(This is the coversheet for the homework. The actual solutions should be worked out neatly on single-sided with each problem labeled (for example: Problem 1, 1.1 #14) and the answer boxed. Those additional sheets are then to be stapled with a metal staple in the upper left corner in such a way that no work is obscured. Failure to follow the style outlined here will result in a loss of credit. The problems refer to Anton and Rorres 10th ed. of *Elementary Linear Algebra: applications version.*)

Problem 1 § 1.1 # 14 (augmented matrix for system linear eqns.)

Problem 2 § 1.2 # 6 (solve via Gauss-Jordan elimination)

Problem 3 § 1.2 # 8 (solve via Gauss-Jordan elimination)

Problem 4 § 1.2 # 30 (solve system in terms of unknown constants a, b, c)

Problem 5 § 1.3 # 6 (computation of transpose, sum, difference and products and trace)

Problem 6 § 1.3 # 10a (matrix-vector product as linear combination)

Problem 7 § 1.3 # 12 (matrix notation for system of linear eqns.)

Problem 8 Let $A, B \in \mathbb{R}^{n\times n}$. Let $trace(A) = \sum_{i=1}^{n} A_{ii}$.

- **a.** If $c \in \mathbb{R}$, show $tr(A + cB) = tr(A) + ctr(B)$.
- **b.** show that $trace(AB) = trace(BA)$.
- **c.** let $[A, B] = AB - BA$, show $trace([A, B]) = 0$.
- **d.** can $[A, B] = I$?

****please use technology to compute matrix operations in the problems that follow.****

Problem 9 § 1.2 # 34 (substitution to make linear)

Problem 10 Find a cubic polynomial whose graph contains the points $(1, 2)$, $(2, 2)$, $(3, 2)$ and $(4, 2)$.

Problem 11 § 1.2 # 38 (circles that fit the triple of points, find all solutions, answer has parameter)

Problem 12 A **probability vector** has entries whose sum is one. A **stochastic matrix** is a square matrix whose columns are probability vectors. Let $A = \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix}$ be a stochastic matrix which models the migration of people to and from the city to the suburbs. In particular, if c_k denotes the number of people (in thousands) in the city in year k and s_k denotes the number of people (in thousands) living in the suburbs in year k then letting $X_k = [c_k, s_k]^T$ we can model the migration of people by the following matrix product:

$$X_{k+1} = AX_k.$$

This model assumes the population stays constant and that there are only two places to live, the city or the suburbs. In addition, the model says that 15% of city people move to suburbs while only 3% of the suburb people move to the city. Suppose that $X_0 = [400, 300]^T$ represents the population in 2000. **Calculate the number of people living in the city and the number of people in suburbs in the years 2001, 2002, ... 2012. Give your answer in a table.**

PROBLEM SET 1 SOLUTION, MATH 221 (ANTON & RORRES 10^{th} Ed.)

PROBLEM 1 (§1.1 #14) find augmented coefficient matrix for system,

a.) $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned} \longrightarrow \boxed{\left[\begin{array}{cc|c} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{array}\right]}$.

b.) $\begin{aligned} 2x_1 + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned} \longrightarrow \boxed{\left[\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & 1 & -1 & 0 \end{array}\right]}$.

c.) $\begin{aligned} x_1 + 2x_2 - x_4 + x_5 &= 1 \\ 3x_2 + x_3 - x_5 &= 2 \\ x_3 + 7x_4 &= 1 \end{aligned} \longrightarrow \boxed{\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 1 & 1 \\ 0 & 3 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 7 & 0 & 1 \end{array}\right]}$

d.) $\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned} \longrightarrow \boxed{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]}$

PROBLEM 2 (§1.2 #6) Solve $\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$ by Gauss-Jordan elimination

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right] \xrightarrow[r_3 - 4r_1]{r_2 + r_1} \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right] \xrightarrow{r_3 + r_2} \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{1}{7}r_2]{\frac{1}{2}r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & 3/7 & -1/7 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We find sol^n $\boxed{x_1 = \frac{-1}{7} - \frac{3}{7}x_3}$ and $\boxed{x_2 = \frac{1}{7} - \frac{4}{7}x_3}$

for all $x_3 \in \mathbb{R}$. Alternatively, $\boxed{\text{sol}^{\underline{n}} \text{ set} = \left\{\left(\frac{-1-3t}{7}, \frac{1-4t}{7}, t\right) \middle| \, t \in \mathbb{R}\right\}}$

PROBLEM 3 (§1.2#8) Solve

$$\begin{aligned} -2b+3c &= 1 \\ 3a+6b-3c &= -2 \\ 6a+6b+3c &= 5 \end{aligned}$$

Solve via row reduction on aug. coeff. mat.

$$\left[\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right] \xrightarrow[r_3-2r_2]{-r_1} \left[\begin{array}{ccc|c} 0 & 2 & -3 & -1 \\ 3 & 6 & -3 & -2 \\ 0 & -6 & 9 & -12 \end{array}\right] \xrightarrow{r_3+3r_1} \left[\begin{array}{ccc|c} 0 & 2 & -3 & -1 \\ 3 & 6 & -3 & -2 \\ 0 & 0 & 0 & -15 \end{array}\right]$$

We're done!

This row $\Rightarrow$ $0a+0b+0c=-15$ which is impossible. $\boxed{\text{No sol}^{\underline{n}}}$

PROBLEM 4 (§1.2#30) Solve

$$\begin{aligned} x_1+x_2+x_3 &= a \\ 2x_1+2x_3 &= b \\ 3x_2+3x_3 &= c \end{aligned}$$

in terms of constants a, b, c

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 2 & 0 & 2 & b \\ 0 & 3 & 3 & c \end{array}\right] \xrightarrow{r_2-2r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & 0 & b-2a \\ 0 & 3 & 3 & c \end{array}\right] \xrightarrow[2r_3]{3r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -6 & 0 & 3b-6a \\ 0 & 6 & 6 & 2c \end{array}\right]$$

$$\xrightarrow[r_3+r_2]{6r_1} \left[\begin{array}{ccc|c} 6 & 6 & 6 & 6a \\ 0 & -6 & 0 & 3b-6a \\ 0 & 0 & 6 & 2c+3b-6a \end{array}\right] \xrightarrow{r_1-r_3} \left[\begin{array}{ccc|c} 6 & 6 & 0 & 12a-2c-3b \\ 0 & -6 & 0 & 3b-6a \\ 0 & 0 & 6 & 2c+3b-6a \end{array}\right]$$

$$\xrightarrow{r_1+r_2} \left[\begin{array}{ccc|c} 6 & 0 & 0 & 6a-2c \\ 0 & -6 & 0 & 3b-6a \\ 0 & 0 & 6 & 2c+3b-6a \end{array}\right] \xrightarrow[\frac{1}{6}r_3]{\frac{1}{6}r_1,\ -\frac{1}{6}r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-\frac{1}{3}c \\ 0 & 1 & 0 & a-\frac{1}{2}b \\ 0 & 0 & 1 & -a+\frac{1}{2}b+\frac{1}{3}c \end{array}\right]$$

thus,

$$\boxed{\begin{aligned} x_1 &= a-c/3 \\ x_2 &= a-b/2 \\ x_3 &= -a+b/2+c/3 \end{aligned}}$$

PROBLEM 5 (§1.3 #6)

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Calculate where possible the quantities listed

(a.) $$(2D^T - E)A = \left[2\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \right] A$$

$$= \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12+3+3 & -6+3 \\ 33+1+2 & -2+2 \\ -1+5 & 2+5 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{bmatrix}}$$

(b.) $(4B)C + 2B$ — $\underbrace{4B}_{2\times 2}\ \underbrace{C}_{2\times 3}\ \underbrace{2B}_{2\times 2}$ — cannot add 2×3 and 2×2. $\boxed{\text{undefined}}$

(c.) $$(-AC)^T + 5D^T = \boxed{\begin{bmatrix} 2 & -10 & 11 \\ 13 & 2 & 5 \\ 4 & -3 & 13 \end{bmatrix}}$$

(you should've shown steps!)

(d.) $$(BA^T - 2C)^T = \left(\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \right)^T$$

$$= \left(\begin{bmatrix} 12 & -6 & 3 \\ 0 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 8 & 4 \\ 6 & 2 & 10 \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} 10 & -14 & -1 \\ -6 & 2 & -8 \end{bmatrix}^T$$

$$= \boxed{\begin{bmatrix} 10 & -6 \\ -14 & 2 \\ -1 & -8 \end{bmatrix}}.$$

PROBLEM 5 continued

(e.) $B^T(CC^T - A^TA) = \boxed{\begin{bmatrix} 40 & 72 \\ 26 & 42 \end{bmatrix}}$

(f.) $\underbrace{D^T E^T}_{(3\times3)(3\times3)} - \underbrace{(ED)^T}_{(3\times3)(3\times3)} = D^TE^T - \underbrace{D^T E^T}_{} = 0 = \boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}.$

just checking.

Th$^{\underline{m}}$ for transpose of product ☺

(again, I expected you to show steps here, this key incomplete here.)

PROBLEM 6 (§1.3 #10a)

express each column of AB as linear combo. of col's of A where $A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$ & $B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$

$AB = A[B_1 | B_2 | B_3] = [AB_1 | AB_2 | AB_3]$ by concatenation proposition.

$$\boxed{\begin{aligned} col_1(AB) &= AB_1 = 6\begin{bmatrix}3\\6\\0\end{bmatrix} + 7\begin{bmatrix}7\\4\\9\end{bmatrix} \\ col_2(AB) &= AB_2 = -2\begin{bmatrix}3\\6\\0\end{bmatrix} + \begin{bmatrix}-2\\5\\4\end{bmatrix} + 7\begin{bmatrix}7\\4\\9\end{bmatrix} \\ col_3(AB) &= AB_3 = 4\begin{bmatrix}3\\6\\0\end{bmatrix} + 3\begin{bmatrix}-2\\5\\4\end{bmatrix} + 5\begin{bmatrix}7\\4\\9\end{bmatrix} \end{aligned}}$$

(I'm using the fact $A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1 col_1(A) + x_2 col_2(A) + x_3 col_3(A)$.)

PROBLEM 7 (§1.3 #12) translate systems given into matrix form

a.) $\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ 0 & -3 & 4 \\ 1 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \\ 5 \end{bmatrix}}_{b}$

b.) $\underbrace{\begin{bmatrix} 3 & 3 & 3 \\ -1 & -5 & -2 \\ 0 & -4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}}_{b}$

PROBLEM 8) Prove a few fun facts about the trace, let $A, B \in \mathbb{R}^{n \times n}$, $c \in \mathbb{R}$

a.) $\mathrm{tr}(A + cB) = \sum_i (A+cB)_{ii}$: def^n of trace.

$= \sum_i (A_{ii} + cB_{ii})$: def^n of matrix add. and scalar mult.

$= \sum_i A_{ii} + c \sum_i B_{ii}$: property of finite sums.

$= \underline{\mathrm{tr}(A) + c\, \mathrm{tr}(B)}$ //

b.) $\mathrm{tr}(AB) = \sum_j (AB)_{jj}$: def^n of trace.

$= \sum_j \sum_k A_{jk} B_{kj}$: def^n of matrix multiplication

$= \sum_j \sum_k B_{kj} A_{jk}$: $A_{jk}, B_{kj} \in \mathbb{R}$ these <u>commute</u>.

$= \sum_k \sum_j B_{kj} A_{jk}$: prop. of finite sums

$= \sum_k (BA)_{kk}$: def^n of mat. mult.

$= \underline{\mathrm{tr}(BA)}$ //

c.) $\mathrm{tr}([A,B]) = \mathrm{tr}(AB - BA)$: def^n of $[A,B]$.

$= \mathrm{tr}(AB) - \mathrm{tr}(BA)$: part. a.

$= \mathrm{tr}(AB) - \mathrm{tr}(AB)$: part. b.

$= \underline{0}$ //

d.) Suppose $[A,B] = I$ note $\mathrm{trace}(I) = n$ for $I \in \mathbb{R}^{n \times n}$. But, $\mathrm{trace}\,[A,B] = 0$ which is a contradiction since $\mathrm{trace}\,[A,B] = 0 = n$.

Thus, $[A,B] \neq I$. //

PROBLEM 9 (§1.2 #34)

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9 \end{aligned}$$

$0 \le \alpha, \beta \le 2\pi$
$0 \le \gamma < \pi$
(solve it.)

$$\begin{array}{cccc} \sin\alpha & \cos\beta & \tan\gamma & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \Rightarrow \sin\alpha = 1 \\ \Rightarrow \cos\beta = -1 \\ \Rightarrow \tan\gamma = 0 \end{array}$$

Thus, $\boxed{\alpha = \pi/2, \ \beta = \pi, \ \gamma = 0}$

PROBLEM 10 find cubic (oops!) polynomial which contains the points $(1,2), (2,2), (3,2), (4,2)$

Let $f(x) = Ax^3 + Bx^2 + Cx + D$

$$\underbrace{\begin{aligned} f(1) &= 2 = A + B + C + D \\ f(2) &= 2 = 8A + 4B + 2C + D \\ f(3) &= 2 = 27A + 9B + 3C + D \\ f(4) &= 2 = 64A + 16B + 4C + D \end{aligned}}_{\text{Solve these by calculating}}$$

$$\text{rref}\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 8 & 4 & 2 & 1 & 2 \\ 27 & 9 & 3 & 1 & 2 \\ 64 & 16 & 4 & 1 & 2 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Thus $\boxed{f(x) = 2}$ (alternatively, no answer possible since data necessarily forces non-cubic polynomial strictly speaking <u>need</u> $A \neq 0$ for cubic here.)

PROBLEM 11 (§1.2 #38)

Find a, b, c, d such that $ax^2 + ay^2 + bx + cy + d = 0$ contains the points $(4,-3), (-4,5), (-2,7)$

$$\begin{aligned} (4,-3):&\quad 16a + 9a + 4b - 3c + d = 0 \\ (-4,5):&\quad 16a + 25a - 4b + 5c + d = 0 \\ (-2,7):&\quad 4a + 49a - 2b + 7c + d = 0 \end{aligned}$$

we find aug. coeff. matrix and use tech. to find rref

$$\text{rref}\left[\begin{array}{cccc|c} 25 & 4 & -3 & 1 & 0 \\ 41 & -4 & 5 & 1 & 0 \\ 53 & -2 & 7 & 1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/29 & 0 \\ 0 & 1 & 0 & -2/29 & 0 \\ 0 & 0 & 1 & -4/29 & 0 \end{array}\right]$$

Hence, for any $\boxed{d \in \mathbb{R}, \ a = -d/29, \ b = 2d/29, \ c = 4d/29}$
(although $d = 0$ makes graph false could remove $d=0$ logically.)

PROBLEM 12 Find # of people in city & suburbs predicted by $X_{k+1} = AX_k$ where $X_k = \begin{bmatrix} C_k \\ S_k \end{bmatrix}$ ← city folk ← suburbites.

where $A = \begin{bmatrix} 0.85 & 0.03 \\ 0.15 & 0.97 \end{bmatrix}$, $X_0 = \begin{bmatrix} 400 \\ 300 \end{bmatrix}$ for 2000.

2001 $X_1 = \begin{bmatrix} 0.85 & 0.03 \\ 0.15 & 0.97 \end{bmatrix}\begin{bmatrix} 400 \\ 300 \end{bmatrix} = \begin{bmatrix} 349 \\ 351 \end{bmatrix}$

2002 $X_2 = \begin{bmatrix} 0.85 & 0.03 \\ 0.15 & 0.97 \end{bmatrix}\begin{bmatrix} 349 \\ 351 \end{bmatrix} = \begin{bmatrix} 307.18 \\ 392.82 \end{bmatrix}$

2003 $X_3 = AX_2 = \begin{bmatrix} 272.89 \\ 427.11 \end{bmatrix}$

2004 $X_4 = AX_3 = A^4 X_0 = \begin{bmatrix} 244.77 \\ 455.23 \end{bmatrix}$

2005 $X_5 = A^5 X_0 = \begin{bmatrix} 221.71 \\ 478.29 \end{bmatrix}$

$\vdots$

2010 $X_{10} = A^{10} X_0 = \begin{bmatrix} 155.61 \\ 544.39 \end{bmatrix}$

2011 $X_{11} = A^{11} X_0 = \begin{bmatrix} 148.6 \\ 551.4 \end{bmatrix}$

2012 $X_{12} = A^{12} X_0 = \begin{bmatrix} 142.85 \\ 557.15 \end{bmatrix}$.