MATH 221 | PROBLEM SET 3 (DUE 9-11)

(This is the coversheet for the homework. The actual solutions should be worked out neatly on single-sided with each problem labeled (for example: Problem 1, 1.1 #14) and the answer boxed. Those additional sheets are then to be stapled with a metal staple in the upper left corner in such a way that no work is obscured. Failure to follow the style outlined here will result in a loss of credit. The problems refer to Anton and Rorres 10th ed. of *Elementary Linear Algebra: applications version.*)

Problem 28 Suppose $M \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times n}$ such that x_1, x_2 are solutions $Mx_1 = y_1$ and $Nx_2 = y_2$. Let A be a block-diagonal matrix of the form:

$$A = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right]$$

If $det(M), det(N) \neq 0$ then solve $Az = w$ where $w = [2y_1, 3y_2]^T$.

Problem 29 § 2.1 # 10 (arrow-technique to calculate 3×3 determinant)

Problem 30 § 2.1 # 16 (equation given by determinant, solve it)

Problem 31 § 2.1 # 32 (determinant by inspection)

Problem 32 § 2.2 # 12 (row-reduction to calculate det)

Problem 33 § 2.2 # 20 (properties of det calculation)

Problem 34 The matrix $A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ is an example of a **Vandermonde matrix**. Use properties about row and column operations to show that $det(A) = (b-a)(c-a)(c-b)$.

Problem 35 Define a **Vandermonde matrix** $V(t)$ as follows:

$$V(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}$$

This matrix provides us a convenient way of creating a cubic polynomial that passes through distinct zeros $(x_1, 0)$, $(x_2, 0)$, $(x_3, 0)$. Define $f(x) = det(V(x))$ and explain why $f(x_1) = f(x_2) = f(x_3) = 0$. Based on an analogy to the preceding problem state an explicit formula for $f(t)$.

Problem 36 § 2.3 # 8 (invertible? use det to decide)

Problem 37 § 2.3 # 18 (choose k to make matrix invertible)

Problem 38 Supplementary exercises, page 117 #26 (Cramer's rule for symbolic problem)

Problem 39 Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(a.) let $p(t) = det(A - tI)$. Calculate the formula for $p(t)$

(b.) show that $p(A) = 0$

Problem 40 Let $\vec{v}_1 = \langle 1,2,3\rangle$, $\vec{v}_2 = \langle 1,1,0\rangle$ and $\vec{v}_3 = \langle 2,1,3\rangle$. Find the volume of a parallel-piped with edges $\vec{v}_1, \vec{v}_3, \vec{v}_3$. Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a right-handed triple?

$\vec{V}_2$

(typo)

PROBLEM 28 Given $M \in \mathbb{R}^{m \times m}$, $N \in \mathbb{R}^{n \times n}$
with sol^{n}'s x_1, x_2 such that $Mx_1 = y_1$ and $Nx_2 = y_2$.
If $A = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right] \in \mathbb{R}^{(m+n)\times(m+n)}$ and $\det M, \det N \neq 0$
then solve $Az = w$ where $w = [2y_1, 3y_2]^T$

Let z_1, z_2 be such that Mz_1, Nz_2 are defined,

$$Az = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} Mz_1 \\ Nz_2 \end{bmatrix} = \begin{bmatrix} 2y_1 \\ 3y_2 \end{bmatrix}$$

we must solve $Mz_1 = 2y_1$ and $Nz_2 = 3y_2$
Note $Mx_1 = y_1$ & $Nx_2 = y_2 \Rightarrow M^{-1}y_1 = x_1$ & $N^{-1}y_2 = x_2$
since $\det M, \det N \neq 0$ and we know M^{-1}, N^{-1} exist.
We find $z_1 = 2M^{-1}y_1 = 2x_1$ & $z_2 = 3N^{-1}y_2 = 3x_2$
Thus $\boxed{z = [2x_1, 3x_2]^T}$

PROBLEM 29 (§2.1 #10)

$$\det\begin{bmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{bmatrix} = (-8)(-42)(240) - 18(32)(140) = \boxed{0}$$

$$\begin{array}{ccc|cc} -2 & 7 & 6 & -2 & 7 \\ 5 & 1 & -2 & 5 & 1 \\ 3 & 8 & 4 & 3 & 8 \end{array}$$

PROBLEM 30 (§2.1 #16) find sol^{n}'s of $\det(A) = 0$.

$$\det\begin{bmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{bmatrix} = (\lambda-4)[\lambda(\lambda-1) - 6]$$
$$= (\lambda-4)(\lambda^2 - \lambda - 6)$$
$$= (\lambda-4)(\lambda-3)(\lambda+2) = 0$$
$$\therefore \boxed{\lambda = 3, 4, -2}$$

PROBLEM 31 (§2.1#32) find det. by inspection.

the matrix $A = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 40 & 10 & -1 & 0 \\ 100 & 200 & -23 & 3 \end{bmatrix}$ is lower triangular

$\therefore \quad \det(A) = -3(2)(-1)(3) = \boxed{18.}$

PROBLEM 32 (§2.2#12) use row-reduction to calculate determinants.

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix} \xrightarrow[r_3 - 5r_1]{r_2 + 2r_1} \begin{bmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{bmatrix} \xrightarrow{r_2 + \frac{2}{13} r_3} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 17/13 \\ 0 & 13 & 2 \end{bmatrix}$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \underbrace{\begin{bmatrix} 1 & -3 & 0 \\ 0 & 13 & 2 \\ 0 & 0 & 17/13 \end{bmatrix}} \cdots \longrightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix} = \text{ref}(A)$$

$$\det(A) = -(13)\left(\frac{17}{13}\right) = \boxed{-17.}$$

because I only did row additions and a row-swap.

Maybe you'll like this notation $\det A = |A|$

$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} \xrightarrow[r_3 - 5r_1]{r_2 + 2r_1} \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} \xrightarrow{r_2 + \frac{2}{13} r_3} \begin{vmatrix} 1 & -3 & 0 \\ 0 & 0 & 17/13 \\ 0 & 13 & 2 \end{vmatrix}$$

$$\xrightarrow{r_2 \leftrightarrow r_3} - \begin{vmatrix} 1 & -3 & 0 \\ 0 & 13 & 2 \\ 0 & 0 & 17/13 \end{vmatrix} \xrightarrow{\frac{1}{13} r_2} \frac{-1}{13} \begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & 2/13 \\ 0 & 0 & 17/13 \end{vmatrix}$$

$$\xrightarrow{\frac{13}{17} r_3} \frac{13}{17}\left(\frac{-1}{13}\right) \begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & 2/13 \\ 0 & 0 & 1 \end{vmatrix} = -17\,(1)(1)(1) = \boxed{-17}$$

PROBLEM 33 (§2.2 #20)

$$\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = - \underbrace{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}_{\text{given to be } -6} = -(-6) = \boxed{6.}$$

PROBLEM 34 You could use row op. do do the whole calculation. This is easier.

$$\det\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = \det\begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix}$$

$$= 1\cdot\left[(b-a)(c^2-a^2) - (c-a)(b^2-a^2)\right]$$

$$= (b-a)(c-a)(c+a) - (c-a)(b+a)(b-a)$$

$$= (b-a)(c-a)\left[c+\cancel{a} - (b+\cancel{a})\right]$$

$$= \underline{(b-a)(c-a)(c-b)}.$$

PROBLEM 35

$$V(t) = \det\begin{bmatrix} 1 & t & t^2 & t^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}.$$ Show $f(x_1) = f(x_2) = f(x_3) = 0$ and find formula for $f(t)$

Clearly $f(x_1) = f(x_2) = f(x_3) = 0$ since

$$V(x_1) = \det\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} = 0$$ repeated row $\Rightarrow$ zero det.

Likewise for x_2 & x_3. We've found

$f(t) = A(x_1-t)(x_2-t)(x_3-t)$, need to find A.

$$f(0) = x_1x_2x_3A = V(0) = \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}$$

$$V(0) = \begin{vmatrix} x_1 & x_1^2 & x_1^3 \\ x_2 & x_2^2 & x_2^3 \\ x_3 & x_3^2 & x_3^3 \end{vmatrix} = x_1(x_2^2x_3^3 - x_2^3x_3^2) - x_1^2(x_2x_3^3 - x_3x_2^3) + x_1^3(x_2x_3^2 - x_3x_2^2$$

$$= x_1x_2x_3\left[\underbrace{x_2^2x_3^3 - x_2^3x_3^2 - x_1x_2x_3^2 + x_1x_2^3x_3 + x_1^2x_2x_3 - x_1^2x_3^2}_{A}\right.$$

$$\therefore \boxed{f(t) = x_2x_3(x_2x_3^2 - x_2^2x_3 - x_1x_3 + x_1x_2^2 + x_1^2 - x_1^2x_2)(x_1-t)(x_2-t)(x_3-t}$$

PROBLEM 36 (§2.3 #8)

$$\det \underbrace{\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}}_{A} = 2(-12) + 3(6) = \underline{-6 \neq 0}$$

$\therefore$ $\boxed{A^{-1} \text{ exists}}$

PROBLEM 37 (§2.3 #18)

$A = \begin{bmatrix} 1 & 2 & 0 \\ k & 1 & k \\ 0 & 2 & 1 \end{bmatrix}$ find k for which A^{-1} exists.

We need $\det(A) \neq 0$. Consider,

$$\det(A) = 1(1-2k) - 2(k) = 1 - 4k = 0 \Rightarrow \underline{k = \tfrac{1}{4}}$$

We find $\boxed{A^{-1} \text{ exists for all } k \neq 1/4}$

PROBLEM 38 (pg. 117 #26)

Solve
$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$
for x', y'. By Cramer's Rule

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$x' = \frac{\det\begin{bmatrix} x & -\sin\theta \\ y & \cos\theta \end{bmatrix}}{\det\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} = \frac{x\cos\theta + y\sin\theta}{\cos^2\theta + \sin^2\theta}$$

$$y' = \frac{\det\begin{bmatrix} \cos\theta & x \\ \sin\theta & y \end{bmatrix}}{1} = y\cos\theta - x\sin\theta$$

$$\therefore \boxed{\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= y\cos\theta - x\sin\theta \end{aligned}}$$

PROBLEM 39 Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

(a.) $P(t) = \det(A - tI)$. find formula,

$$P(t) = \det \begin{bmatrix} 1-t & 1 \\ 1 & 2-t \end{bmatrix}$$

$$= (1-t)(2-t) - 1$$

$$= (t-1)(t-2) - 1$$

$$= t^2 - 3t + 2 - 1$$

$$= \boxed{t^2 - 3t + 1 = P(t)}$$

(b.) $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

$$A^2 - 3A + I = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$(P(A) = \det(A - AI) = \det(0) = 0$ also good$)$

PROBLEM 40 edges $\vec{V}_1, \vec{V}_2, \vec{V}_3$ give what ||-piped volume? is $\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ right-handed?

$$\det(\vec{V}_1|\vec{V}_2|\vec{V}_3) = \det\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix} = -6 \quad \therefore \boxed{Vol = 6}$$

this $\det(\vec{V}_1|\vec{V}_2|\vec{V}_3) < 0 \Rightarrow \{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ is <u>not</u> right-handed.