(This is the coversheet for the homework. The problems refer to Anton and Rorres 10th ed. of *Elementary Linear Algebra: applications version.* See Problem Sets 1, 2 or 3 for further formatting.)

Problem 104 § 6.1 # 8 (inner product on P_2)

Problem 105 § 6.1 # 30 (inner product for functions on $[0, 1]$)

Problem 106 § 6.2 # 10 (find unit-normals to hyperplane in $\mathbb{R}^4$)

Problem 107 § 6.2 # 16 (find basis for orthogonal complement of span)

Problem 108 § 6.3 # 12 (finding coordinates w.r.t. orthonormal basis)

Problem 109 § 6.3 # 14 (projection of 4-vector onto plane in $\mathbb{R}^4$)

Problem 110 § 6.3 # 22 (gram-schmidt on 3 vectors in $\mathbb{R}^3$)

Problem 111 § 6.3 # 24 (gram-schmidt for basis of subspace... oh noes, it's a trap)

Problem 112 § 6.3 # 32 (Legendre polynomials in action)

Problem 113 § 6.3 # 32 (gram-schmidt on $\{1, x, x^2\}$ on $[0, 1]$)

Problem 114 § 6.4 # 4a and 6a (least squares solution and its error)

Problem 115 § 6.4 # 12 (finding orthogonal projection via Anton's equation (10) of page 372)

PROBLEM 104 (§6.1 #8) Calculate $\langle p, q \rangle$ for a.) $p = -2 + x + 3x^2$, $q = 4 - 7x^2$ and (b.) $p = -5 + 2x + x^2$, $q = 3 + 2x - 4x^2$ where $\langle p, q \rangle = \sum_{i=0}^{2} p_i q_i$

a.) $\langle -2 + x + 3x^2, 4 - 7x^2 \rangle = -8 + 0 - 21 = \boxed{-29}$

b.) $\langle -5 + 2x + x^2, 3 + 2x - 4x^2 \rangle = -15 + 4 - 4 = \boxed{-15}$

PROBLEM 105 (§6.1 #30) Let $\langle f, g \rangle = \int_0^1 f(x) g(x) dx$. Calculate $\langle f, g \rangle$ for (a.) $f = \cos 2\pi x$, $g = \sin 2\pi x$ (b.) $f = x$, $g = e^x$ (c.) $f = \tan \frac{\pi x}{4}$, $g = 1$

(a.) $\langle f, g \rangle = \int_0^1 \cos 2\pi x \sin 2\pi x \, dx = \frac{1}{2} \int_0^1 \sin(4\pi x) dx = \frac{-1}{8\pi} \cos \pi x \Big|_0^1 = \boxed{0.}$

(b.) $\langle f, g \rangle = \int_0^1 x e^x dx = (xe^x - e^x) \Big|_0^1 = (e - e) - (0 \cdot 1 - 1) = \boxed{1.}$

(c.) $\langle f, g \rangle = \int_0^1 \tan \frac{\pi x}{4} dx = \int_0^1 \frac{\sin \pi x/4 \, dx}{\cos \frac{\pi x}{4}} = \int_1^{1/\sqrt{2}} \frac{-4}{\pi} \frac{du}{u} = \frac{-4}{\pi} \ln|u| \Big|_1^{1/\sqrt{2}}$

$u = \cos \frac{\pi x}{4}$
$du = -\frac{\pi}{4} \sin \frac{\pi x}{4} dx$

$= \frac{-4}{\pi} \ln(1/\sqrt{2})$

$= \frac{4}{\pi} \ln(\sqrt{2}) \qquad (2^{\frac{1}{2}})^4 = 4$

$= \boxed{\frac{1}{\pi} \ln(4)}$

PROBLEM 106 (§6.2 #10) Find two unit vectors which are $\perp$ to $u = (2, 1, -4, 0)$, $v = (-1, -1, 2, 2)$ and $w = (3, 2, 5, 4)$

We seek $x = (x_1, x_2, x_3, x_4)$ such that $x \cdot x = 1$ (unit-length) and

$u \cdot x = 0$, $v \cdot x = 0$, $w \cdot x = 0$

$u^T x = 0$, $v^T x = 0$, $w^T x = 0$

We solve these simultaneously, I'll use technology to row reduce,

$$\text{rref} \begin{bmatrix} 2 & 1 & -4 & 0 \\ -1 & -1 & 2 & 2 \\ 3 & 2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 34/11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 6/11 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= \tfrac{-34}{11} x_4 \\ x_2 &= 4x_4 \\ x_3 &= \tfrac{-6}{11} x_4 \end{aligned} \Rightarrow x = k \begin{bmatrix} -34 \\ 44 \\ -6 \\ 11 \end{bmatrix}, k \neq 0$$

gives $\perp$ vector.

Now, choose k to normalize x: $\|(-34, 44, -6, 11)\| = 57$

thus $\boxed{x = \frac{\pm 1}{57}(-34, 44, -6, 11)}$

$\boxed{\text{PROBLEM 107 (§6.2#16) Find basis for orthogonal complement of subspace of } \mathbb{R}^n \text{ spanned by the vectors given below.}}$ In each case to find orthogonal vectors to $\{V_1, V_2, .., V_k\}$ we calculate $\text{rref}\begin{bmatrix} V_1^T \\ \vdots \\ V_n^T \end{bmatrix}$ to ascertain sol^{n}s to $V_1 \cdot X = 0$, $V_2 \cdot X = 0$, .., $V_n \cdot X = 0$. (see my Example 7.3.6 for another Ex)

(a.) $\text{rref}\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & 4 \\ 7 & -6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} X_1 = 16X_3 \\ X_2 = 19X_3 \\ X_3 = X_3 \end{matrix} \therefore \boxed{\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}}$ forms basis for $\text{Null}\begin{bmatrix} V_1^T \\ V_2^T \\ V_3^T \end{bmatrix}$

and hence is orthogonal to $\{V_1, V_2, V_3\}$.

(b) $\text{rref}\begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$

Hence $X_1 = \frac{1}{2}X_3$, X_2 is free. Thus $x = \begin{bmatrix} \frac{1}{2}X_3 \\ X_2 \\ X_3 \end{bmatrix} = X_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}X_3\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Thus $\boxed{\{(0,1,0), (1,0,2)\}}$ forms basis to $\{V_1, V_2\}^\perp$

(c.) $\text{rref}\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2/7 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} -X_3 + \frac{2}{7}X_4 \\ -X_3 - \frac{4}{7}X_4 \\ X_3 \\ X_4 \end{bmatrix} = X_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{X_4}{7}\begin{bmatrix} 2 \\ -4 \\ 0 \\ 7 \end{bmatrix}$

Thus $\{V_1, V_2, V_3\}^\perp$ has basis $\boxed{\{(-1,-1,1,0), (2,-4,0,7)\}}$

(d.) $\text{rref}\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} -X_3 - 2X_4 - X_5 \\ -X_3 - X_4 - 2X_5 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}$

Thus $\boxed{\{(-1,-1,1,0,0), (-2,-1,0,1,0), (-1,-2,0,0,1)\}}$

forms basis to $\{V_1, V_2, V_3, V_4\}^\perp$.

PROBLEM 108 (§6.3 #12) Find coordinates given an orthonormal basis.

(a.) $u_1 = \frac{1}{\sqrt{2}}\langle 1,-1\rangle$, $u_2 = \frac{1}{\sqrt{2}}\langle 1,1\rangle$, $w = \langle 3,7\rangle$. $\beta = \{u_1, u_2\}$

$$[w]_\beta = (w\cdot u_1,\ w\cdot u_2) = \boxed{\left(-4/\sqrt{2},\ 10/\sqrt{2}\right)}.$$

(b.) $u_1 = \frac{1}{3}(2,-2,1)$, $u_2 = \frac{1}{3}(2,1,-2)$, $u_3 = \frac{1}{3}(1,2,2)$
$w = (2,0,5)$ and $\beta = \{u_1,u_2,u_3\}$ find $[w]_\beta$,

$$[w]_\beta = (w\cdot u_1,\ w\cdot u_2,\ w\cdot u_3) = \left(\tfrac{9}{3}, \tfrac{-6}{3}, \tfrac{12}{3}\right) = \boxed{(3,-2,4)}$$

PROBLEM 109 (§6.3 #14) a.) $W = \text{span}\{(1,1,1,1),\ (1,1,-1,-1)\}$ find $\text{Proj}_W(x)$
b.) $W = \text{span}\{(0,1,-4,-1),\ (3,5,1,1)\}$ find $\text{Proj}_W(x)$
where $x = (1,2,0,-2)$

$$\text{(a.)}\ \text{Proj}_W(x) = \left[\frac{(1,1,1,1)\cdot(1,2,0,-2)}{(1,1,1,1)\cdot(1,1,1,1)}\right](1,1,1,1) + \left[\frac{(1,1,-1,-1)\cdot(1,2,0,-2)}{(1,1,-1,-1)\cdot(1,1,-1,-1)}\right](1,1,-1,-1)$$
$$= \tfrac{1}{4}(1,1,1,1) + \tfrac{5}{4}(1,1,-1,-1)$$
$$= \boxed{\left(3/2,\ 3/2,\ -1,\ -1\right)}$$

$$\text{(b.)}\ \text{Proj}_W(x) = \left[\frac{(0,1,-4,-1)\cdot(1,2,0,-2)}{\|(0,1,-4,-1)\|^2}\right](0,1,-4,-1) + \left[\frac{(3,5,1,1)\cdot(1,2,0,-2)}{\|(3,5,1,1)\|^2}\right](3,5,1,1)$$
$$= \tfrac{4}{18}(0,1,-4,-1) + \tfrac{11}{36}(3,5,1,1)$$
$$= \boxed{\left(11/12,\ 7/4,\ -7/12,\ 1/12\right)}$$

<u>Remark</u>: The calculations above work because the given spanning sets for W are <u>orthogonal</u>. If they were not orthogonal then we'd need to apply Gram Schmidt or some other technique before calculating the projections.

$\boxed{\text{PROBLEM 110 } (\S 6.3 \# 22)}$ Apply Gram-Schmidt to orthonormalize the sets:

(a.) $u_1 = (1,1,1)$, $u_2 = (-1,1,0)$, $u_3 = (1,2,1)$

(b.) $u_1 = (1,0,0)$, $u_2 = (3,7,-2)$, $u_3 = (0,4,1)$

(a.) $V_1 = \frac{1}{\sqrt{3}}(1,1,1)$

$V_2 = \frac{1}{\sqrt{2}}(-1,1,0)$ (note: $V_1 \perp V_2$ clear, just needed to normalize)

$$\begin{aligned}
\overline{V_3} &= u_3 - (u_3 \cdot V_1)V_1 - (u_3 \cdot V_2)V_2 \\
&= (1,2,1) - \tfrac{1}{3}\left[(1,2,1)\cdot(1,1,1)\right](1,1,1) - \tfrac{1}{2}\left[(1,2,1)\cdot(-1,1,0)\right](-1,1,0) \\
&= (1,2,1) - \tfrac{4}{3}(1,1,1) - \tfrac{1}{2}(-1,1,0) \\
&= (1/6, 1/6, -1/3) \\
&= \tfrac{1}{6}(1,1,-2) \quad \Rightarrow \quad V_3 = \tfrac{1}{\sqrt{6}}(1,1,-2)
\end{aligned}$$

Hence $\boxed{\left\{ \frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(-1,1,0), \frac{1}{\sqrt{6}}(1,1,-2) \right\}}$

(b.) $V_1 = (1,0,0)$

$$\begin{aligned}
\overline{V_2} &= u_2 - (u_2 \cdot V_1)V_1 \\
&= (3,7,-2) - 3(1,0,0) \\
&= (0,7,-2) \quad \Rightarrow \quad V_2 = \tfrac{1}{\sqrt{53}}(0,7,-2)
\end{aligned}$$

$$\begin{aligned}
\overline{V_3} &= (0,4,1) - \left[\frac{(0,4,1)\cdot(1,0,0)}{1}\right](1,0,0) - \left[\frac{(0,4,1)\cdot(0,7,-2)}{53}\right](0,7,-2) \\
&= (0,4,1) - \tfrac{26}{53}(0,7,-2) = \left(0, \tfrac{30}{53}, \tfrac{105}{53}\right) = \tfrac{15}{53}(0,2,7) \\
&\Rightarrow V_3 = \tfrac{1}{\sqrt{53}}(0,2,7)
\end{aligned}$$

Hence, $\boxed{\left\{ (1,0,0), \frac{1}{\sqrt{53}}(0,7,-2), \frac{1}{\sqrt{53}}(0,2,7) \right\}}$

PROBLEM III (§6.3#24) Find orthonormal basis for subspace spanned by $(0,1,2)$, $(-1,0,1)$, $(-1,1,3)$

I'll use a two-step procedure this time.

1.) set $V_1 = (0,1,2)$

2.) set $V_2 = (-1,0,1) - \left[\dfrac{(-1,0,1)\cdot(0,1,2)}{(0,1,2)\cdot(0,1,2)}\right](0,1,2)$

$$= (-1,0,1) - \frac{2}{5}(0,1,2)$$

$$= (-1, -2/5, 1/5) = \frac{1}{5}(-5,-2,1)$$

3.) set $V_3 = (-1,1,3) - \left[\dfrac{(-1,1,3)\cdot(0,1,2)}{5}\right](0,1,2) - \left[\dfrac{(-1,1,3)\cdot(-1,\frac{-2}{5},\frac{1}{5})}{1+\frac{4}{25}+\frac{1}{25}}\right](-1,\frac{-2}{5},\frac{1}{5})$

$$= (-1,1,3) - \frac{7}{5}(0,1,2) - \frac{5}{6}\left[1 + \left(\frac{-2}{5}\right) + \frac{3}{5}\right](-1,\frac{-2}{5},\frac{1}{5})$$

$$= (-1,1,3) - (0, 7/5, 14/5) - (-1, -2/5, 1/5)$$

$$= (-1+1, 1 - 7/5 + 2/5, 3 - 14/5 - 1/5)$$

$$= (0, 1-1, 3-3)$$

$$= (0,0,0).$$

ah ha! this makes sense. Why? Because $V_3 \in \text{span}\{V_1, V_2\}$.

Observe $V_1 \cdot V_2 = 0$, $V_1 \cdot V_3 = 0$

Normalizing we find that

$$\boxed{\left\{ \frac{1}{\sqrt{5}}(0,1,2), \frac{1}{\sqrt{30}}(-5,-2,1) \right\} = \beta}$$

By the way, note $(-1,1,3)\cdot \frac{1}{\sqrt{5}}(0,1,2) = \frac{7}{\sqrt{5}}$

and $(-1,1,3)\cdot\frac{1}{\sqrt{30}}(-5,-2,1) = \frac{6}{\sqrt{30}}$

We find

$$\left(\frac{7}{\sqrt{5}}\right)\frac{1}{\sqrt{5}}(0,1,2) + \left(\frac{6}{\sqrt{30}}\right)\frac{1}{\sqrt{30}}(-5,-2,1) = (0,\tfrac{7}{5},\tfrac{14}{5}) + (-1,\tfrac{-2}{5},\tfrac{1}{5}) = (-1,1,3).$$

This shows $(-1,1,3) \in \text{span}\,\beta$.

PROBLEM 112 (§6.3 #32) Express the following as linear combinations of the 1^{st} three Legendre Polynomials

(a.) $1 + x + 4x^2$

(b.) $2 - 7x^2$

(c.) $4 + 3x$

$$\underbrace{V_1 = 1, \quad V_2 = x, \quad V_3 = x^2 - \tfrac{1}{3}}$$

these are orthogonal, but, not orthonormal with respect to $\langle p, q\rangle = \int_{-1}^{1} p(x)\, q(x)\,dx$

Th^{m} (6.3.4) $\mathrm{Proj}_{P_2}(f) = \left(\dfrac{\langle 1, f\rangle}{\|1\|^2}\right) 1 + \left(\dfrac{\langle x, f\rangle}{\|x\|^2}\right) x + \left(\dfrac{\langle x^2 - \frac{1}{3}, f\rangle}{\|x^2 - \frac{1}{3}\|^2}\right)(x^2 - \tfrac{1}{3})$

applied to $W = P_2$ with $\{V_1, V_2, V_3\}$. Now apply this to solve a, b, c. But, first calculate the quantities:

$\langle 1, 1\rangle = \int_{-1}^{1} dx = 2 \qquad \langle 1, x\rangle = \int_{-1}^{1} x\,dx = 0$

$\langle x, x\rangle = \int_{-1}^{1} x^2 dx = \frac{2}{3} \qquad \langle 1, x^2\rangle = \langle x, x\rangle = \frac{2}{3}$

$\langle x^2, x^2\rangle = \int_{-1}^{1} x^4 dx = \frac{2}{5} \qquad \langle x^2, x\rangle = 0 \qquad \langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle = \frac{2}{5} - \frac{4}{9} + \dots$

The basic integrals above paired with $\langle , \rangle$ properties allow: $\frac{8}{45}$

(a.) $\mathrm{Proj}_{P_2}(f) = \frac{1}{2}\langle 1, 1 + x + 4x^2\rangle + \frac{3}{2}\langle x, 1 + x + 4x^2\rangle x + \frac{45}{8}\langle x^2 - \frac{1}{3}, 1 + x + 4x^2\rangle \wedge$ $(x^2 - \frac{1}{3}$

$= \frac{1}{2}(2 + 0 + \frac{8}{3}) + \frac{3}{2}(0 + \frac{2}{3})x + \frac{45}{8}\left(\frac{2}{3} + \frac{8}{5} - \frac{2}{3} - \frac{8}{9}\right)(x^2 - \frac{1}{3})$

$= \frac{1}{2}\left(\frac{14}{3}\right) + x + \frac{45}{8}\left(\left(\frac{10 + 24 - 10}{15}\right)\cdot\frac{3}{3} - \frac{40}{45}\right)(x^2 - \frac{1}{3})$

$= \frac{7}{3} + x + \frac{45}{8}\cdot\frac{32}{45}(x^2 - \frac{1}{3})$

$= \boxed{7/3 + x + 4(x^2 - \tfrac{1}{3})}$

(b.) Alternatively, calculate the $\int$'s directly,

$\langle 1, 2 - 7x^2\rangle = \int_{-1}^{1}(2 - 7x^2)dx = 4 - \frac{7x^3}{3}\Big|_{-1}^{1} = 4 - \frac{14}{3} = \frac{-2}{3}.$

$\langle x, 2 - 7x^2\rangle = \int_{-1}^{1}(2x - 7x^3)dx = 0.$

$\langle x^2 - \frac{1}{3}, 2 - 7x^2\rangle = \int_{-1}^{1}(x^2 - \frac{1}{3})(2 - 7x^2)dx$

$= \int_{-1}^{1}\left(2x^2 + \frac{7}{3}x^2 - \frac{2}{3} - 7x^4\right)dx$

$= \int_{-1}^{1}\left(\frac{13}{3}x^2 - \frac{2}{3} - 7x^4\right)dx$

$= \frac{26}{9} - \frac{4}{3} - \frac{14}{5} = \frac{-56}{45}$

Adjust for $\langle 1, 1\rangle = 2$ and $\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle = 8/45$ notice $\frac{-56}{45}\cdot\frac{45}{8} = \frac{-56}{8} = -7$,

$\boxed{\mathrm{Proj}_{P_2}(2 - 7x^2) = \frac{-1}{3} - 7(x^2 - \tfrac{1}{3})}$ (make sense?)

Problem 112 continued: I notice an easier way. (just a trick for this one)

(c.) $\boxed{4+3x}$ well, ok. not so interesting.

(b.) $2-7x^2 = 2-7\left(x^2-\frac{1}{3}+\frac{1}{3}\right)$ (added zero)

$= 2-7\left(x^2-\frac{1}{3}\right)-\frac{7}{3}$

$= \underbrace{\frac{-1}{3}-7\left(x^2-\frac{1}{3}\right)}$

see how $2-7x^2$ is linear combination of $\{1, x, x^2-\frac{1}{3}\}$.

PROBLEM 113 (§6.3#32) Gram Schmidt on $\{1,x,x^2\}$ for $[0,1]$ with $\langle f,g\rangle = \int_0^1 fg\,dx$.

$\langle 1,1\rangle = \int_0^1 1\,dx = 1$ hence $\|1\| = 1$.

$\langle x,x\rangle = \int_0^1 x^2dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$ hence $\|x\| = \sqrt{\frac{1}{3}}$, but $\langle 1,x\rangle \neq 0$ so this is beside the point anyways.

$\langle x,1\rangle = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$.

Apply Gram-Schmidt

$\text{Orth}_1(x) = x - \left(\frac{\langle 1,x\rangle}{\langle 1,1\rangle}\right)1 = x - \left(\frac{1/2}{1}\right)1 = \underline{x-\frac{1}{2}}$.

observe $\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle = \int_0^1 \left(x-\frac{1}{2}\right)^2dx = \int_{-1/2}^{1/2} u^2du = \frac{u^3}{3}\Big|_{-1/2}^{1/2} = \frac{1/8}{3} - \frac{-1/8}{3}$

thus $\|x-\frac{1}{2}\|^2 = \frac{1}{12}$ $\therefore$ $\sqrt{12}\left(x-\frac{1}{2}\right)$ is normalized.

Finally, note $\langle \sqrt{12}\left(x-\frac{1}{2}\right), x^2\rangle = \sqrt{12}\int_0^1 x^3dx - \frac{\sqrt{12}}{2}\int x^2dx$

$= \sqrt{12}/4 - \sqrt{12}/6$

$= \sqrt{12}\left(\frac{3-2}{12}\right)$

$= \frac{1}{\sqrt{12}}$

Then,

$V_3 = x^2 - \langle x^2,1\rangle 1 - \cancel{\langle x^2, \sqrt{12}\left(x-\frac{1}{2}\right)\rangle \sqrt{12}}\left(x-\frac{1}{2}\right)$

$= x^2 - \frac{1}{3} - \left(x-\frac{1}{2}\right)$

$= x^2 - x + \frac{1}{6}$ normalized? $\|x^2-x+\frac{1}{6}\|^2 = \int_0^1\left(x^2-x+\frac{1}{6}\right)^2dx = \frac{1}{180}$

Thus $\boxed{\left\{1, \sqrt{12}\left(x-\frac{1}{2}\right), \sqrt{180}\left(x^2-x+\frac{1}{6}\right)\right\}}$

PROBLEM 114 (§6.4 #4a & 6a) consider $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{bmatrix}$ & $b = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$
find the least squares "sol$^{\underline{n}}$" to $Av = b$ and
find the error $e = b - Av$ for the "sol$^{\underline{n}}$" and show $e \in \mathrm{Col}(A)^{\perp}$.

Set-up the normal eq$^{\underline{n}}$'s by multiplying by A^T:

$$A^TAv = A^Tb \Rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 2 & -4 & 10 \\ -1 & 3 & -7 \end{bmatrix}\begin{bmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{bmatrix} v = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -4 & 10 \\ -1 & 3 & -7 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 12 & -7 \\ 12 & 120 & -84 \\ -7 & -84 & 59 \end{bmatrix} v = \begin{bmatrix} 5 \\ 22 \\ -15 \end{bmatrix}$$

$$\mathrm{rref}\left[A^TA \,|\, A^Tb\right] = \mathrm{rref}\left[\begin{array}{ccc|c} 11 & 12 & -7 & 5 \\ 12 & 120 & -84 & 22 \\ -7 & -84 & 59 & -15 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1/7 & 2/7 \\ 0 & 1 & -5/7 & 13/84 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In this case there is a whole family of least square sol$^{\underline{n}}$'s. For each $t \in \mathbb{R}$, $v_1 = \frac{2}{7} - \frac{t}{7}$, $v_2 = \frac{13}{84} + \frac{5}{7}t$

$$\boxed{v = \left(\frac{2}{7} - \frac{t}{7},\ \frac{13}{84} + \frac{5t}{7},\ t\right)}$$

For example, $t = 0$ gives $v = (2/7, 13/84, 0)$.

We find $e = b - Av = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{bmatrix}\begin{bmatrix} \frac{2}{7} - \frac{t}{7} \\ \frac{13}{84} + \frac{5t}{7} \\ t \end{bmatrix} = \begin{bmatrix} 5/6 \\ -5/3 \\ -5/6 \end{bmatrix}$

Even though there are many least-squares sol$^{\underline{n}}$'s they all have same error.
Recall $\mathrm{Col}(A)^{\perp} = \mathrm{Null}\, A^T$ so we check $A^Te = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -4 & 10 \\ -1 & 3 & -7 \end{bmatrix}\begin{bmatrix} 5/6 \\ -5/3 \\ -5/6 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Problem 115 (§6.4#12) Use formula (10) to find orthogonal projection $P : \mathbb{R}^2 \to \mathbb{R}^2$ onto
(a.) the x-axis (b.) the y-axis
Formula (10) says take any basis for subspace W and calculate $[P] = A(A^TA)^{-1}A^T$ as the matrix for $P = Proj_W$

(a.) $W = \text{span}\{e_1\}$, $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A^T = [1, 0]$

$A^TA = [1,0]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$ thus $(A^TA)^{-1} = \frac{1}{1} = 1.$

hence $[P] = A(1)A^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}[1, 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$

$$Proj_{x\text{-axis}}(a,b) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} = (a, 0)$$ YEP. makes sense.

(b.) $W = \text{span}\{e_2\}$

$$[P] = e_2(e_2^Te_2)^{-1}e_2^T = e_2e_2^T = E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

hence

$$Proj_{y\text{-axis}}(a,b) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} = (0, b).$$

Remark: the power of formula (10) is better revealed for something like $W = \text{span}\{(1,1,0), (1,1,1)\}$ it allows us to formulate $Proj_W(V)$ w/o performing Gram-Schmidt on the basis for W. When $\dim(W) \geq 3$ this could be very nice to know. (I can easily solve 3x3, 4x4 etc.. linear systems by calculator, website etc... however, I do not at present have an automatic Gram Schmidt-ing machine.... see Matlab Prj: 2)