(This is the coversheet for the homework. The problems refer to Anton and Rorres 10th ed. of *Elementary Linear Algebra: applications version.* See Problem Sets 1, 2 or 3 for further formatting.)

Problem 72 § 4.10 #4 (standard matrix, relation of composition and multiplication)

Problem 73 § 4.10 #6 (rotations, reflections, compositions thereof in $\mathbb{R}^2$)

Problem 74 § 4.10 #10 (rotations, reflections, compositions thereof in $\mathbb{R}^3$)

Problem 75 § 4.10 #14 (injective? If so, find inverse)

Problem 76 § 4.11 #4 (please sketch the transformed unit-square for each, let this suffice for words)

Problem 77 § 4.6 #2 (coordinate vectors relative to nonstandard basis of $\mathbb{R}^3$)

Problem 78 § 4.6 #4 (coordinate vectors for polynomials)

Problem 79 § 4.6 #6 (transition matrix for bases of $\mathbb{R}^2$)

Problem 80 Let $T(f(x)) = x^2 f''(x)$ define a linear transformation on P_2. Find the matrix of T with respect to the usual basis $\{1, x, x^2\}$ for P_2.

Problem 81 Let $V = span\{\cos(x), \sin(x), e^x\}$ and suppose $S, T : V \to V$ are defined by $T(f) = df/dx$ and $S(f) = P(\int_0^x f(t)\, dt)$ for all $f \in V$. Find the matrices for S and T relative the natural basis $\beta = \{\cos(x), \sin(x), e^x\}$. Compute the product of the matrices $[S]_{\beta,\beta}$ and $[T]_{\beta,\beta}$ and comment on the significance of the result. *Hint: if you've taken calculus and you don't know this, I kill you.*

added P to kill constants. See Sol'n. $P(a\cos x + b\sin x + ce^x + d) = a\cos x + b\sin x + ce^x$ (d squashed)

Problem 82 Suppose V denotes three-dimensional physical space. Furthermore, suppose $T : V \to V$ is a rotation by angle θ. Consider this, we can **choose** coordinates on V which make the axis of rotation the z-axis. It then follows that the standard-matrix of T has the form

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose we choose any other basis for V, say β, and let $[T]_{\beta,\beta} = R$. Calculate $trace(R)$.

Problem 83 Suppose $R = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$ is a rotation matrix on $\mathbb{R}^3$. By what angle does R rotate? What is the axis of rotation? Find a vector in the plane which takes the axis as its normal and show that the vector rotates as claimed.

Note: the axis is not hard to find here if you think about how a rotation acts on the axis. There is a formula to find the axis in general for an arbitrary rotation matrix R. See #28 of § 4.9

Problem Set 6 Solution:

Problem 72 (§4.10 #4)

$$T_1(x_1, x_2, x_3) = \begin{bmatrix} 4x_1 \\ -2x_1 + x_2 \\ -x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

$$T_2(x_1, x_2, x_3) = \begin{bmatrix} x_1 + 2x_2 \\ -x_3 \\ 4x_1 - x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(a.) $[T_1] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}$ (standard matrices for T_1 & T_2.)

(b.) $$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 2 & 0 \\ 1 & 3 & 0 \\ 17 & 3 & 0 \end{bmatrix} = [T_2 \circ T_1]}$$

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & 8 & 0 \\ -2 & -4 & -1 \\ -1 & -2 & 3 \end{bmatrix} = [T_1 \circ T_2]}$$

(c.) $$(T_2 \circ T_1)(x_1, x_2, x_3) = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 3 & 0 \\ 17 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{\begin{bmatrix} 2x_2 \\ x_1 + 3x_2 \\ 17x_1 + 3x_2 \end{bmatrix}}.$$

$$(T_1 \circ T_2)(x_1, x_2, x_3) = \begin{bmatrix} 4 & 8 & 0 \\ -2 & -4 & -1 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{\begin{bmatrix} 4x_1 + 8x_2 \\ -2x_1 - 4x_2 - x_3 \\ -x_1 - 2x_2 + 3x_3 \end{bmatrix}}.$$

<u>Remark</u>: the definition of matrix multiplication is chosen precisely so that $[T_2 \circ T_1] = [T_2][T_1]$. I derive how this works explicitly in my lecture notes. See pages 162 – 163 or §5.5.1.

PROBLEM 73 (§4.10 # 6) Find standard matrix for $T: \mathbb{R}^2 \to \mathbb{R}^2$ where

(a.) T rotates by 60°, projects orthogonally on x-axis then reflects about line $y = x$,

60° rotation : $\begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix}$

projection onto x-axis : $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

reflects about line $y = x$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

} see §4.9 for complete list

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$ ← this happens 1st according to problem statement.

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 0 & 0 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}}$$

(b.) dilation by $k = 2$, rotation by 45°, reflection across y-axis.

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

↑ just factor out

$$= \boxed{\frac{1}{\sqrt{2}}\begin{bmatrix} -2 & 2 \\ 2 & 2 \end{bmatrix}}$$

(c.) $R(60°)R(105°)R(15°) = R(60° + 105° + 15°)$ ← rotations are nice!

$$= R(180°)$$

$$= \begin{pmatrix} \cos 180 & -\sin 180 \\ \sin 180 & \cos 180 \end{pmatrix}$$

$$= \boxed{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$$

PROBLEM 74 (§4.10 #10) Determine if $T_1 \circ T_2 = T_2 \circ T_1$ for T_1 & T_2 given below:

(a.) $T_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ dilates by k

$T_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ rotates about z-axis by θ

$$[T_1] = k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [T_1]A = A[T_1] \quad \text{for any } A \in \mathbb{R}^{3\times 3}$$

so certainly it commutes with $[T_2] = \left[\begin{array}{c|c} R(\theta) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 1 \end{array}\right]$.

It follows $\boxed{T_1 \circ T_2 = T_2 \circ T_1}$.

(b.) T_1 rotation about x-axis by θ_1

T_2 rotation about z-axis by θ_2

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow [T_1 \circ T_2] = \left[\begin{array}{c|c|c} \cos\theta_2 & -\sin\theta_2 & 0 \\ \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_1 \\ \sin\theta_1\sin\theta_2 & \sin\theta_1\cos\theta_2 & \cos\theta_1 \end{array}\right]$$

On the other hand,

$$[T_2 \circ T_1] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} = \left[\begin{array}{c|c|c} \cos\theta_2 & -\cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2\cos\theta_1 & -\sin\theta_1\cos\theta_2 \\ 0 & \cos\theta_1 & \cos\theta_1 \end{array}\right]$$

clearly $[T_1 \circ T_2] \neq [T_2 \circ T_1]$ and it follows $\boxed{T_1 \circ T_2 \neq T_2 \circ T_1}$.

PROBLEM 75 (§4.10 #14) For $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined below is T 1-1, if so, find $[T^{-1}]$ and find $T^{-1}(w_1, w_2, w_3)$

__Approach__: write $T(v) = [T]v$ then calculate $\text{rref}([T] \mid I)$ and if possible read off $[T]^{-1}$. Since $[T^{-1}] = [T]^{-1}$ we find all we need to answer the question. Alternatively, we can calculate $\det([T])$ to check 1-1.

If $\det([T]) = 0$ then T is not 1-1 since $[T]x = 0$ has more than $x = 0$ as a sol$^{\underline{n}}$.

continued ↷

PROBLEM 75 continued (§4.10 #14)

Comment: finding inverse function is algebra problem.

(a.)
$$W_1 = X_1 - 2X_2 + 2X_3$$
$$W_2 = 2X_1 + X_2 + X_3$$
$$W_3 = X_1 + X_2$$

If we can solve for X_1, X_2, X_3 →
$$X_1 = W_1 - 2W_2 + 4W_3$$
$$X_2 = -W_1 + 2W_2 - 3W_3$$
$$X_3 = -W_1 + 3W_2 - 5W_3$$
these give the inverse function.

Our matrix tools do the work for us

$$\text{rref}\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & -1 & 3 & -5 \end{array}\right] \Rightarrow \boxed{[T]^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ -1 & 2 & -3 \\ -1 & 3 & -5 \end{bmatrix}}$$

T^{-1} exists hence T is 1-1.

Hence $\boxed{T^{-1}(W_1, W_2, W_3) = (W_1 - 2W_2 + 4W_3,\ -W_1 + 2W_2 - 2W_3,\ -W_1 + 3W_2 - 5W_3)}$

(b.) $[T] = \begin{bmatrix} 1 & -3 & 4 \\ -1 & 1 & 1 \\ 0 & -2 & 5 \end{bmatrix}$: clearly $\det[T] = 0$ since the 3rd row is sum of top two rows. Hence T is not 1-1 and $\boxed{\text{consequently } T^{-1} \text{ d.n.e.}}$

(c.) $[T] = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix}$: no obvious dependence between rows/columns. I'll use rref$([T] \mid I)$ to check for sure and also to calculate $[T]^{-1}$.

$$\text{rref}\,[[T] \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & -3/2 & 11/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -3/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & -1/2 \end{array}\right]$$

∴ T is 1-1 and we find $[T]^{-1}$

$$\boxed{T^{-1}(W_1, W_2, W_3) = \tfrac{1}{2}(-3W_1 - 3W_2 + 11W_3,\ W_1 + W_2 - 3W_3,\ -W_1 + W_2 - W_3)}$$

(d.) $[T] = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 4 \\ 7 & 4 & -5 \end{bmatrix}$

$$\text{rref}\,[[T] \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -7/5 & 0 & -4/15 & 1/5 \\ 0 & 1 & 6/5 & 0 & 7/15 & 2/15 \\ 0 & 0 & 0 & 1 & -2/3 & -1/3 \end{array}\right]$$

(left block: not I)

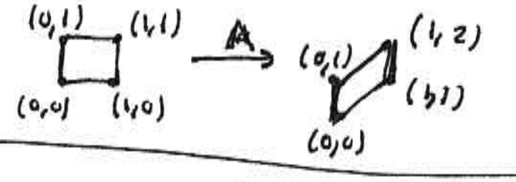
⇒ columns of $[T]$ not LI set

⇒ $[T]$ not 1-1

⇒ $\boxed{T^{-1} \text{ d.n.e.}}$

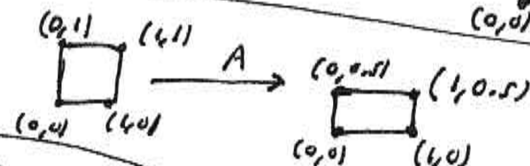
PROBLEM 76 (§4.11 #4) just sketch unit square transformed by A.

(a.) $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$, $A\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}=\begin{bmatrix}1&1&0\\1&2&1\end{bmatrix}$

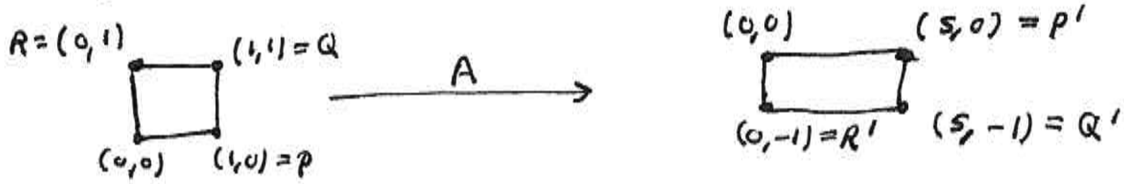


(b.) $A=\begin{bmatrix}1&0\\0&5\end{bmatrix}$, $A\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}=\begin{bmatrix}1&1&0\\0&5&5\end{bmatrix}$

(c.) $A=\begin{bmatrix}1&0\\0&1/2\end{bmatrix}$, $A\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}=\begin{bmatrix}1&1&0\\0&0.5&0.5\end{bmatrix}$



(d.) $A=\begin{bmatrix}5&0\\0&-1\end{bmatrix}$, $A\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}=\begin{bmatrix}5&5&0\\0&-1&-1\end{bmatrix}$



CONCEPT: I'm putting the vectors which point to corners (1,0), (1,1) and (0,1) into matrix $\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}$ and multiplying by $A\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}$ because I know the result is same as $\left[\underbrace{A\begin{bmatrix}1\\0\end{bmatrix}}_{P'} \middle| \underbrace{A\begin{bmatrix}1\\1\end{bmatrix}}_{Q'} \middle| \underbrace{A\begin{bmatrix}0\\1\end{bmatrix}}_{R'}\right]$

Moreover $A\vec{0}=\vec{0}$ so I don't need to calculate the origin, it's free.

Sorry my pictures are sloppy ... I hope yours were nicer ☺.

PROBLEM 77 (§4.6 #2) Find $[V]_\beta$.

(a.) $V = (2,-1,3)$, $\beta = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\2\\0\end{bmatrix}, \begin{bmatrix}3\\3\\3\end{bmatrix} \right\}$

$$[V]_\beta = [\beta]^{-1} V = \boxed{\begin{bmatrix}3\\-2\\1\end{bmatrix}}$$

← $\text{rref}[[\beta]|V] = [I|[V]_\beta]$ — how I calculated this time.

(b.) $V = (5,-12,3)$

$V_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $V_2 = \begin{pmatrix}-4\\5\\6\end{pmatrix}$, $V_3 = \begin{pmatrix}7\\-8\\9\end{pmatrix}$

$$[V_1|V_2|V_3]^{-1} = \begin{bmatrix}31/80 & 13/40 & -1/80\\ -7/40 & -1/20 & 11/120\\ -1/80 & -3/40 & 13/240\end{bmatrix} = [\beta]^{-1}$$

$$[V]_\beta = [\beta]^{-1}\begin{pmatrix}5\\-12\\3\end{pmatrix} = \boxed{\begin{bmatrix}-2\\0\\1\end{bmatrix}}$$

(oh, I actually used $\text{rref}[[\beta]|V] = [I|[V]_\beta]$ again, the matrix multiplication looked tedious.)

PROBLEM 78 (§4.6 #4)

$S' = \{A_1, A_2, A_3, A_4\} = \left\{ \begin{bmatrix}-1&1\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix} \right\}$

find $[A]_{S'}$ for $A = \begin{bmatrix}2&0\\-1&3\end{bmatrix}$

Solve $A = V_1A_1 + V_2A_2 + V_3A_3 + V_4A_4$ ✩ to obtain $[A]_{S'} = \begin{pmatrix}V_1\\V_2\\V_3\\V_4\end{pmatrix}$

$$\text{rref}\left[\begin{array}{cccc|c}-1&1&0&0&2\\1&1&0&0&0\\0&0&1&0&-1\\0&0&0&1&3\end{array}\right] = \left[\begin{array}{cccc|c}1&0&0&0&-1\\0&1&0&0&1\\0&0&1&0&-1\\0&0&0&1&3\end{array}\right]$$

← this comes from equating the 11, 12, 21 and 22 component of ✩ - eq$^{\underline{n}}$.

Observe $A = -A_1 + A_2 - A_3 + 3A_4$

$$\therefore \boxed{[A]_{S'} = \begin{bmatrix}-1\\1\\-1\\3\end{bmatrix}}$$

PROBLEM 79 (§4.6 # 6)

Consider $\beta = \{u_1, u_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\beta' = \{u_1', u_2'\} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$

find (a.) transition matrix from β to β'

(b.) from β' to β

(c.) $w = (3, -5)$ calculate $[w]_\beta$ and $[w]_{\beta'}$ by (10.)

(d.) compute $[w]_{\beta'}$ directly.

a.) $\text{rref}[\beta' | \beta] = \text{rref}\left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 4/11 & 3/11 \\ 0 & 1 & -1/11 & 2/11 \end{array}\right]$

$$P_{\beta \to \beta'} = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$$

b.) $P_{\beta' \to \beta} = (P_{\beta \to \beta'})^{-1} = \frac{11}{8+3}\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} = P_{\beta' \to \beta}}$

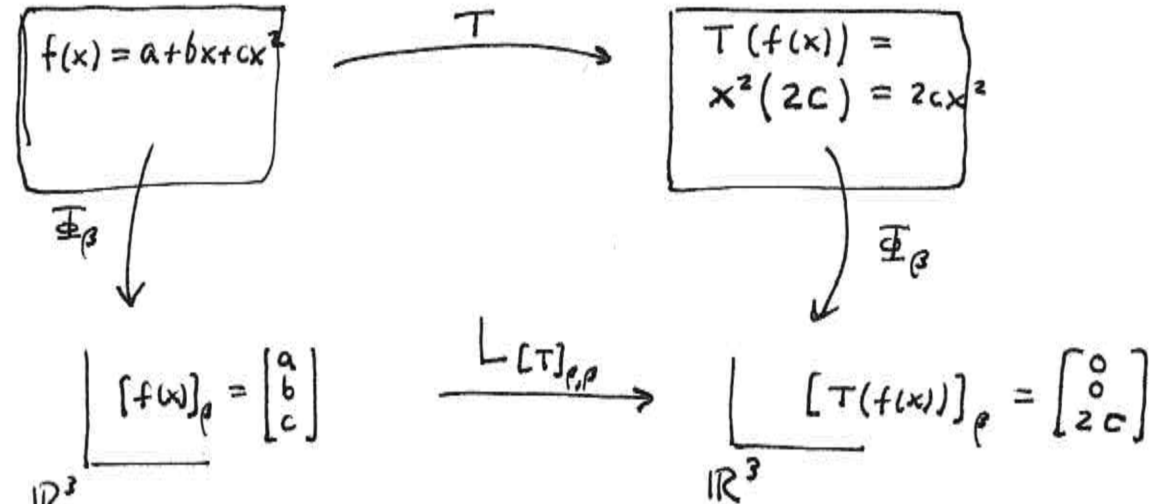
c.) $\text{rref}\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -5 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -5 \end{array}\right]$ just being silly obviously

$[w]_\beta = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ because $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is the standard basis!

$\underbrace{[w]_{\beta'} = P_{\beta \to \beta'} [w]_\beta}_{\text{actually (12) makes better sence here.}} = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \boxed{\frac{1}{11}\begin{bmatrix} -3 \\ -13 \end{bmatrix}}$

d.) $\text{rref}\left[\begin{array}{cc|c} 2 & -3 & 3 \\ 1 & 4 & -5 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -3/11 \\ 0 & 1 & -13/11 \end{array}\right] \Rightarrow \boxed{[w]_{\beta'} = \begin{bmatrix} -3/11 \\ -13/11 \end{bmatrix}}$ ✓

PROBLEM 80 Let $T(f(x)) = x^2 f''(x)$ define $T: P_2 \to P_2$
find $[T]_{\beta,\beta}$ where $\beta = \{1, x, x^2\}$



$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2c \end{bmatrix} \quad \text{hence} \quad [T]_{\beta,\beta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

PROBLEM 81 $V = \text{span}\{\cos x, \sin x, e^x\}$
and $T(f) = f'$ and $S(f) = P\left(\int_0^x f(t)\,dt\right)$ for all $f \in V$
find $[T]_{\beta,\beta}$ & $[S]_{\beta,\beta}$ where $\beta = \{\cos x, \sin x, e^x\}$

where $P(a\cos x + b\sin x + ce^x + d) = a\cos x + b\sin x + ce^x$
P kills constant term.

$$f(x) = a\cos x + b\sin x + ce^x$$
$$f'(x) = -a\sin x + b\cos x + ce^x$$

$T(f) = f' = -a\sin x + b\cos x + ce^x$. (I)

$$\int_0^x f(t)\,dt = \int_0^x (a\cos t + b\sin t + ce^t)\,dt = \left(a\sin t - b\cos t + ce^t\right)\Big|_0^x = a\sin x - b\cos x + ce^x + b - c$$

$P(f(x)) = -b\cos(x) + a\sin x + ce^x$ (II)

$$\text{(I)}: [T(f)]_\beta = \begin{bmatrix} b \\ -a \\ c \end{bmatrix} \Rightarrow [T]_{\beta\beta} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{(II)}: [S(f)]_\beta = \begin{bmatrix} -b \\ a \\ c \end{bmatrix} \Rightarrow [S]_{\beta,\beta} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus we calculate $[S]_{\beta\beta}[T]_{\beta,\beta} =$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

☺

PROBLEM 82

$$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \text{ has } [T] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If β is another basis for $\mathbb{R}^3$

find $\text{trace}([T]_{\beta,\beta}) = \text{trace}([\beta]^{-1}[T][\beta])$

$= \text{trace}([\beta][\beta]^{-1}[T])$ prop. of trace.

$= \text{trace}([T])$ since $[\beta][\beta]^{-1} = I$.

$= \cos\theta + \cos\theta + 1$

$= \boxed{2\cos\theta + 1}$

(Problem 83 has typo, moved to PROBLEM SET 7.)