(This is the coversheet for the homework. The actual solutions should be worked out neatly on single-sided with each problem labeled (for example: Problem 1, 1.1 #14) and the answer boxed. Those additional sheets are then to be stapled with a metal staple in the upper left corner in such a way that no work is obscured. Failure to follow the style outlined here will result in a loss of credit. The problems refer to Anton and Rorres 10th ed. of *Elementary Linear Algebra: applications version.*)

Problem 13 Suppose that $AB = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$. Calculate A.

Problem 14 § 1.4 # 8 (hint: $\cos(-\theta) = \cos(\theta), \sin(-\theta) = -\sin(\theta)$ makes answer neat)

Problem 15 § 1.4 # 30 (symbolic matrix algebra problem)

Problem 16 § 1.5 # 14 (find the inverse, do not use technology for solution, show work)

Problem 17 § 1.5 # 26a (find the inverse, do not use technology for solution, show work)

Problem 18 § 1.5 # 32 (inverse is product of elementary matrices)

Problem 19 § 1.6 # 4 (you can use technology to find the inverse)

Problem 20 § 1.6 # 12 (you can use technology to calculate the rref)

Problem 21 § 1.6 # 14 (find conditions on b_1, b_2 to make $Ax = b$ consistent)

Problem 22 § 1.7 # 34 (matrix algebra problem, involves powers of diagonal matrices)

Problem 23 § 1.8 # 2 (you can use tech. to find an rref here)

Problem 24 § 1.8 # 8 (you can use tech. to find an rref here)

Problem 25 § 1.8 # 10 (you can use tech. to find an rref here)

Problem 26 supplementary exercise # 16 on page 92 (you can use tech. to find an rref here)

Problem 27 Let $Z = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Recall that there are no real solutions of the equation $x^2 + 1 = 0$.

The same is not true for matrices. Show that $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ solves $Z^2 + I = 0$. Let $a, b, c, d \in \mathbb{R}$ and define $Z = aI + bJ$ and $W = cI + dJ$. Calculate ZW and interpret what this calculation represents. Hint: use your *imagination.*

PROBLEM SET 2 SOLUTION, MATH 221

PROBLEM 13 $AB = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$, find A.

$$A = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix} B^{-1}, \quad B^{-1} = \frac{1}{7-6}\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -3 & 13 \\ -8 & 27 \end{bmatrix}}.$$

(check: $\begin{bmatrix} -3 & 13 \\ -8 & 27 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} = \left[\begin{array}{c|c} -21+26 & -9+13 \\ -56+54 & -24+27 \end{array}\right] \overset{\checkmark}{=} \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$)

PROBLEM 14 (§1.4 #8) Find inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

rotation by $-\theta$ gives the inverse matrix, make sense? $\longrightarrow$ $= \boxed{\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}}$

PROBLEM 15 (§1.4 #30) Assume A, B, C, D are invertible. Solve $ABC^TDBA^TC = AB^T$ for D.

$$ABC^TDBA^TC = AB^T \Rightarrow (C^T)^{-1}B^{-1}A^{-1}(ABC^TDBA^TC) = (C^T)^{-1}B^{-1}A^{-1}AB^T$$

$$\Rightarrow DBA^TC = (C^T)^{-1}B^{-1}B^T$$

$$\Rightarrow (DBA^TC)C^{-1}(A^T)^{-1}B^{-1} = (C^T)^{-1}B^{-1}B^TC^{-1}(A^T)^{-1}B^{-1}$$

$$\Rightarrow \boxed{D = (C^T)^{-1}B^{-1}B^TC^{-1}(A^T)^{-1}B^{-1}}$$

PROBLEM 16 (§1.5 #14) find inverse of $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = A$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{3r_1} \\ \xrightarrow{2r_2} \\ \xrightarrow{3r_3} \end{array} \left[\begin{array}{ccc|ccc} 3 & 6 & 0 & 3 & 0 & 0 \\ 0 & -6 & 4 & -4 & 2 & 0 \\ 0 & 6 & 3 & 0 & 0 & 3 \end{array}\right]$$

$$\begin{array}{c} \xrightarrow{r_1 + r_2} \\ \xrightarrow{r_3 + r_2} \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 4 & -1 & 2 & 0 \\ 0 & -6 & 4 & -4 & 2 & 0 \\ 0 & 0 & 7 & -4 & 2 & 3 \end{array}\right] \begin{array}{c} \xrightarrow{7r_1} \\ \xrightarrow{7r_2} \\ \xrightarrow{4r_3} \end{array} \left[\begin{array}{ccc|ccc} 21 & 0 & 28 & -7 & 14 & 0 \\ 0 & -42 & 28 & -28 & 14 & 0 \\ 0 & 0 & 28 & -16 & 8 & 12 \end{array}\right] \begin{array}{c} \xrightarrow{r_1 - r_3} \\ \xrightarrow{r_2 - r_3} \end{array} \left[\begin{array}{ccc|ccc} 21 & 0 & 0 & 9 & 6 & -12 \\ 0 & -42 & 0 & -12 & 6 & -12 \\ 0 & 0 & 28 & -16 & 8 & 12 \end{array}\right]$$

$$\Rightarrow \text{rref}(A|I) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9/21 & 6/21 & -12/21 \\ 0 & 1 & 0 & -12/42 & -6/42 & 12/42 \\ 0 & 0 & 1 & -16/28 & 8/28 & 12/28 \end{array}\right] \Rightarrow \boxed{A^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 2 & -4 \\ -2 & -1 & 2 \\ -4 & 2 & 3 \end{bmatrix}}$$

PROBLEM 17 (§1.5 # 26a) Find inverse of $\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$

Observe $\begin{bmatrix} 0 & 0 & 0 & 1/k_4 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 1/k_1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Thus, $\boxed{\begin{bmatrix} & & & k_1 \\ & & k_2 & \\ & k_3 & & \\ k_4 & & & \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/k_4 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 1/k_1 & 0 & 0 & 0 \end{bmatrix}}$

To derive this you could do row reduction by $E_{r_1 \leftrightarrow r_4}$, $E_{r_2 \leftrightarrow r_3}$, $E_{\frac{1}{k_4} r_1}$, $E_{\frac{1}{k_3} r_2}$, $E_{\frac{1}{k_2} r_3}$ $E_{\frac{1}{k_1} r_4}$.

$$\underbrace{E_{r_4/k_1} E_{r_3/k_2} E_{r_2/k_3} E_{r_1/k_4} E_{r_2 \leftrightarrow r_3} E_{r_1 \leftrightarrow r_4}}_{\text{inverse, same as I guessed by examination.}} \begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

PROBLEM 18 (§1.5 # 32) write $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ as product of elementary matrices.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{r_1 - r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence,

(☆) $E_{1 \leftrightarrow 2} E_{r_1 - r_3} E_{r_3 - r_2} E_{r_2 - r_1} A = I$

$$\Rightarrow \boxed{A = \underbrace{E_{r_2 + r_1} E_{r_3 + r_2} E_{r_1 + r_3} E_{r_1 \leftrightarrow r_2}}_{\text{these are inverses of those in ☆. I solved for A.}}}$$

PROBLEM 19 (§1.6 #4) Solve
$$\begin{aligned}5x_1 + 3x_2 + 2x_3 &= 4\\ 3x_1 + 3x_2 + 2x_3 &= 2\\ x_2 + x_3 &= 5\end{aligned}$$
by multiplication by inverse matrix of coeff. matrix

$$\underbrace{\begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}}_{b}$$

Calculate $rref(A|I) = (I|A^{-1}) \Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -3 & 5 & -4 \\ 3 & -5 & 6 \end{bmatrix}$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -3 & 5 & -4 \\ 3 & -5 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ -22 \\ 32 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}}$$

PROBLEM 20 (§1.6 #12) Solve systems by aug. coeff. technique

$$rref\left[\begin{array}{ccc|ccc} & & & (i) & (ii.) & (iii.) \\ 1 & 3 & 5 & 1 & 0 & -1 \\ -1 & -2 & 0 & 0 & 1 & -1 \\ 2 & 5 & 4 & -1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 18 & -23 & 5 \\ 0 & 1 & 0 & -9 & 11 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

Therefore, I find sol's to systems (i.), (ii.) and (iii) all at once!

(i.) $x_1 = 18$, $x_2 = -9$, $x_3 = 2$ (ii.) $x_1 = -23$, $x_2 = 11$, $x_3 = -2$ (iii.) $x_1 = 5$, $x_2 = -2$, $x_3 = 0$

PROBLEM 21 (§1.6 #14) Find conditions on b_1, b_2 to solve
$$\begin{aligned}6x_1 - 4x_2 &= b_1\\ 3x_1 - 2x_2 &= b_2\end{aligned}$$
if it is possible!

$$\left[\begin{array}{cc|c} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{array}\right] \xrightarrow{r_2 - \frac{1}{2}r_1} \left[\begin{array}{cc|c} 6 & -4 & b_1 \\ 0 & 0 & b_2 - \frac{1}{2}b_1 \end{array}\right]$$

thus, for consistency we must require $\boxed{b_2 - \frac{1}{2}b_1 = 0}$

PROBLEM 22 (§1.7 #34) Find all 3×3 diagonal matrices solving $A^2-3A-4I=0$

Let $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ then $A^2 = \begin{bmatrix} x^2 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & z^2 \end{bmatrix}$

$$A^2 - 3A - 4I = \left[\begin{array}{c|c|c} x^2-3x-4 & 0 & 0 \\ \hline 0 & y^2-3y-4 & 0 \\ \hline 0 & 0 & z^2-3z-4 \end{array}\right] = 0$$

We find $(x-4)(x+1)=0$, $(y-4)(y+1)=0$, $(z-4)(z+1)=0$

thus $x, y, z \in \{-1, 4\}$.

$$\boxed{\begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} -1 & & \\ & -1 & \\ & & 4 \end{bmatrix}, \begin{bmatrix} -1 & & \\ & 4 & \\ & & 4 \end{bmatrix}, \begin{bmatrix} 4 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} 4 & & \\ & -1 & \\ & & 4 \end{bmatrix}, \begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix}}$$

Or, $\begin{bmatrix} i & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & k \end{bmatrix}$ where $i, j, k = -1, 4$.

PROBLEM 23 (§1.8 #2) See text for problem.

a.)

Conservation of flows.

$$\boxed{\begin{aligned} x_1 + x_3 &= 200 \\ x_2 &= x_1 + 25 \\ x_4 + x_5 &= x_3 + 150 \\ x_2 + x_4 &= x_6 + 175 \\ x_5 + x_6 &= 200 \end{aligned}}$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 200 \\ -1 & 1 & 0 & 0 & 0 & 0 & 25 \\ 0 & 0 & -1 & 1 & 1 & 0 & 150 \\ 0 & 1 & 0 & 1 & 0 & -1 & 175 \\ 0 & 0 & 0 & 0 & 1 & 1 & 200 \end{array}\right] \longrightarrow \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & -1 & 150 \\ 0 & 1 & 0 & 1 & 0 & -1 & 175 \\ 0 & 0 & 1 & -1 & 0 & 1 & 50 \\ 0 & 0 & 0 & 0 & 1 & 1 & 200 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

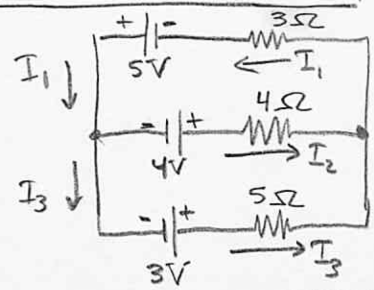
b.)

$$\boxed{\begin{aligned} x_1 &= -t + s + 150 \\ x_2 &= -t + s + 175 \\ x_3 &= t - s + 50 \\ x_5 &= -s + 200 \\ &\text{where } x_4 = t,\ x_6 = s \end{aligned}}$$

c.) Given $\underbrace{\boxed{x_4 = 50}}_{t=50}$ and $\underbrace{\boxed{x_6 = 0}}_{s=0}$ the flow rates are

$$\boxed{\begin{aligned} x_1 &= 100 \\ x_2 &= 125 \\ x_3 &= 100 \\ x_5 &= 200 \end{aligned}}$$

PROBLEM 24 (§1.8 #8)



$$5 - 3I_1 + 4 - 4I_2 = 0$$
$$4 - 4I_2 + 5I_3 - 3 = 0$$
$$I_1 = I_2 + I_3$$

$$\text{rref}\left[\begin{array}{ccc|c} 3 & 4 & 0 & 9 \\ 0 & 4 & -5 & 1 \\ 1 & -1 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 77/47 \\ 0 & 1 & 0 & 48/47 \\ 0 & 0 & 1 & 29/47 \end{array}\right] \Rightarrow \boxed{\begin{array}{l} I_1 = \frac{77}{47}\,A \\ I_2 = \frac{48}{47}\,A \\ I_3 = \frac{29}{47}\,A \end{array}}$$

PROBLEM 25 (§1.8 #10)

Balance $C_6H_{12}O_6 \longrightarrow CO_2 + C_2H_5OH$

We have C, H, O atoms, we need to find x_1, x_2, x_3

$$x_1 \begin{bmatrix} 6 \\ 12 \\ 6 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} \qquad (x_1\, C_6H_{12}O_6 \to x_2\, CO_2 + x_3\, C_2H_5OH)$$

$$\text{rref}\left[\begin{array}{ccc|c} 6 & -1 & -2 & 0 \\ 12 & 0 & -6 & 0 \\ 6 & -2 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_1 = \frac{1}{2}x_3 \\ x_2 = x_3 \end{array}$$

We seek integer sol'ns. choose $\boxed{x_3 = 2}$.

Hence $\boxed{x_1 = 1}$ and $\boxed{x_2 = 2}$.

$$\boxed{C_6H_{12}O_6 \longrightarrow 2CO_2 + 2C_2H_5OH}$$

PROBLEM 26 (Supp. ex. #16 on pg. 92)

Find a, b, c such that $p(x) = ax^2 + bx + c$ passes through $(-1, 0)$ and has $\underbrace{\text{horizontal tangent at } (2,-9)}$

$P(-1) = 0$

$P(2) = -9$

$P'(2) = 0$

implies $y = -9$ for $x = 2$ in $y = P(x)$ and $\underline{P'(2) = 0}$ horizontal.

Note, $P'(x) = 2ax + b$. We find,

$$a - b + c = 0$$
$$4a + 2b + c = -9$$
$$4a + b = 0$$

$$\text{rref}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 2 & 1 & -9 \\ 4 & 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -5 \end{array}\right] \Rightarrow \boxed{P(x) = x^2 - 4x - 5}$$

PROBLEM 27

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow J^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I \therefore \underline{J^2 + I = 0.}$$

Let $Z = aI + bJ$ and $W = cI + dJ$

Thus $Z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $W = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$

$$ZW = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -ad - bc \\ bc + ad & -bd + ac \end{bmatrix}$$
$$= (ac - bd)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (ad + bc)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note
$$(a + ib)(c + id) = ac + iad + ibc + i^2 bd$$
$$= ac - bd + i(ad + bc)$$

The matrices Z and W correspond to complex numbers $a+ib$ and $c+id$ respective. Moreover ZW likewise corresponds to $(a+ib)(c+id)$.

<u>Remark</u>: $\mathbb{C} \xrightarrow{\psi} M < \mathbb{R}^{2\times 2}$ $\psi(x+iy) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ makes ψ an <u>isomorphism</u>. Same algebra, different notation.