x^2}{k}\right)\cos kx + \frac{2}{k^2}\underbrace{\sin kx}_{0}\right]\Bigg|_0^{2\pi}$$

$$= \left(\cancel{\frac{2}{k^3}} - \frac{4\pi^2}{k}\right) - \cancel{\frac{2}{k^3}} \Rightarrow b_k = \frac{1}{\pi}\left(\frac{-4\pi^2}{k}\right) = \underline{\frac{-4\pi}{k}}.$$

continued →

(note, $n=0$, $\frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{x^3}{3\pi}\Big|_0^{2\pi} = \frac{8\pi^3}{3\pi} = \frac{8\pi^2}{3}$

<u>PROBLEM 119 (§6.6 #2) continued</u> found $a_n = \frac{4}{n^2}$, $b_n = \frac{-4\pi}{n}$ for $n \geq 1$

(a.) $x^2 \sim \frac{a_0}{2} + \sum_{n=1}^{3}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$

$$\boxed{x^2 \sim \frac{4\pi}{3} + 4\cos x - 4\pi\sin x + \cos 2x - 2\pi\sin 2x + \frac{4}{9}\cos 3x - \frac{4\pi}{3}\sin 3x}$$

(b.) $\boxed{x^2 \sim \frac{4\pi}{3} + \sum_{n=1}^{n}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)}$

As we take $n \longrightarrow \infty$ we obtain the <u>Fourier</u> Series for x^2.

PROBLEM 120 (§6.6 #4) a.) Find least squares approx. of e^x on $[0,1]$ by a polynomial of the form $a_0 + a_1 x$ (b.) find mean square error.

(a.) I assume $(f, g) = \int_0^1 fg\, dx$ defines length for fncts.
we showed $\{1, \sqrt{12}(x - \frac{1}{2})\}$ orthonormal basis for P_1 on $[0,1]$ (we actually did P_2, but this suffices here).
The projection of e^x onto P_1 is given by

$$Proj_{P_1}(e^x) = \langle e^x, 1\rangle 1 + \langle e^x, \sqrt{12}(x-\tfrac{1}{2})\rangle(\sqrt{12}(x-\tfrac{1}{2}))$$

Calculate the integrals,

$$\langle e^x, 1\rangle = \int_0^1 e^x dx = e - 1$$

$$\langle e^x, x - \tfrac{1}{2}\rangle = \int_0^1 (xe^x - \tfrac{1}{2}e^x)dx = \left(xe^x - \tfrac{3}{2}e^x\right)\Big|_0^1 = \tfrac{-1}{2}e + \tfrac{3}{2}.$$

Thus,

$$\begin{aligned} Proj_{P_1}(e^x) &= e - 1 + 12\left(\tfrac{3}{2} - \tfrac{e}{2}\right)(x - \tfrac{1}{2}) \\ &= e - 1 + 6(3-e)(x - \tfrac{1}{2}) \\ &= e - 1 + 3(e-3) + 6(3-e)x \\ &= \boxed{4e - 10 + 6(3-e)x} \end{aligned}$$

aka <u>$0.873 + 1.69x$ ($\approx$)</u>

(b.) mean-square error $= \int_0^1 \underbrace{\left(e^x - Proj_{P_1}(e^x)\right)^2}_{} dx \approx \boxed{0.00394}$.

(I omit details here)

you plug-in the approx from (a) and square this difference to calculate the $\|e^x - Proj_{P_1}(e^x)\|^2$.

$\underline{\text{PROBLEM 121}}$ (§7.1 #2) Show $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix}$ is orthogonal

(b.) Find $T(x)$ for $x = (-2, 3, 5)$ and verify for $T(x) = Ax$ it is indeed true that $\|T(x)\| = \|x\|$.

$$(a.) \quad A A^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I.$$

$$(b.) \quad T(x) = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2+6+10 \\ -4-6+5 \\ 4-3+10 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 14 \\ -5 \\ 11 \end{bmatrix}$$

$$\|T(x)\| = \frac{1}{3}\sqrt{(14)^2 + (25) + 121} = \frac{1}{3}\sqrt{342} = \sqrt{38}.$$

$$\|x\| = \sqrt{4+9+25} = \sqrt{38} \quad \therefore \quad \|T(x)\| = \|x\| \checkmark$$

(of course we knew this as
$$\begin{aligned} \|Ax\| &= \sqrt{(Ax)^T Ax} \\ &= \sqrt{x^T A^T A x} \\ &= \sqrt{x^T I x} \\ &= \sqrt{x^T x} \\ &= \|x\| \;) \end{aligned}$$

a simple, but important problem.

$\underline{\text{PROBLEM 122}}$ (§7.2 #2) Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ find P which orthogonally diagonalizes A and determine $P^{-1}AP$

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = (\lambda-3)^2 - 1 = (\lambda-3-1)(\lambda-3+1) = (\lambda-4)(\lambda-2) = 0$$

$$0 = (A - 4I)\vec{u}_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\vec{u}_1 \Rightarrow \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$0 = (A - 2I)\vec{u}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\vec{u}_2 \Rightarrow \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$$

In this case $P^T = P$ and $P^{-1} = P$ as it happens.

We can calculate $\boxed{P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}}$.

PROBLEM 123 (§7.2 #6) orthogonally diagonalize $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Find P and $P^{-1}AP$.

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \det(-\lambda)$$
$$= \left[(\lambda-1)^2 - 1\right](-\lambda)$$
$$= (\lambda-2)(\lambda)(-\lambda)$$
$$= \lambda^2(\lambda-2) \longrightarrow A\vec{u} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence $u + v = 0$, w free

$$\vec{u} = \begin{bmatrix} u \\ -u \\ w \end{bmatrix} = u\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + w\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

orthonormalizing, we define $\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$(A - 2I)\vec{u}_3 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \begin{matrix} u - v = 0 \\ -2w = 0 \end{matrix} \Rightarrow \vec{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Therefore set $\boxed{P = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}}$ note: $P^{-1} = P^T$ here.

We can calculate that $P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

PROBLEM 124 (§7.2 #16) Find the spectral decomposition of the matrices a.) $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ b.) $\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$ c.) $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$ d.) $\begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$

As explained on pg. 400 of the text,

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$$

for (a.) we recall our work from Problem 122,

a.)
$$A = 4u_1u_1^T + 2u_2u_2^T$$
$$= 4\,\tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} + \tfrac{2}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}$$
$$= \boxed{4\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + 2\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}}.$$

PROBLEM 124 continued: I omit details, but for b, c, d we need to compute e-values and orthonormal e-basis as we can since $A^T = A$ for each given matrix

(b.) $\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} = 2\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} + 7\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$.

(c.) $\begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix} = -4\begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 4\begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 1/3 & -1/3 \\ -1/3 & -1/3 & 1/3 \end{bmatrix} + 2\begin{bmatrix} 1/6 & 1/6 & 1/3 \\ 1/6 & 1/6 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix}$.

(d.) $\begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix} = 25\begin{bmatrix} 16/25 & 0 & -12/25 \\ 0 & 0 & 0 \\ -12/25 & 0 & 9/25 \end{bmatrix} - 3\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 50\begin{bmatrix} 9/25 & 0 & 12/25 \\ 0 & 0 & 0 \\ 12/25 & 0 & 16/25 \end{bmatrix}$.

Remark: parts (c.) & (d.) written out with details are easily 2 pages in total. It would be nice to find Matlab commands to extract the spectral decomposition w/o all this tedius calculation

PROBLEM 125 (§7-2#20) If $\{u_1, u_2, \dots, u_n\}$ is orthonormal basis for $\mathbb{R}^n$ and $A = c_1 u_1 u_1^T + c_2 u_2 u_2^T + \dots + c_n u_n u_n^T$ then A is symmetric with eigenvalues $c_1, c_2, \dots, c_n$ (Prove this).

Proof:
$$\begin{aligned} A^T &= (c_1 u_1 u_1^T + \dots + c_n u_n u_n^T)^T \\ &= c_1 u_1^{T^T} u_1^T + \dots + c_n u_n^{T^T} u_n^T \\ &= c_1 u_1 u_1^T + \dots + c_n u_n u_n^T \\ &= A \quad \text{hence } A \text{ is symmetric.} \end{aligned}$$

Note $u_i^T u_j = \delta_{ij}$ by orthonormality of the basis. Consider,
$$\begin{aligned} A u_j &= c_1 u_1 u_1^T u_j + \dots + c_n u_n u_n^T u_j \\ &= c_1 u_1 \delta_{1j} + \dots + c_n u_n \delta_{nj} \\ &= c_j u_j \end{aligned}$$
thus $u_j \neq 0$ (since $\|u_j\| = 1$) is a e-vector with e-value c_j.

Finally, we find $u_1, u_2, \dots, u_n$ have e-values $c_1, c_2, \dots, c_n$ and no more are possible.//

PROBLEM 126 (§7.3#6) Change coordinates to uncouple the quad. form

$$Q = 5x_1^2 + 2x_2^2 + 4x_3^2 + 4x_1x_2$$

$$[Q] = A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad \det(A - \lambda I) = -(\lambda-1)(\lambda-4)(\lambda-6)$$

Hence $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 6 \Rightarrow \vec{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Thus $P = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & \sqrt{5} & 0 \end{bmatrix}$ gives desired change of basis matrix.

$\vec{y} = P^{-1}\vec{x}$ however $P^{-1} = P^T = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & \sqrt{5} \\ 2 & 1 & 0 \end{bmatrix}$ on the other hand $\vec{x} = P\vec{y} = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & \sqrt{5} & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}}(-y_1 + 2y_3) \\ \frac{1}{\sqrt{5}}(2y_1 + y_3) \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

this usually fails when you try to do the algebra ... this is my 3rd attempt. ☺

$$5x_1^2 = (-y_1 + 2y_3)^2 = y_1^2 - 4y_1y_3 + 4y_3^2$$

$$2x_2^2 = \frac{2}{5}(2y_1 + y_3)^2 = \frac{2}{5}(4y_1^2 + 4y_1y_3 + y_3^2) = \frac{8}{5}y_1^2 + \frac{8}{5}y_1y_3 + \frac{2}{5}y_3^2$$

$$4x_3^2 = 4y_2^2$$

$$4x_1x_2 = \frac{4}{5}(-y_1 + 2y_3)(2y_1 + y_3) = \frac{4}{5}(-2y_1^2 + 3y_1y_3 + 2y_3^2)$$

Thus, due to several cancellations above

$$\boxed{Q = y_1^2 + 4y_2^2 + 6y_3^2}$$

Better way

$$D = P^TAP \qquad PDP^T = A$$

$$\begin{aligned} Q(x) &= x^TAx \\ &= x^TPDP^Tx \\ &= (P^Tx)D(P^Tx) \\ &= \vec{y}^TD\vec{y} \\ &= [y_1, y_2, y_3]\begin{bmatrix} 1 & & \\ & 4 & \\ & & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\ &= \boxed{y_1^2 + 4y_2^2 + 6y_3^2} \end{aligned}$$

PROBLEM 127 (§7.3 #14) Rotate coordinates by some θ to place $9=5x^2+4xy+5y^2$ in $\bar{x},\bar{y}$ for which the transformed equation has no crossterms. Find θ.

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\det(A-\lambda I) = \det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} = (\lambda-5)^2-2^2 = (\lambda-7)(\lambda-3)$$

$\underline{\lambda = 7}$ $\quad 0 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\vec{u}_1 \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{\lambda = 3}$ $\quad 0 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\vec{u}_2 \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\tilde{P} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ oops. this has $\det\tilde{P} = -1$
use $\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ instead!

$$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow \boxed{\theta = 45^\circ}$$

We find $\boxed{9 = 7\bar{x}^2 + 3\bar{y}^2}$ for $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$.
(an ellipse)

PROBLEM 128 (§7.3 #16) $x^2+xy+y^2 = \frac{1}{2}$ remove xy by coord. change

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \rightarrow \det\begin{bmatrix} 1-\lambda & 1/2 \\ 1/2 & 1-\lambda \end{bmatrix} = (\lambda-1)^2 - \left(\tfrac{1}{2}\right)^2 = \left(\lambda - \tfrac{3}{2}\right)\left(\lambda - \tfrac{1}{2}\right)$$

$\left(A - \frac{3}{2}I\right)\vec{u}_1 = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\vec{u}_1 \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\left(A - \frac{1}{2}I\right)\vec{u}_2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\vec{u}_2 \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ← to make $[\vec{u}_1 | \vec{u}_2]$ have positive determinant.

$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ again rotation with $\boxed{\theta = 45^\circ}$

and $\boxed{\frac{3}{2}\bar{x}^2 + \frac{1}{2}\bar{y}^2 = \frac{1}{2}}$ aka $3\bar{x}^2 + \bar{y}^2 = 1$ or $\dfrac{\bar{x}^2}{\frac{1}{3}} + \dfrac{\bar{y}^2}{1} = 1$
ellipse.

$\boxed{\text{PROBLEM 129}}$ (§7.3 #28) Find all values for which k makes $Q = 3x_1^2 + x_2^2 + 2x_3^2 - 2x_1x_3 + 2kx_2x_3$ positive definite.

<u>Recall</u>: Q is positive definite if $Q(x) > 0 \quad \forall x \neq 0$.
Furthermore, we derived $Q(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_n x_n^2$ in eigen coordinates hence we need $\lambda_1, \lambda_2, \dots, \lambda_n > 0$.

$$\det([Q] - \lambda I) = \det \begin{bmatrix} 3-\lambda & 0 & -1 \\ 0 & 1-\lambda & k \\ -1 & k & 2-\lambda \end{bmatrix}$$

$$= (3-\lambda)\left[(1-\lambda)(2-\lambda) - k^2\right] - 1 \cdot \left[0 \cdot k + 1(1-\lambda)\right]$$

$$= (3-\lambda)\left[(\lambda-1)(\lambda-2) - k^2\right] + (\lambda - 1)$$

$$= (3-\lambda)\left[\lambda^2 - 3\lambda + 2 - k^2\right] + \lambda - 1$$

$$= -\lambda^3 + 6\lambda^2 + (k^2 - 9)\lambda + 6 - 3k^2 + \lambda - 1$$

$$= -\lambda^3 + 6\lambda^2 + (k^2 - 8)\lambda + 5 - 3k^2$$

$$= -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

We have $\lambda_1 \lambda_2 \lambda_3 = 5 - 3k^2$ if $\lambda_1, \lambda_2, \lambda_3 > 0$ then this implies $5 - 3k^2 > 0 \Rightarrow k^2 < 5/3 \Rightarrow \underline{k \in (-\sqrt{5/3}, \sqrt{5/3})}$.

Alternative, $\text{Trace}(Q) = \lambda_1 + \lambda_2 + \lambda_3$ and $\det Q = \lambda_1 \lambda_2 \lambda_3$

$$[Q] = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & k \\ -1 & k & 2 \end{bmatrix} \begin{array}{l} \longrightarrow \text{Trace } Q = 6 \\ \longrightarrow \det Q = 3(2-k^2) - 1(1) = 5 - 3k^2 \end{array}$$

Hence $\underbrace{\lambda_1 + \lambda_2 + \lambda_3 = 6}$ and $\lambda_1 \lambda_2 \lambda_3 = 5 - 3k^2$

$$\lambda_1 = 6 - \lambda_2 - \lambda_3 \Rightarrow (6 - \lambda_2 - \lambda_3)(\lambda_2 \lambda_3) = 5 - 3k^2$$

$$\Rightarrow \lambda_2 + \lambda_3 < 6 \quad \text{ok.}$$

well, my best answer at the moment is $\boxed{-\sqrt{\tfrac{5}{3}} < k < \sqrt{\tfrac{5}{3}}}$

PROBLEM 130 (§7.4#2)

Find extrema of $f(x,y) = xy$ on the unit-circle $x^2+y^2=1$

Consider $Q = [x,y]\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = xy$

$$\det\begin{bmatrix} -\lambda & 1/2 \\ 1/2 & -\lambda \end{bmatrix} = \lambda^2 - \left(\tfrac{1}{2}\right)^2 = \left(\lambda - \tfrac{1}{2}\right)\left(\lambda + \tfrac{1}{2}\right)$$

Thus $\exists$ eigencoordinates $\bar{x}, \bar{y}$ such that

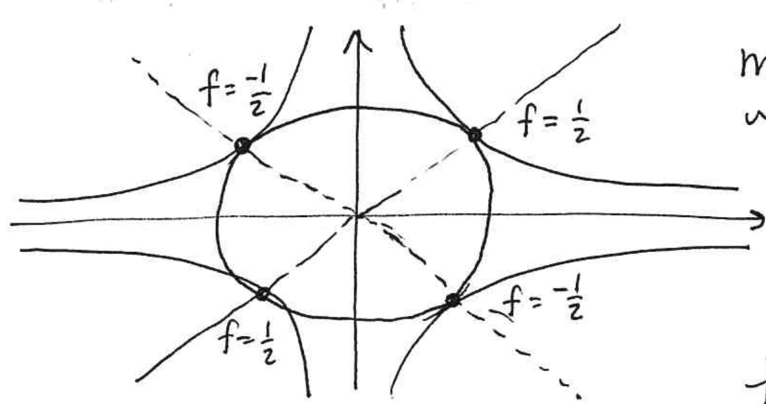
$$Q = \tfrac{1}{2}\bar{x}^2 - \tfrac{1}{2}\bar{y}^2$$

It is then obvious that $\boxed{f_{max} = \tfrac{1}{2} \text{ and } f_{min} = \tfrac{-1}{2}}$ for $\bar{x}^2+\bar{y}^2 = 1$.

(*Remark*: I know $[Q]^T = Q$ hence the eigen coordinates are obtained by an orthonormal basis and $P^T = P$ insures that the level set $x^2+y^2=1$ transform to the same $\bar{x}^2+\bar{y}^2=1$ in the rotated eigencoordinate system)

(I don't have to find $\bar{x}, \bar{y}$ explicitly to make these above claims,

but the text also asked us where the min/max were attained.



min/max where normals to level curves of f match that of the constraint $x^2+y^2=1$.

The precise locales of these min/max values are at the eigenvectors

$$\pm\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for } f = \tfrac{1}{2} \quad\text{and}\quad \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for } f = \tfrac{-1}{2}$$

To give a complete answer,

Max: $f = \frac{1}{2}$ reached at $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$

Min: $f = \frac{-1}{2}$ attained at $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$

PROBLEM 131 (§7.4#6)

$$Q = 2x^2 + y^2 + z^2 + 2xy + 2xz$$

Find min/max and where they're attained by Q on the unit-sphere $x^2+y^2+z^2=1$

$$[Q] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \det(Q-\lambda I) = \det\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}$$

$$= (2-\lambda)\left[(1-\lambda)^2\right] - 1(1-\lambda) + 1\cdot\left[-(1-\lambda)\right]$$

$$= (\lambda-1)^2(2-\lambda) - 2(1-\lambda)$$

$$= (\lambda-1)\left[(\lambda-1)(2-\lambda)+2\right]$$

$$= (\lambda-1)\left[2\lambda - \cancel{2} + \lambda - \lambda^2 + \cancel{2}\right]$$

$$= (\lambda-1)\left[-\lambda^2+3\lambda\right]$$

$$= -\lambda(\lambda-1)(\lambda-3) \therefore \underline{\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 3}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

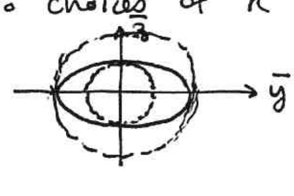
$v = -u$, $w = -u$, $v + w \neq -2u$; $u = 0$, $v + w = 0$, $w = -v$; $v = \frac{1}{2}u$, $w = \frac{1}{2}u$

$$\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \vec{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$(\lambda_1 = 0)$ $(\lambda_2 = 1)$ $(\lambda_3 = 3)$

Thus we find minimum value of $Q = 0$ at $\left(\frac{\pm 1}{\sqrt{3}}, \frac{\mp 1}{\sqrt{3}}, \frac{\mp 1}{\sqrt{3}}\right)$, and maximum value of $Q = 3$ at $\left(\frac{\pm 2}{\sqrt{6}}, \frac{\pm 1}{\sqrt{6}}, \frac{\pm 1}{\sqrt{6}}\right)$

<u>Remark</u>: $Q = \bar{y}^2 + 3\bar{z}^2 = k$ give cylinders (elliptical) and we're finding the two choices of k which give matching normals with the unit-sphere.



I'd need to graph to check on the picture here