MATH 221 | **PROBLEM SET 8 (DUE 10-30)**

(This is the coversheet for the homework. The problems refer to Anton and Rorres 10th ed. of *Elementary Linear Algebra: applications version.* See Problem Sets 1, 2 or 3 for further formatting.)

~~Problem 83 modified was moved to here.~~ Set $R = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$ and move to Problem Set 9. Sorry ☺.

Problem 91 § 5.2 #2 (show the matrices are not similar)

Problem 92 § 5.2 #12 (diagonalize it)

Problem 93 § 5.2 #16 (diagonalize it?)

Problem 94 § 5.2 #17 (diagonalize it?)

Problem 95 For the problem(s) above which was(were) not diagonalizable, find a Jordan-basis and perform a similarity transformation to reveal the Jordan form of the given matrix. See § 6.4 of my lecture notes for related discussion.

Problem 96 § 5.2 #24c (computation of powers via diagonalization)

Problem 97 Solve $d\vec{x}/dt = A\vec{x}$ for A given in problems 92, 93, 94. This should not require much, if any, computation. You need only understand § 6.5.1 or perhaps it's better to read Chapter 8 of my notes. For the impatient reader, § 8.4 has the examples you desire.

Problem 98 § 5.3 #18 (complex e-vectors)

Problem 99 § 5.3 #24 (similarity transformation to scale/rotation matrix C)

Problem 100 § 5.3 #29 (but, let's be smart about this, use blocks and start by studying products of the 2×2 Pauli matrices)

Problem 101 Solve $d\vec{x}/dt = A\vec{x}$ for A given in problems 98, 99. This should not require much, if any, computation. You need only understand § 8.5 my notes.

Problem 102 § 5.4 #3 (system of differential equations with initial conditions)

Problem 103 Suppose A, B are square matrices which are both diagonalizable. Furthermore, suppose you wish to diagonalize both of them with the same similarity transformation P. Let's say $P^{-1}AP = D_A$ and $P^{-1}BP = D_B$ where D_A and D_B are diagonal matrices. **What condition must hold for the commutator $[A, B] = AB - BA$ in order that the simultaneous diagonalization is possible?**

note: in quantum mechanics this math explains why we cannot simultaneous measure position $\mathbf{x}$ *and momentum* $\mathbf{p}$. *A simple calculation shows* $[\mathbf{x}, \mathbf{p}] = i\hbar$. *This is the source of Heisenberg's uncertainty principle. I mention some tidbits about quantum mechanics because some of you aspire to be electrical engineers and quantum mechanics is relevant to many circuits designed today... who knows what awaits you in your career.*

$\boxed{\text{PROBLEM 91 (§ 5.2 #2)}}$

Show $A = \begin{bmatrix} 4 & -1 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}$ are not similar.

well, $\det A = 16+2 = 18$ & $\det B = 16-2 = 14 \neq \det(A)$.

$\boxed{\text{PROBLEM 92 (§ 5.2 #12)}}$

Let $A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$. Diagonalize A by find P s.t $P^{-1}AP$ = diag.

$$\det(A-\lambda I) = \det\begin{bmatrix} -14-\lambda & 12 \\ -20 & 17-\lambda \end{bmatrix} = (\lambda+14)(\lambda-17)+12(20) = \lambda^2-3\lambda+2 = (\lambda-1)(\lambda-2)$$

$\lambda = 1$)

$$\begin{bmatrix} -15 & 12 \\ -20 & 16 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-15u + 12v = 0 \hookrightarrow v = \frac{15}{12}u \Rightarrow \underline{\vec{u}_1 = \begin{bmatrix} 12 \\ 15 \end{bmatrix}}.$

$\lambda = 2$)

$$\begin{bmatrix} -16 & 12 \\ -20 & 15 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-16u + 12v = 0 \hookrightarrow v = \frac{16}{12}u \Rightarrow \underline{\vec{u}_2 = \begin{bmatrix} 12 \\ 16 \end{bmatrix}}.$

Hence, use $P = \begin{bmatrix} 12 & 12 \\ 15 & 16 \end{bmatrix}$ to cause $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(other answers possible since choice of $\vec{u}_1$ & $\vec{u}_2$ not unique)

PROBLEM 93 (§ 5.2 #16)

$A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$ has $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 2$ with $\vec{u}_2 = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$

we find $\lambda_1 = \lambda_2 = 1$ hence 1 has algebraic multiplicity 2.
However, only $\vec{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$ is e-vector with e-value 1.
thus geometric multiplicity of 1 is 1 and we learn A is <u>not</u> diagonalizable.

PROBLEM 94 (§5.2 #17)

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \quad \text{has} \quad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{matrix} \quad \text{with } \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \ \vec{u}_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

It follows $\boxed{P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix} \text{ has } P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$

PROBLEM 95 (find Jordan form by $P^{-1}AP$ for A not diagonalizable, this turns out to concern A from PROBLEM 93 alone.)

$$A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix} \quad \text{has } \lambda_1 = 1 = \lambda_2 \text{ with } \vec{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

and $\lambda_3 = 2$ with $\vec{u}_3 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$.

Seek $\vec{u}_2$ such that

$$(A - I)\vec{u}_2 = \vec{u}_1 \quad \Rightarrow \quad \begin{bmatrix} 18 & -9 & -6 \\ 25 & -12 & -9 \\ 17 & -9 & -5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

$$\text{rref}\left(A - I \,\middle|\, \begin{smallmatrix} 3 \\ 4 \\ 3 \end{smallmatrix}\right) = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -4/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} u = w \\ v = \frac{4}{3}w - \frac{1}{3} \end{matrix}$$

Let $w = 0$ to select $\vec{u}_2 = (0, -1/3, 0)$.

Set $P = \begin{bmatrix} 3 & 0 & 3 \\ 4 & -1/3 & 3 \\ 3 & 0 & 4 \end{bmatrix}$ and you can

calculate that $\boxed{P^{-1}AP = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$

PROBLEM 96 §5.2 #24c

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{has } \lambda_1 = -1 = \lambda_2, \ \lambda_3 = 1 \Rightarrow P = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$\vec{u}_1 = (1,1,0)$, $\vec{u}_2 = (-4,0,1)$, $\vec{u}_3 = (1,0,0)$

note $P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$ (by technology)

we calculate from $P^{-1}AP = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}$

$$A^{2301} = P D^{2301} P^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$

PROBLEM 97 (Solve $\frac{d\vec{x}}{dt} = A\vec{x}$ for A from PROB. 92, 93, 94)

(92.) $\underline{\vec{x}(t) = c_1 e^t \begin{bmatrix} 12 \\ 15 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 12 \\ 16 \end{bmatrix}}$. (use e-values & e-vectors to write e-sol$^{\underline{n}}$ to $\vec{x}' = A\vec{x}$.)

(93) & (95) we obtain two e-vector type sol$^{\underline{n}}$'s from $\vec{u}_1$ & $\vec{u}_3$ however $\vec{u}_2$ has $(A - I)\vec{u}_2 = \vec{u}_1$ where $(A - I)\vec{u}_1 = 0$.

Note

$$e^{At}\vec{u}_2 = e^t\left(I + t(A-I) + \frac{t^2}{2}(A-I)^2 + \cdots\right)\vec{u}_2$$

$$= e^t\left(\vec{u}_2 + t(A-I)\vec{u}_2 + \frac{t^2}{2}(A-I)\vec{u}_1 + \cdots\right)$$

$$= e^t\left(\vec{u}_2 + t\vec{u}_1\right)$$

(see my notes! chapter 8 explains)

Hence,

$$\underline{\vec{x}(t) = c_1 e^t \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} + c_2 e^t \left(\begin{bmatrix} 0 \\ -1/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} \right) + c_3 e^{2t} \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}}.$$

(94) all e-vector sol$^{\underline{n}}$'s, easy:

$$\underline{\vec{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}}.$$

PROBLEM 98 (§5.3 #18): Let $A = \begin{bmatrix} 8 & 6 \\ -3 & 2 \end{bmatrix}$ find e-values & e-bases for A

$$\det(A - \lambda I) = \det\begin{bmatrix} 8-\lambda & 6 \\ -3 & 2-\lambda \end{bmatrix} = (\lambda - 8)(\lambda - 2) + 18 = \lambda^2 - 10\lambda + 34$$

$$\Rightarrow (\lambda - 5)^2 + 9 = 0 \Rightarrow \boxed{\lambda = 5 \pm 3i}$$

$\underline{\lambda = 5 + 3i}$

$$\begin{bmatrix} 8 - (5+3i) & 6 \\ -3 & 2 - (5+3i) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3-3i & 6 \\ -3 & -3-3i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, setting $(3 - 3i)u + 6v = 0 \therefore v = \frac{1}{6}(3i - 3)u$

let $u = 2$ hence $v = i - 1$ $\therefore$

$$\boxed{\vec{u}_1 = \begin{bmatrix} 2 \\ i-1 \end{bmatrix},\ \lambda = 5 + 3i \quad \Big/ \quad \vec{u}_2 = \begin{bmatrix} 2 \\ -i-1 \end{bmatrix},\ \lambda = 5 - 3i}$$

note 2nd e-vector obtained by conjugation!

PROBLEM 99 (§5.3 #24)

$A = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$ find P s.t. $A = PCP^{-1}$ for $C = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 4-\lambda & -5 \\ 1 & -\lambda \end{pmatrix} = \lambda(\lambda - 4) + 5$$
$$= \lambda^2 - 4\lambda + 5$$
$$= (\lambda - 2)^2 + 1 \Rightarrow \underline{\lambda = 2 \pm i}.$$

Consider then,

$$(A - (2+i)I)\vec{u} = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow u - (2+i)v = 0$$
$$u = (2+i)v$$

Let $v = 1$ thus $\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Set $P = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ note $P^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$

has
$$P^{-1}AP = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}}_{C} \Rightarrow A = PCP^{-1}. \quad \boxed{P = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}}$$

Notice $C = \sqrt{5}\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

dilation by $\sqrt{5}$

rotation by θ with $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{-1}{\sqrt{5}}$

PROBLEM 100 (§5.3 #29)

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Define DIRAC 4×4 matrices by

$$\beta = \begin{bmatrix} I_2 & \\ & -I_2 \end{bmatrix}, \quad \alpha_x = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}, \quad \alpha_y = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}, \quad \alpha_z = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix}$$

$$\left.\begin{aligned}
\sigma_1\sigma_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \\
\sigma_2\sigma_2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \\
\sigma_3\sigma_3 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = I_2 \\
\sigma_1\sigma_2 &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i\sigma_3 \\
\sigma_2\sigma_1 &= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = -i\sigma_3 \\
\sigma_1\sigma_3 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = i\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -i\sigma_2 \\
\sigma_3\sigma_1 &= i\sigma_2 \\
\sigma_2\sigma_3 &= i\sigma_1 \\
\sigma_3\sigma_2 &= -i\sigma_1
\end{aligned}\right\} \quad \boxed{\sigma_i\sigma_j = \delta_{ij} I_2 + i\,\epsilon_{ijk}\sigma_k}$$

(sum over k)

(a.) $\beta^2 = \alpha_x^2 = \alpha_y^2 = \alpha_z^2$ (show this)

$$\beta^2 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I_4$$

$$\alpha_x^2 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix} = \begin{bmatrix} I_2 & \\ & I_2 \end{bmatrix} = I_4$$

$$\alpha_y^2 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I_4$$

$$\alpha_z^2 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_3^2 & 0 \\ 0 & \sigma_3^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I_4$$

Thus $\beta^2 = \alpha_x^2 = \alpha_y^2 = \alpha_z^2 \neq I_4$.

PROBLEM 100 continued

§5.3 # 29b) Show that $\{A, B\} = AB + BA = 0 \quad \forall A, B \in \{\alpha_x, \alpha_y, \alpha_z, \beta\}$

Note: $\beta\alpha = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right]\left[\begin{array}{c|c} 0 & \sigma \\ \hline \sigma & 0 \end{array}\right] = \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix}$

thus $\beta\alpha_j = \begin{bmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{bmatrix}$ where $j = x, y, z$.

$\alpha_j\beta = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} 0 & -\sigma \\ \sigma & 0 \end{bmatrix}$ thus $\alpha_j\beta = \begin{bmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{bmatrix}$

Thus we find $\{\alpha_j, \beta\} = \alpha_j\beta + \beta\alpha_j = \begin{bmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{bmatrix} \neq 0.$

Furthermore, note, for $i, j \in \{x, y, z\}$,

$$\alpha_i\alpha_j = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix} = \begin{bmatrix} \sigma_i\sigma_j & 0 \\ 0 & \sigma_i\sigma_j \end{bmatrix}$$

$$\alpha_j\alpha_i = \begin{bmatrix} \sigma_j\sigma_i & 0 \\ 0 & \sigma_j\sigma_i \end{bmatrix}$$

$$\begin{aligned} \{\alpha_i, \alpha_j\} &= \alpha_i\alpha_j + \alpha_j\alpha_i \\ &= \left[\begin{array}{c|c} \sigma_i\sigma_j + \sigma_j\sigma_i & 0 \\ \hline 0 & \sigma_i\sigma_j + \sigma_j\sigma_i \end{array}\right] \end{aligned}$$

However, $\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i$

$$\begin{aligned} &= \delta_{ij} I + \sum_k i\epsilon_{ijk}\sigma_k + \delta_{ji} I + \sum_k i\epsilon_{jik}\sigma_k \\ &= 2\delta_{ij} I_2 + i\cancel{\sum_k \epsilon_{ijk}\sigma_k} - i\cancel{\sum_k \epsilon_{ijk}\sigma_k} \\ &= 2\delta_{ij} I_2 . \end{aligned}$$

Thus $\{\alpha_i, \alpha_j\} = \left[\begin{array}{c|c} 2\delta_{ij} I_2 & 0 \\ \hline 0 & 2\delta_{ij} I_2 \end{array}\right] = 2\delta_{ij} I_4 .$

In summary, the DIRAC MATRICES anticommute with each other, but not themselves ($\{\alpha_x, \alpha_x\} = 2I_4$)

$$\boxed{\begin{array}{ll} \{\beta, \alpha_x\} = \{\beta, \alpha_y\} = \{\beta, \alpha_z\} = 0 & \text{whereas } \{\alpha_i, \alpha_i\} = 2I_4 \\ \{\alpha_x, \alpha_y\} = \{\alpha_x, \alpha_z\} = \{\alpha_y, \alpha_z\} = 0 & \text{and } \{\beta, \beta\} = I_4 \end{array}}$$

PROBLEM 101 (Solve $\frac{d\vec{x}}{dt} = A\vec{x}$ for A from 98 & 99

(98) $\vec{x}(t) = e^{\lambda t}\vec{u}$ is complex sol^{n} since $\lambda = 5+3i$ and $\vec{u} = \begin{bmatrix} 2 \\ i-1 \end{bmatrix}$. This provides two real sol^{n}s. Why? See:

$$e^{(5+3i)t}\begin{bmatrix} 2 \\ i-1 \end{bmatrix} = e^{5t}e^{3it}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= e^{5t}(\cos 3t + i\sin 3t)\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= e^{5t}\left(\cos 3t\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \sin 3t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + i\left(e^{5t}\left[\sin 3t\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \cos 3t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]\right)$$

Hence, the real general sol^{n} is

$$\boxed{\vec{x}(t) = c_1 e^{5t}\begin{bmatrix} 2\cos 3t \\ -\cos 3t - \sin 3t \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 2\sin 3t \\ -\sin 3t + \cos 3t \end{bmatrix}}$$

(which can be written other ways for full-credit) and you could just write this down on the basis of my Chapter 8 Lecture Notes)

(99) Getting to the point directly, again the calculation to justify is same as (98)

$$\boxed{\vec{x}(t) = c_1 e^{2t}\left(\cos t\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + c_2 e^{2t}\left(\sin t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)}$$

PROBLEM 102 (§5.4 #3)

a.) Solve
$$y_1' = 4y_1 + y_3$$
$$y_2' = -2y_1 + y_2$$
$$y_3' = -2y_1 + y_3$$

b.) Solve with initial conditions $y_1(0) = -1$, $y_2(0) = 1$, $y_3(0) = 0$

(a.) $\vec{x}' = \underbrace{\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}_{A}\vec{x}$ has $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ with $\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$$\therefore \boxed{\vec{x}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

$$(b.)\ \vec{x}(0) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{rref}\left[\begin{array}{ccc|c} 0 & -1 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \therefore \boxed{\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - e^{2t}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + 2e^{3t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

which gives $\underline{y_1 = e^{2t} - 2e^{3t},\ y_2 = e^{t} - 2e^{2t} + 2e^{3t},\ y_3 = -2e^{2t} + 2e^{3t}}$.

PROBLEM 103 (see problem page, the question is when is it possible to simultaneously diagonalize A & B?

Suppose $P^{-1}AP = D_A$ and $P^{-1}AP = D_B$

where $D_A = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix}$ and $D_B = \begin{bmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{bmatrix}$

note $D_A D_B = \begin{bmatrix} a_1 b_1 & & & \\ & a_2 b_2 & & \\ & & \ddots & \\ & & & a_n b_n \end{bmatrix}$

$= \begin{bmatrix} b_1 a_1 & & & \\ & b_2 a_2 & & \\ & & \ddots & \\ & & & b_n a_n \end{bmatrix}$

$= D_B D_A$ $\therefore$ diagonal matrices commute.

Consider then,

$$\begin{aligned} P^{-1}[A,B]P &= P^{-1}(AB - BA)P \\ &= P^{-1}ABP - P^{-1}BAP \\ &= P^{-1}APP^{-1}BP - P^{-1}BPP^{-1}AP \\ &= D_A D_B - D_B D_A \\ &= 0 \implies \underline{[A,B] = 0}. \end{aligned}$$

We need A, B commute if we hope to simultaneously diagonalize A & B.

(this is a necessary but not sufficient condition since $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is commuting with $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ however A is not diagonalizable!)