MATH 221	PROBLEM SET 7 (DUE 10-23)

(This is the coversheet for the homework. The problems refer to Anton and Rorres 10th ed. of *Elementary Linear Algebra: applications version.* See Problem Sets 1, 2 or 3 for further formatting.)

Problem 84 § 5.1 #2 (check that it's an e-vector) ✓

Problem 85 § 5.1 #6a,c,f (find characteristic equations for 3×3 matrices) ✓

Problem 86 § 5.1 #8a,c,f (find bases for e-spaces of the matrices in previous problem.)

Problem 87 § 5.1 #12 (find e-values by inspection)

Problem 88 § 5.1 #14 (on relation of e-value/vectors of A and A^k)

Problem 89 § 5.1 #18 (characteristic equation for 2×2 fixed by trace and det)

Problem 90 § 5.1 #26 (on e-vector modification tricks)

PROBLEM SET 7 SOLUTION:

PROBLEM 84 (§5.1#2)

$$Ax = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\text{X is e-vector with } \lambda = 0 \text{ for A.}}$$

PROBLEM 85 (§5.1 # 6a, c, f) find characteristic eq$^{\underline{n}}$ for A.

(a) $$\det(A - \lambda I) = \det\begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\left((4-\lambda)(1-\lambda)+2\right)$$
$$= -(\lambda-1)\left((\lambda-4)(\lambda-1)+2\right)$$
$$= -(\lambda-1)(\lambda^2-5\lambda+6)$$
$$= \boxed{-(\lambda-1)(\lambda-3)(\lambda-2) = 0}$$

(c.) $$\det(A-\lambda I) = \det\begin{bmatrix} -2-\lambda & 0 & 1 \\ -6 & -2-\lambda & 0 \\ 19 & 5 & -4-\lambda \end{bmatrix}$$
$$= -(\lambda+2)\left[(\lambda+2)(\lambda+4)\right] + 1\left[-30+19(\lambda+2)\right]$$
$$= \boxed{-(\lambda+2)\ (\lambda+4) + 19\lambda + 8 = 0}$$ ← (can simplify further if you wish...)

(f.) $$\det(A-\lambda I) = \det\begin{bmatrix} 5-\lambda & 6 & 2 \\ 0 & -1-\lambda & -8 \\ 1 & 0 & -2-\lambda \end{bmatrix}$$
$$= \boxed{(5-\lambda)\left[(\lambda+1)(\lambda+2)\right] - 6(8) + 2(\lambda+1) = 0}$$

PROBLEM 86 (§5.1 # 8a, c, f) find bases for e-spaces of matrices in previous problem

(a.) $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

(b.) $\lambda_1 = -8$, $\lambda_2 = \pm i$ → $\lambda_2 = i$

$$\vec{u}_1 = \begin{bmatrix} -1 \\ -1 \\ 6 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 10-5i \\ -18+24i \\ 25 \end{bmatrix}$$

(c.) $\lambda_1 = -4$, $\lambda_2 = 3$, $\lambda_3 = 3$

$$\vec{u}_1 = \begin{bmatrix} -6 \\ 8 \\ 3 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix},$$

(not diagonalizable no $\vec{u}_3$ here)

(Used technology to calculate these e-values/vectors.)

PROBLEM 87 (§5.1 #12): find e-values by inspection:

(a.) $\begin{bmatrix} -1 & 6 \\ 0 & 5 \end{bmatrix}$ $\boxed{\lambda_1 = -1,\ \lambda_2 = 5}$

(b.) $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}$ $\boxed{\lambda_1 = 3,\ \lambda_2 = 7,\ \lambda_3 = 1}$

(c.) $\begin{bmatrix} -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$ $\boxed{\begin{array}{l} \lambda_1 = -\frac{1}{3} = \lambda_2 \\ \lambda_3 = 1,\ \lambda_4 = \frac{1}{2} \end{array}}$

PROBLEM 88 (§5.1 #14): Given $A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ find e-values and e-spaces for A^{25}

Notice $Av = \lambda v \Rightarrow A^{25}v = A^{24}Av = A^{24}\lambda v = \cdots = \lambda^{25}v$
thus find e-values for A and extrapolate. By technology,

$$\det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & -2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{pmatrix}$$

$$= -\lambda^3 + \lambda^2 + \lambda - 1$$

$$= (\lambda+1)(\lambda^2 - 2\lambda + 1)$$

$$= (\lambda+1)(\lambda-1)^2 \Rightarrow$$

(because $\lambda^{25} = (\pm 1)^{25} = \pm 1$.)

$$\boxed{\begin{array}{l} \lambda_1 = -1,\quad \lambda_2 = \lambda_3 = 1 \\ \vec{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\ \vec{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{array}}$$

PROBLEM 89 (§5.1 #18)
Show that $\det(\lambda I - A) = \lambda^2 - tr(A)\lambda + \det A = 0$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so $tr(A) = a+d$ and $\det A = ad - bc$.

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix}$$

$$= (\lambda - a)(\lambda - d) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

$$= \lambda^2 - tr(A)\lambda + \det(A) = 0.$$

PROBLEM 90 (§5.1 #26)

Find e-values and e-spaces' basis for $A = \begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix}$ and then use results of Exercises

[#23 (A^{-1} has $\frac{1}{\lambda}$ e-value if $Av = \lambda v$)

#24 (If A has $Av = \lambda v$ then $(A - sI)v = (\lambda - s)v$)]

to find bases for the e-spaces of

(a.) A^{-1}, (b.) $A - 3I$, (c.) $A + 2I$

$$\det(A - \lambda I) = \det\begin{bmatrix} -2-\lambda & 2 & 3 \\ -2 & 3-\lambda & 2 \\ -4 & 2 & 5-\lambda \end{bmatrix}$$

$$= -(\lambda+2)\left[(\lambda-3)(\lambda-5)-4\right] - 2\left(2(\lambda-5)+8\right) + 3\left(-4+4(3-\lambda)\right)$$

$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

$$= -(\lambda-1)(\lambda-2)(\lambda-3) \quad \therefore \quad \underline{\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3}.$$

$$(A - I)\vec{u}_1 = \begin{bmatrix} -3 & 2 & 3 \\ -2 & 2 & 2 \\ -4 & 2 & 4 \end{bmatrix}\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}.$$

$$(A - 2I)\vec{u}_2 = \begin{bmatrix} -4 & 2 & 3 \\ -2 & 1 & 2 \\ -4 & 2 & 3 \end{bmatrix}\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{\vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}.$$

$$(A - 3I)\vec{u}_3 = \begin{bmatrix} -5 & 2 & 3 \\ -2 & 0 & 2 \\ -4 & 2 & 2 \end{bmatrix}\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}.$$

Thus,

(a.) A^{-1} has $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$, $\lambda_3 = \frac{1}{3}$ with the same $\vec{u}_1, \vec{u}_2, \vec{u}_3$ e-vectors as A.

(b.) $A - 3I$ has $\lambda_1 = 1 - 3 = -2$, $\lambda_2 = -1$, $\lambda_3 = 0$ has e-vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$.

(c.) $A + 2I$ has $\lambda_1 = 3$, $\lambda_2 = 4$, $\lambda_3 = 5$ once more with $\vec{u}_1, \vec{u}_2, \vec{u}_3$.