Math 221 — Problem Set 4 (due 9-25)

(This is the coversheet for the homework. The problems refer to Anton and Rorres 10th ed. of *Elementary Linear Algebra: applications version.* See Problem Set 1, 2 or 3 for further formatting.)

****you are free to use technology to do row-reductions etc... however, you still must show your work by explaining where the matrix came from and writing sentences about why you are doing what you do****

Problem 41 Is $(1, 2, 3) \in span\{v_1, v_2, v_3\}$ given that $v_1 = (1, 1, 1), v_2 = (2, 2, 2), v_3 = (-1, 0, 1)$? If it is then find the linear combination $\{v_1, v_2, v_3\}$ which yields $(1, 2, 3)$.

Problem 42 Let v_1, v_2, v_3 be as above. Is $\{v_1, v_2, v_3\}$ linearly independent? If it is linearly dependent then use the CCP to find a linear dependence between the vectors.

Problem 43 § 4.2 # 8 (span of 3-vectors)

Problem 44 § 4.3 # 2 (test for LI)

Problem 45 § 4.3 # 8 (on linear dependence)

Problem 46 § 4.4 # 2 (basis for $\mathbb{R}^2$?)

Problem 47 § 4.4 # 8 (find coordinate vectors)

Problem 48 § 4.5 # 4 (find dimension of solution space)

Problem 49 § 4.5 # 8 (4 dimensional geometry)

Problem 50 § 4.5 # 12 (extending linearly dependent set to make basis)

Problem 51 What is the least number of linear equations it takes to determine a

(a.) a line in $\mathbb{R}^3$

(b.) a line in $\mathbb{R}^4$

(c.) a plane in $\mathbb{R}^4$

(d.) a plane in $\mathbb{R}^5$

Problem 52 Suppose $W = span\{v_1, v_2, \ldots v_k\} \subseteq \mathbb{R}^n$. Is this a k-dimensional subspace of $\mathbb{R}^n$? Justify your answer, a simple yes or no is not sufficient here. I would like you to outline some strategy or calculation that would allow you to ascertain the dimension of this subspace.

Problem 53 Given k-points in n-dimensional space where $k > n$ what is the dimension of the subspace on which the points reside? I'll give this question more concretely; what is the dimension of the subspace of $\mathbb{R}^4$ which contains

$$(1, 2, 3, 4, 0),\ (0, -1, 1, 1, 3),\ \cancel{(1, 1, 1, 1, 1),}\ (3, 5, 7, 9, 1),\ (3, 4, 8, 10, 4)\ ?$$

Problem 54 § 3.4 # 6 (find point and direction vector for a line)

Problem 55 § 3.4 # 12 (find parametrization of plane)

PROBLEM SET 4 Sol^{n}

PROBLEM 41 Is $(1,2,3) \in \text{span}\{V_1, V_2, V_3\}$ where $V_1 = (1,1,1)$, $V_2 = (2,2,2)$, $V_3 = (-1,0,1)$. If YES then how?

$$\text{rref}\begin{array}{c}\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 3 \end{array}\right] \\ \begin{array}{ccc} V_1 & V_2 & V_3 \end{array}\end{array} = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2V_1 + V_3} \text{ by CCP.}$$

$\exists$ many other correct answers since $V_2 = 2V_1$ by CCP or inspection.

PROBLEM 42 Is $\{V_1, V_2, V_3\}$ LI?

No. As we noted $V_2 = 2V_1$ hence $\{V_1, V_2, V_3\}$ is not LI.

PROBLEM 43 (§4.2 # 8)

Express the vectors below as linear combo. of ~~...~~ $u = (2,1,4)$, $v = (1,-1,3)$, $w = (3,2,5)$

(a.) $(-9,-7,-15)$ (c.) $(0,0,0)$

(b.) ~~...~~ $(6,11,6)$ (d.) $(7,8,9)$

The easiest way to answer a, b, c, d is all at once!

$$\text{rref}\begin{array}{c}\left[\begin{array}{ccc|cccc} 2 & 1 & 3 & -9 & 6 & 0 & 7 \\ 1 & -1 & 2 & -7 & 11 & 0 & 8 \\ 4 & 3 & 5 & -15 & 6 & 0 & 9 \end{array}\right] \\ \begin{array}{ccc} u & v & w \end{array}\end{array} = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -2 & 4 & 0 & 0 \\ 0 & 1 & 0 & 1 & -5 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 & 0 & 3 \end{array}\right]$$

By CCP in view of computation above we find by inspection,

$$\boxed{\begin{array}{ll} \text{(a.)} & (-9,-7,-15) = -2u + v - 2w \\ \text{(b.)} & (6,11,6) = 4u - 5v + w \\ \text{(c.)} & (0,0,0) = 0\cdot u + 0\cdot v + 0\cdot w \\ \text{(d.)} & (7,8,9) = -2v + 3w \end{array}}$$

PROBLEM 44 (§4.3 #2) Check for LI of sets of vectors below:

(a.) $\{(4,-1,2), (-4,10,2)\}$ is not linearly dependent. $[(4,-1,2) \neq k(-4,10,2)$ for any k

(b.) $\{(-3,0,4), (5,-1,2), (1,1,3)\} = S$

consider $\text{rref}[S] = \text{rref}\begin{bmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\therefore$ $\boxed{S \text{ is LI}}$

(S not linearly dependent.)

Remark: (for (b.) could utilize $\det[S] \neq 0 \Rightarrow$ LI of S) continued ↷

Problem 44 continued (§4.3#2)

(c.) $T = \{(8,-1,3), (4,0,1)\}$ note $rref[T] = rref\begin{bmatrix} 8 & 4 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \therefore$ T is LI.

(d.) $T_2 = \{(-2,0,1), (3,2,5), (6,-1,1), (7,0,-2)\}$

note $rref[T_2] = rref\begin{bmatrix} -2 & 3 & 6 & 7 \\ 0 & 2 & -1 & 0 \\ 1 & 5 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -79/29 \\ 0 & 1 & 0 & 3/29 \\ 0 & 0 & 1 & 6/29 \end{bmatrix}$

thus $(7,0,-2) = -\frac{79}{29}(-2,0,1) + \frac{3}{29}(3,2,5) + \frac{6}{29}(6,-1,1)$

Clearly T_2 is linearly dependent set.
(this could be anticipated w/o calculation!
we can have at most 3 LI vectors in $\mathbb{R}^3$)

PROBLEM 45 (§4.3#8)

(a.) $V_1 = (1,2,3,4)$, $V_2 = (0,1,0,-1)$, $V_3 = (1,3,3,3)$
show $\{V_1, V_2, V_3\}$ is not LI.

(b.) express V_1, V_2, V_3 as Linear combo of (V_2,V_3), (V_1,V_3), (V_1,V_2) respective.

(a.) $rref[V_1|V_2|V_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ hence $V_3 = V_1 + V_2$
$\Rightarrow \boxed{\{V_1, V_2, V_3\} \text{ is not LI}}$

(b.) $\boxed{V_3 = V_1 + V_2, \quad V_1 = V_3 - V_2, \quad V_2 = V_3 - V_1}$ (algebra!)

PROBLEM 46 (§4.4#2) which of the following are bases for $\mathbb{R}^2$?
I'll use determinants for each since we're testing if two vectors in $\mathbb{R}^2$ are LI.

(a.) $\beta_1 = \{(2,1), (3,0)\}$ has $\det\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = -3 \neq 0$ $\therefore$ β_1 is LI set of 2-vectors for $\mathbb{R}^2$ $\Rightarrow$ β_1 is basis.

(b.) $\beta_2 = \{(4,1), (-7,-8)\}$ has $\det\begin{bmatrix} 4 & -7 \\ 1 & -8 \end{bmatrix} = -32+7 \neq 0$
Hence β_2 is LI and consequently a basis.

(c.) $\beta_3 = \{(0,0), (1,3)\}$ is linearly dependent since it contains the zero vector $\therefore$ β_3 is not a basis for $\mathbb{R}^2$.

(d.) $\beta_4 = \{(3,9), (-4,-12)\}$ has $\det\begin{bmatrix} 3 & -4 \\ 9 & -12 \end{bmatrix} = -36+36 = 0$ $\therefore$ β_4 is not LI
$\Rightarrow$ β_4 not a basis.

PROBLEM 47 | (§4.4#8)

Find coordinate vector of w relative to $S' = \{u_1, u_2\}$ of $\mathbb{R}^2$

(a.) $u_1 = (1,-1)$, $u_2 = (1,1)$ and $w = (1,0)$ (method: inspection.)

Note: $w = \frac{1}{2}(u_1 + u_2) \therefore \underline{[w]_{S'} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}$.

(b.) $u_1 = (1,-1)$, $u_2 = (1,1)$ and $w = (0,1)$

$$\text{rref}\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \end{array}\right] \Rightarrow \underline{w = -\tfrac{1}{2}u_1 + \tfrac{1}{2}u_2}$$

$$\therefore \underline{[w]_{S'} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}}.$$

(c.) $u_1 = (1,-1)$, $u_2 = (1,1)$, $w = (1,1)$

can use parts (a.), (b.) note

$$(1,1) = (1,0) + (0,1) \Rightarrow [w]_{S'} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \underline{\begin{bmatrix} 0 \\ 1 \end{bmatrix} = [w]_{S'}}.$$

or, again,

$$\text{rref}\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

(you can also find $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and use this to calculate coordinates... maybe next problem...)

PROBLEM 48 (§4.5#4) | find dimension of sol$^{\underline{n}}$ space for $x_1 - 3x_2 + x_3 = 0$, $2x_1 - 6x_2 + 2x_3 = 0$, $3x_1 - 9x_2 + 3x_3 = 0$

$$\text{rref}\underbrace{\begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}}_{A}\left|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] = \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = 3x_2 - x_3$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 3x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(Thus $\text{Null}(A) = \text{span}\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} = \{x \mid Ax = 0\}$) ← not asked for here.

$\therefore$ $\boxed{\text{dimension of sol}^{\underline{n}} \text{ space is two}}$

$\boxed{\text{PROBLEM 49 (§4.5 \#8)}}$

(a.) $\{(a,b,c,0) \mid a,b,c \in \mathbb{R}\} = \text{span}\{e_1, e_2, e_3\}$ is $\boxed{3\text{-dim'l}}$

(b.) $W = \{(a,b,c,d) \mid d = a+b,\ c = a-b\}$

$$\Rightarrow W = \{(a, b, a-b, a+b) \mid a, b \in \mathbb{R}\}$$
$$= \text{span}\{e_1 + e_3 + e_4,\ e_2 - e_3 + e_4\} \quad \therefore \boxed{W \text{ is } 2\text{-dim'l}}$$

(c.) $\{(a,b,c,d) \mid a = b = c = d \in \mathbb{R}\} = \text{span}\{(1,1,1,1)\} \therefore \boxed{1\text{-dim'l}}$

PROBLEM 50 (§4.5 #12)

Extend $\{V_1, V_2\}$ to make $\{V_1, V_2, V_3\}$ a basis for $\mathbb{R}^3$

Note: if $\{V_1, V_2, V_3\}$ is basis for $\mathbb{R}^3$ then $\det[V_1|V_2|V_3] \neq 0$ so we could use det as a guide for selecting V_3.

(a.) $V_1 = (-1, 2, 3)$, $V_2 = (1, -2, -2)$ let $V_3 = (a, b, c)$

$$\begin{bmatrix} -1 & 1 & a \\ 2 & -2 & b \\ 3 & -2 & c \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & a \\ 0 & 0 & b+2a \\ 0 & 1 & c+3a \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -c-2a \\ 0 & 0 & b+2a \\ 0 & 1 & c+3a \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2a+c \\ 0 & 1 & 3a+c \\ 0 & 0 & b+2a \end{bmatrix}$$

we find any (a,b,c) such that $b + 2a \neq 0$ will force $\underbrace{\text{LI of } \{V_1, V_2, V_3\}}_{\text{Basis}}$.

Of course you can guess V_3, it's by far the best sol$^{\underline{n}}$. Just guess and check for LI.

(b.) $V_1 = (1, -1, 0)$, $V_2 = (3, 1, -2)$ guess $V_3 = (1, 1, 0)$.

Check. $\det \begin{bmatrix} 1 & 3 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2(2) = 4 \neq 0.$

$\therefore \boxed{V_3 = (1,1,0) \text{ extends } \{V_1, V_2\} \text{ to a basis.}}$

$\exists\ \infty$ly other correct answers!

PROBLEM 51 | What is the least # of linear eq^n's to define:

(a.) a line in $\mathbb{R}^3$ needs a 1-dim'l sol^t space
to $Ax = 0$ $\Rightarrow$ Null(A) should be 1-dim'l
$\Rightarrow$ Col(A) should be 2-dim'l (rank(A) + nullity(A) = 3)
$\Rightarrow$ Row(A) should be 2-dim'l (dim(Row(A)) = dim(Col(A)))
$\Rightarrow$ $\boxed{\text{at least 2 linear eq}^n\text{'s}}$

btw, these are the "symmetric eq^n's for a line" in the standard calc. III texts.

(b.) a line in $\mathbb{R}^4$ will be fixed by at least $\boxed{3 \text{ eq}^n\text{'s}}$

(c.) plane in $\mathbb{R}^4$ needs at least $\boxed{2 \text{ eq}^n\text{'s}}$

(d.) plane in $\mathbb{R}^5$ needs at least $\boxed{3 \text{ eq}^n\text{'s}}$

Remark: to parametrize a plane we always need 2 LI vectors in the plane. To write eq^n(s) for the plane in $\mathbb{R}^n$ we need $(n-2)$ - eq^n's. For $n=3$ we get $3-2=1$ which is nice.

PROBLEM 52 | No, span$\{V_1, V_2, .., V_k\} = W$ isn't necessarily k-dim'l because $\{V_1, V_2, .., V_k\}$ can have some linear dependence.
To find dim(W) we simply calculate

$$\text{rref}\,[V_1 | V_2 | \cdots | V_k]$$

and count pivot columns to see dim(W).
(Moreover, the pivot V_j's can be used to form a basis for W.)

PROBLEM 53 (modified as discussed in lecture)

Find dim (span (S)) where
$S = \{(1,1,1), (0,0,1), (2,2,3), (0,0,2), (1,1,0)\}$

$$\text{rref}[S] = \text{rref}\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

thus $\text{span}\, S = \text{span}\,\{\underbrace{(1,1,1), (0,0,1)}_{LI}\} \Rightarrow \boxed{\dim(\text{Span}(S)) = 2}$

PROBLEM 54 (§3.4 #6) Given $(x,y,z) = (4t, 7, 4+3t)$ parametrizes a line find a pt. on this line and its direction vector.

$(x,y,z) = (0,7,4) + t(4,0,3) \Rightarrow$ $(0,7,4)$ is pt. on line
$(4,0,3)$ is direction vector.

PROBLEM 55 (§3.4 #12) Find parametrization of plane with point $P = (0,5,-4)$ and vectors $V_1 = (0,0,-5)$ and $V_2 = (1,-3,-2)$

$$\vec{r}(s,t) = P + sV_1 + tV_2$$
$$= \boxed{(t, 5-3t, -4-5s-2t)}$$

vector parametric

$$\left.\begin{aligned} x &= t \\ y &= 5-3t \\ z &= -4-5s-2t \end{aligned}\right\}$$

scalar parametric.