MATH 221 — PROBLEM SET 5 (DUE 10-2)

(This is the coversheet for the homework. The problems refer to Anton and Rorres 10th ed. of *Elementary Linear Algebra: applications version.* See Problem Sets 1, 2 or 3 for further formatting.)

Problem 56 § 4.1 # 8 (is it a vector space?) ✓

Problem 57 § 4.2 # 2 (is it a vector space?) ✓

Problem 58 § 4.2 # 4 (on function space subspaces) ✓

Problem 59 § 4.2 # 14 (hint: review your trig. identities) ✓

Problem 60 § 4.3 # 4 (hint: convert it to a matrix problem) ✓

Problem 61 § 4.4 # 4 (basis for P_2 ?) ✓

Problem 62 § 4.7 # 4 (solution set, parameters) ✓

Problem 63 § 4.7 # 6 (find basis for null space of A) ✓

Problem 64 § 4.7 # 10 (find basis for row space of A with rows of A) ✓

Problem 65 For the matrices from the previous pair of problems, find a basis for the column space (use the same a,b,c,d,e labels) ✓

Problem 66 § § 4.7 # 12 (find basis for the span, express linear dependencies) ✓

Problem 67 § 4.7 # 16 (geometry of null space, line, plane, point) ✓

Problem 68 § 4.8 # 2 (rank-nullity theorem) ✓

Problem 69 § 4.8 # 9 (over-determined) ✓

Problem 70 § 4.8 # 10 (full-rank from subdeterminants) ✓

Problem 71 § 4.8 # 14 (uses result of previous problem) ✓

PROBLEM SET 5 Sol$^{\underline{n}}$

PROBLEM 56 (§4.1 #8) Let $S = \{A \in \mathbb{R}^{2\times2} \mid A^{-1} \text{ exists}\}$. Is this a vector space w.r.t. usual matrix addition/scalar mult.?

Answer, No. Reasons include:

1.) 0^{-1} d.n.e. since $\det(0) = 0$ it follows 0 matrix is not invertible.

2.) If A^{-1} exists then $\det(A) \neq 0$. However, $-A$ is also invertible since $\det(-A) = \pm\det(A) \neq 0$. Observe $-A + A = 0$ thus the sum of invertible matrices need not be invertible.

3.) If A is invertible the $0 \cdot A = 0$ is not invertible! S not closed under scalar multiplication.

PROBLEM 57 (§4.2 #2) In each case below is W a vector space? I'll give one reason why. If you gave multiple good reasons we gave some extra credit here.

(a.) $W = \{\text{diagonal matrices}\}$. Note $0 \in W$ hence $W \neq \phi$. Let $A = \text{diag}(a_1, a_2, \dots, a_n)$ and $B = \text{diag}(b_1, b_2, \dots, b_n)$ and $c \in \mathbb{R}$. Notice

$$A + cB = \begin{pmatrix} a_1 + cb_1 & & & \\ & a_2 + cb_2 & & \\ & & \ddots & \\ & & & a_n + cb_n \end{pmatrix} \in W$$

Hence W is closed under $+$ and $\cdot$ and $\therefore \underline{W \leq \mathbb{R}^{n\times n}}$.

(b.) Notice $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W = \{A \in \mathbb{R}^{2\times2} \mid \det(A) = 0\}$
However, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$
hence $\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \notin W$ and we find W is not a vector space
(one counter-example strikes down $n = 2$, but clearly this generalizes)

Problem S7 continued (§4.2c, d, e, f, g)

(c.) $W = \{A \in \mathbb{R}^{n\times n} \mid \text{trace}(A) = tr(A) = 0\}$

Let $A, B \in W$ and $c \in \mathbb{R}$. Recall from early hwk,

$$tr(A + cB) = tr(A) + c\,tr(B) = 0 + c(0) = 0$$

thus $A + cB \in W$. Moreover, $tr(0) = 0$ hence $W \neq \emptyset$ and by Th$\underline{m}$ 4.2.1 we find $W \leq \mathbb{R}^{n\times n}$.

(d.) $W = \{A \in \mathbb{R}^{n\times n} \mid A^T = A\}$

Let us note $0^T = 0$ hence W has at least one element.

Take $A, B \in W$ and consider by prop. of transpose,

$$(A+cB)^T = A^T + cB^T$$
$$= A + cB \quad \therefore A + cB \in W.$$

Hence $W \leq \mathbb{R}^{n\times n}$ by $\underbrace{\text{Th}\underline{\text{m}}\ 4.2.1}_{\text{"a.k.a. two-step subspace test."}}$ once more.

(e.) same argument as d. except $(A+cB)^T = A^T + cB^T$
(Yes antisymmetric matrices form subspace) $= -A + c(-B)$
$= -(A + cB)$.

(f.) The set of all $n\times n$ matrices A for which $Ax = 0$ has only the trivial sol$\underline{n}$ is same as $\boxed{\text{Problem S6}}$. Note $Ax = 0 \iff x = 0$ implies A^{-1} exists so $W = \{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\}$ and we argued $W \not\leq \mathbb{R}^{n\times n}$ previously in ~~Prob S6~~ PROB. S6.

(g.) $W = \{A \in \mathbb{R}^{n\times n} \mid AB = BA \text{ for some } B \in \mathbb{R}^{n\times n}\}$.

Note $0 \in W$ since $0B = B0 = 0$ hence $W \neq \emptyset$.

Suppose $A_1, A_2 \in W$ and take $c \in \mathbb{R}$. Consider,

$$(A_1 + cA_2)B = A_1B + cA_2B = BA_1 + cBA_2 = B(A_1 + cA_2)$$

hence $A_1 + cA_2 \in W \Rightarrow W \leq \mathbb{R}^{n\times n}$.

(interesting example.)

PROBLEM 58 (§4.2#4) Which of the following are subspaces of $\mathcal{F}(-\infty, \infty)$?

(a.) $W_a = \{f \in \mathcal{F} \mid f(0) = 0\} \leq \mathcal{F}$.

(b.) $W_b = \{f \in \mathcal{F} \mid f(0) = 1\} \not\leq \mathcal{F}$

note $f_1(0) = 1,\ f_2(0) = 1 \Rightarrow (f_1 + f_2)(0) = 1 + 1 = 2 \neq 1$

Hence $f_1, f_2 \in W_b \not\Rightarrow f_1 + f_2 \in W_b$.

(c.) $W_c = \{f \in \mathcal{F} \mid f(x) = f(-x)\} \leq \mathcal{F}$

(d.) $W_d = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\} \leq \mathcal{F}$

▶ The proof of a, c and d. follows from the 2-step subspace test Th$^{\underline{m}}$ and those who did that work earned extra credit.

PROBLEM 59 §4.2#14 Let $f = \cos^2 x$ and $g = \sin^2 x$ which of the functions $\cos 2x$, $3 + x^2$, 1, $\sin x$, 0 are found in span$\{f, g\}$?

(a.) Note $f = \frac{1}{2}(1 + \cos 2x)$ and $g = \frac{1}{2}(1 - \cos 2x)$

hence $f - g = \cos 2x \in \text{span}\{f, g\}$.

(b.) $x^2 \notin \text{span}\{f, g\}$. This is clear.

(c.) $f + g = 1 \in \text{span}\{f, g\}$

(d.) $\sin(x) \notin \text{span}\{f, g\}$. One way to show this is to calculate $\det\begin{bmatrix} f & g & \sin x \\ f' & g' & \cos x \\ f'' & g'' & -\sin x \end{bmatrix}$ and show it's nonzero.

(e.) $0 \cdot f + 0 \cdot g = 0 \in \text{span}\{f, g\}$.

PROBLEM 60 (§4.3#4) TEST linear dependence / LI of sets of polynomials below:

Remark: I use $P_2 = \text{span}\{1, x, x^2\}$ to convert given $f(x) = a + bx + cx^2$ to $[f(x)]_\beta = [a, b, c]^T$.
Moreover a Th$^{\underline{m}}$ which we know states
$\Phi_\beta(S') \text{ is LI} \iff S \text{ is LI}$
Thus we can check LI of S by checking for LI of the corresponding set of coord. vectors $\Phi_\beta(S')$.

LOGIC

(a.) $f_1 = 2 - x + 4x^2$
$f_2 = 3 + 6x + 2x^2$
$f_3 = 2 + 10x - 4x^2$

$$\xrightarrow{} \text{rref}\begin{bmatrix} 2 & 3 & 4 \\ -1 & 6 & 2 \\ 4 & 10 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $\{f_1, f_2, f_3\}$ is LI since $\{[f_1]_\beta, [f_2]_\beta, [f_3]_\beta\}$ is clearly LI by inspection of the matrix and CCP.
▸ you could also use $\det\begin{bmatrix} 2 & 3 & 4 \\ -1 & 6 & 2 \\ 4 & 10 & -4 \end{bmatrix} = \underline{-212}$.

(b.) $$\text{rref}\begin{bmatrix} 3 & 2 & 4 \\ 1 & -1 & 0 \\ 1 & 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \Rightarrow \underline{\text{LI set of polynomials.}}$$

(c.) $$\text{rref}\begin{bmatrix} 6 & 1 \\ 0 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$ thus LI set in (c.)

(alternatively: argue that $6 - x^2 \neq k(1 + x + 4x^2)$ hence not linearly dependent!)

(d.) $$\text{rref}\begin{bmatrix} 1 & 1 & 5 & 7 \\ 3 & 0 & 6 & 2 \\ 3 & 4 & 3 & -1 \end{bmatrix}$$ is definitely not going to be LI. Even without any calculation we can conclude the set is linearly dependent.

PROBLEM 61/ (§4.4 #4) which of given $\{f_1, f_2, f_3\}$ are bases for P_2 ?

Remark: once more we use Logic of Problem 60. Note, if $\{f_1, f_2, f_3\}$ is LI then it follows from one of our dimension Th$^{\underline{m}}$'s that it must be a basis for P_2 since $\{1, x, x^2\}$ is a basis of P_2.

(a.) $\text{rref}\begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\boxed{\text{Not a basis.}}$ ($f_3 = 2f_1 - f_2$ ∴ lin. dep.)

(b.) $\text{rref}\begin{bmatrix} 4 & -1 & 5 \\ 6 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\boxed{\text{Not a basis}}$ ($f_3 = f_1 - f_2$ is linear dep.)

(c.) $\text{rref}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\boxed{\text{is a basis}}$

(d.) $\text{rref}\begin{bmatrix} -4 & 6 & 8 \\ 1 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\boxed{\text{is a basis}}$

PROBLEM 62 (§4.7 #4)/ Given $(-1, 2, 4, -3)$ solves $Ax = b$ and that $Ax = 0$ has sol$^{\underline{n}}$ parametrically given by $x_1 = -3r + 4s$, $x_2 = r - s$, $x_3 = r$, $x_4 = s$. Find vector form of $Ax = 0$ sol$^{\underline{n}}$, and vector form of general $Ax = b$ sol$^{\underline{n}}$

a.) $Ax = 0 \Rightarrow \boxed{x = r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$ b.) $Ax = b \Rightarrow \boxed{x = r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 4 \\ -3 \end{bmatrix}}$

Of course, you can simplify to $x = \begin{bmatrix} -1 - 3r + 4s \\ 2 + r - s \\ 4 + r \\ -3 + s \end{bmatrix}$ but I like my answers ☺

[PROBLEM 63 (§4.7 #6)] Find basis for null(A) given A.] The sol^{n}'s below show rref(A) = [ ] and we can read the sol^{n} to $Ax=0$ from this data and consequently we can write the sol^{n} as span of fundamental sol^{n}'s to $Ax=0$ which we'll identify as the basis for Null(A). I hope you can see the method clearly from what follows here:

(a.) $$\text{rref}\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 16x_3$
$x_2 = 19x_3$
x_3 free

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} \quad \therefore \boxed{\beta = \left\{ \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} \right\}}$$

(b.) $$\text{rref}\begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{2}x_3$
x_2, x_3 free

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \boxed{\beta = \{(0,1,0), (1/2, 0, 1)\}}$$

(c.) $$\text{rref}\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2/7 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} -x_3 + \frac{2}{7}x_4 \\ -x_3 - \frac{4}{7}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{x_4}{7}\begin{bmatrix} 2 \\ -4 \\ 0 \\ 7 \end{bmatrix}$$

$$\boxed{\beta = \{(-1,-1,1,0), (2,-4,0,7)\}}$$

Problem 63 continued,

(d.) $\text{rref}\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$X = \begin{bmatrix} -X_3 - 2X_4 - X_5 \\ -X_3 - X_4 - 2X_5 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = X_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + X_4\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + X_5\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\beta = \{(-1,-1,1,0,0),\ (-2,-1,0,1,0),\ (-1,-2,0,0,1)\}}$$

(e.) $\text{rref}\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2 & 4/3 \\ 0 & 1 & 0 & 0 & -1/6 \\ 0 & 0 & 1 & 0 & -5/12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$X = \begin{pmatrix} -2x_4 - \frac{4}{3}x_5 \\ \frac{1}{6}x_5 \\ \frac{5}{12}x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_4\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -4/3 \\ 1/6 \\ 5/12 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \beta = \{(-2,0,0,1,0),\ \underbrace{\left(-\tfrac{4}{3}, \tfrac{1}{6}, \tfrac{5}{12}, 0, 1\right)}_{(-16,\ 2,\ 5,\ 0,\ 12)}\}$$

$(-16, 2, 5, 0, 12)$
nice replacement
from factoring
out $\frac{1}{12}$.

$\boxed{\text{PROBLEM 64 (§4.7 #10)}}$ find basis for Row(A) from rows of A.

Strategy: use rref$[A^T]$ to see columns of A^T which form basis for Col(A^T). Note $(\text{Col}(A^T))^T = \text{Row}(A)$ and thus we find the desired rows of A,

(a.) $\text{rref}\begin{bmatrix} 1 & 5 & 7 \\ -1 & -4 & -6 \\ 3 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

pivot columns of A^T $\Rightarrow$ Row (A) = span$\{[1,-1,3],[5,-4,-4]\}$

$\boxed{\beta = \{[1,-1,3],[5,-4,-4]\}}$

(b.) $\text{rref}\begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \\ -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{\beta = \{[2,0,-1]\}}$

(c.) $\text{rref}\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{\beta = \{[1,4,5,2],[2,1,3,0]\}}$

(d.) $\text{rref}\begin{bmatrix} 1 & 3 & -1 & 2 \\ 4 & -2 & 0 & 3 \\ 5 & 1 & -1 & 5 \\ 6 & 4 & -2 & 7 \\ 9 & -1 & -1 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1/7 & 0 \\ 0 & 1 & -2/7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\beta = \{[1,4,5,6,9], [3,-2,1,4,-1], [2,3,5,7,8]\}$

(e.) $\text{rref}\begin{bmatrix} 1 & 0 & 2 & 3 & -2 \\ -3 & 3 & -3 & -6 & 9 \\ 2 & 6 & -2 & 0 & 2 \\ 2 & 0 & 4 & 6 & -4 \\ 1 & -3 & 4 & 5 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\boxed{\beta = \{[1,-3,2,2,1],[0,3,6,0,-3],[2,-3,-2,4,4]\}}$

PROBLEM 65 (§4.7 #6 8 10 continued) for the matrices studied in PROBLEM 63 and 64 find basis for Col(A) (call it β_{col})

Sol$^{\underline{n}}$: select the pivot columns from A in each case. to see these we consult the rref(A) given in 63.

(a.) $\beta_{col} = \left\{ \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -6 \end{bmatrix} \right\} = \boxed{\{ (1,5,7), (-1,-4,-6) \}}$.

(b.) $\beta_{col} = \boxed{\left\{ \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \right\}}$

(c.) $\beta_{col} = \boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\}}$

(d.) $\beta_{col} = \boxed{\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 3 \end{bmatrix} \right\}}$

(e.) $\beta_{col} = \boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -3 \\ -6 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -2 \\ 0 \\ 2 \end{bmatrix} \right\}}$

PROBLEM 66 (§4.7 #12) Find basis β for $\{V_1, V_2, V_3 .., V_n\}$ and write each non-basis V_j as a linear combination of β.

(a.)
$$V_1 = (1,0,1,1)$$
$$V_2 = (-3,3,7,1)$$
$$V_3 = (-1,3,9,3)$$
$$V_4 = (-5,3,5,-1)$$

$$\Rightarrow \text{rref}\begin{bmatrix} V_1 | V_2 | V_3 | V_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

use CCP.

$\boxed{\text{Let } \beta = \{V_1, V_2\} \text{ note } V_3 = 2V_1 + V_1 \text{ \& } V_4 = -2V_1 + V_2}$

Problem 66 continued: calculate rref $[V_1|V_2|\cdots|V_n]$ and use the CCP to answer this question efficiently,

(b.) $\text{rref}[V_1|V_2|V_3|V_4] = \text{rref}\begin{bmatrix} 1 & 2 & -1 & 0 \\ -2 & -4 & 1 & -1 \\ 0 & 0 & 2 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\boxed{\text{Let } \beta = \{V_1, V_3\} \text{ note by CCP } V_2 = 2V_1 \text{ and } V_4 = V_1 + V_3}$

(c.) $\text{rref}[V_1|V_2|V_3|V_4|V_5] = \text{rref}\begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ -1 & 3 & -5 & 4 & 18 \\ 5 & 1 & 9 & 2 & 2 \\ 2 & 0 & 4 & -3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Let $\beta = \{V_1, V_2, V_4\}$ note by CCP $V_3 = 2V_1 - V_2$ and ~~$V_5 = -V_1+3V_2+2V_4$~~
$V_5 = -V_1 + 3V_2 + 2V_4$

PROBLEM 67 (§4.7#16) Find a 3×3 matrix whose null space is a (a.) point (b.) line (c.) plane

(a.) $Ax = 0$ has unique sol^{n} $x = 0$ iff A^{-1} exists.
Any matrix with $\det(A) \neq 0$ will do. For example $\boxed{A = I.}$

(b.) Need $\dim(\text{Null}(A)) = 1 \Rightarrow \dim(\text{Col}(A)) = 2$
easy way take any 2 LI vectors and repeat one to make 3×3 A ; $\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$

(c.) Need $\dim(\text{Null}(A)) = 2 \Rightarrow \dim(\text{col}(A)) = 1$ so take any $v \neq 0$ and make $A = [v|v|v]$.
Or Row(A) is one dim'l make $A = \begin{bmatrix} v \\ v \\ v \end{bmatrix}$ or $A = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}$
may make many ways. $\boxed{A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$ $\Leftarrow$ Null(A) is (x,y,z) on plane $x+y+z=0$.

PROBLEM 68 (§4.8 #2) check the rank/nullity theorem by computing $\underbrace{\text{rank}(A)}_{rk(A)}$, $\underbrace{\text{nullity}(A)}_{\nu(A)}$ and verifying $\text{rank}(A) + \text{nullity}(A) = n$

$rk(A) = \dim(\text{Col}(A))$
$\nu(A) = \dim(\text{Null } A)$

(a.) $A = \begin{bmatrix} 5 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$ $\quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A^{-1}$ exists

$\therefore Ax = 0 \Rightarrow x = 0$

$\therefore \nu(A) = 0$.

Also rank$(A) = 3$ since $\{col_1(A), col_2(A), col_3(A)\}$ are LI by CCP. Observe that

$rk(A) + \nu(A) = 3 + 0 \neq 3$

(b.) $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ is same as (b.) of Problem 63. likewise for (c.), (d.), (e.) we refer reader to Problem 63 and 65

found $\nu(A)$ found $rk(A)$

(count vectors from 63 & 65) to find

$rk(A) = 1$

$\nu(A) = 2$

$rk(A) + \nu(A) = 1 + 2 \neq 3$

(c.) $\nu(A) = 2$, $rk(A) = 2 \Rightarrow \nu(A) + rk(A) = 2 + 2 \neq 4$

A is 3×4, $\underline{n = 4}$.

(d.) $\nu(A) = 3$, $rk(A) = 2 \Rightarrow \nu(A) + rk(A) = 3 + 2 \neq 5$

4×5 matrix, $\underline{n = 5}$

(e.) $\nu(A) = 2$, $rk(A) = 3 \Rightarrow \nu(A) + rk(A) = 2 + 3 \neq 5$

5×5 matrix, $n = 5$

PROBLEM 69
§ 4.8 #9

What conditions must be placed on b_1, b_2, b_3, b_4, b_5 to consistently solve

$$\begin{aligned} x_1 - 3x_2 &= b_1 \\ x_1 - 2x_2 &= b_2 \\ x_1 + x_2 &= b_3 \\ x_1 - 4x_2 &= b_4 \\ x_1 + 5x_2 &= b_5 \end{aligned} \longrightarrow Ax = b, \quad A = \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & 1 \\ 1 & -4 \\ 1 & 5 \end{bmatrix}$$

$$[A|b] = \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 1 & -2 & b_2 \\ 1 & 1 & b_3 \\ 1 & -4 & b_4 \\ 1 & 5 & b_5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 4 & b_3 - b_1 \\ 0 & -1 & b_4 - b_1 \\ 0 & 8 & b_5 - b_1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 - b_1 - 4(b_2 - b_1) \\ 0 & 0 & b_4 - b_1 + (b_2 - b_1) \\ 0 & 0 & b_5 - b_1 - 8(b_2 - b_1) \end{array}\right]$$

$$\text{rref}\,[A|b] = \left[\begin{array}{cc|c} 1 & 0 & b_1 + 3(b_2 - b_1) \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 - 4b_2 + 3b_1 \\ 0 & 0 & b_4 - 2b_1 + b_2 \\ 0 & 0 & b_5 - 8b_2 + 7b_1 \end{array}\right]$$

We require
$$\boxed{\begin{aligned} b_3 - 4b_2 + 3b_1 &= 0 \\ b_4 - 2b_1 + b_2 &= 0 \\ b_5 - 8b_2 + 7b_1 &= 0 \end{aligned}}$$

PROBLEM 70 (§ 4.8 #10) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$.
Show $rk(A) = 2$ iff ~~each~~ one or more possible sub 2×2 determinant is nonzero.

~~Assume $rk(A) = 2$. Suppose $\{v_1, v_2\}$ is basis for Col(A)~~
~~then det[~~
~~Suppose Col~~

over

Problem 70 continued

$\Rightarrow$) Suppose $rk(A) = 2$. It follows $\exists j,k$ such that $\{Col_j(A), Col_k(A)\}$ is LI spanning set for $Col(A)$. Furthermore, note $[Col_j(A) \,|\, Col_k(A)]$ is 2×2 matrix and by LI we have $\det\begin{pmatrix} A_{1j} & A_{1k} \\ A_{2j} & A_{2k}\end{pmatrix} \neq 0$. This completes the $\Rightarrow$) part of the proof.

$\Leftarrow$) Suppose one or more of

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}, \quad \det\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23}\end{bmatrix}, \quad \det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23}\end{bmatrix}$$

are nonzero. Then at least one pair of columns of A are LI, thus $rk(A) \geq 2$. Furthermore, if $A \in \mathbb{R}^{m\times n}$ $Col(A) \leq \mathbb{R}^m$ thus $Col\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{pmatrix} \leq \mathbb{R}^2$ consequently $rk(A) \leq 2$. Thus $rk(A) = 2$.

(You could say less, but certainly the connection between $\det[M] \neq 0$ and LI of columns of M should be mentioned in argument)

PROBLEM 71 Show set of points $(x,y,z) \in \mathbb{R}^3$ for which $M(x,y,z) = \begin{bmatrix} x & y & z \\ 1 & x & y\end{bmatrix}$ has rank 1 is a curve with parametric equations $x = t$, $y = t^2$, $z = t^3$

If $\det\begin{pmatrix} x & y \\ 1 & x\end{pmatrix} = 0$ and $\det\begin{pmatrix} x & z \\ 1 & y\end{pmatrix} = 0$ and $\det\begin{pmatrix} y & z \\ x & y\end{pmatrix} = 0$ then $rk(M) \neq 2$. Consider then $rk(M) \neq 0$ due to 1 in 1st column hence $rk(M) = 1$ when all three determinants are zero.

$$x^2 - y = 0, \quad xy - z = 0, \quad y^2 - xz = 0$$

$$y = x^2, \quad z = xy = x^3, \quad z = \frac{1}{x}y^2 = \frac{1}{x}(x^2) \measuredangle x^3$$

Let $x = t$ and it follows $y = t^2$, $z = t^3$ as claimed.