Show your work. There are three problems.

Problem 1 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ and let $B = \begin{bmatrix} -1 & 5 & -2 \\ 0 & 2 & -2 \\ 1 & -3 & 2 \end{bmatrix}$

(a.) multiply AB

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 5 & -2 \\ 0 & 2 & -2 \\ 1 & -3 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}} = 2I \Rightarrow A\left(\tfrac{1}{2}B\right) = I$$

(b.) calculate A^{-1} (if you are doing row reduction here then think again)

$$\boxed{A^{-1} = \frac{1}{2}B} = \begin{bmatrix} -1/2 & 5/2 & -1 \\ 0 & 1 & -1 \\ 1/2 & -3/2 & 1 \end{bmatrix}$$

Problem 2 Suppose A is a 3×3 matrix and I is the 3×3 identity matrix. Furthermore, it is given that

$$rref[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Given this information, determine the following:

(a.) is A invertible?

No, $rref(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq I$

(b.) all solutions of $Av = e_1$ where $v = [x, y, z]^T$.

rref(A|I) given $\Rightarrow$ $\boxed{\begin{aligned} x &= z+1 \\ y &= -z+2 \\ z &= z \end{aligned}}$ or (better) $\boxed{\begin{aligned} x &= t+1 \\ y &= 2-t \\ z &= t \quad \text{for } t \in \mathbb{R} \end{aligned}}$

(c.) all solutions of $Av = e_2$ where $v = [x, y, z]^T$.

rref(A|I) given $\Rightarrow$ $\begin{aligned} x &= z+2 \\ y &= -z \\ z &= z \end{aligned}$ or $\boxed{\begin{aligned} x &= t+2 \\ y &= -t \\ z &= t, \quad t \in \mathbb{R} \end{aligned}}$

(d.) all solutions of $Av = e_3$ where $v = [x, y, z]^T$.

$\boxed{\text{No sol}^{\underline{\text{s}}}}$, the system is inconsistent

$$rref[A|e_3] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Problem 3 Find all a, b, c such that

$$x + 2y + z = a, \qquad y + 2z = b, \qquad x + y - z = c$$

is a consistent system of equations.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 2 & b \\ 1 & 1 & -1 & c \end{array}\right] \xrightarrow{r_3 - r_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 2 & b \\ 0 & -1 & -2 & c-a \end{array}\right]$$

$$\xrightarrow{r_3 + r_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 2 & b \\ 0 & 0 & 0 & b+c-a \end{array}\right]$$

further row reduction will not change

we learn $\boxed{b + c - a = 0}$

is necessary for consistency. Equivalently

$\boxed{a, b, c \in \mathbb{R} \text{ such that } a = b + c}$