Show your work. Pick any three problems. I will grade three.

Problem 1 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ and suppose $AB = \begin{bmatrix} -1 & 5 & -2 \\ 0 & 2 & -2 \\ 1 & -3 & 2 \end{bmatrix}$. Calculate $det(B)$.

$$\det(A) = 1(1) - 2(1-1) + 3(-1) = -2$$

$$\det(AB) = -1(4-6) - 5(2) - 2(-2) = 2 - 10 + 4 = -4$$

$$\det(AB) = \det(A)\det(B) \Rightarrow \det(B) = \frac{\det(AB)}{\det(A)} = \frac{-4}{-2} = \boxed{2.}$$

Problem 2 Find real numbers k for which the following system of equations has a unique solution:

$$kx + 2y = 3 \qquad 50x + ky = 2$$

$$\underbrace{\begin{bmatrix} k & 2 \\ 50 & k \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\det(A) = k^2 - 100 = 0 \Rightarrow k^2 = 100 \Rightarrow k = \pm 10$$

When $k \neq 0$ then A^{-1} exists $\Rightarrow Av = b$ has $v = A^{-1}b$ which is a single (or unique) sol$^{\underline{n}}$. $\therefore \boxed{k \neq \pm 10}$

(in fact, $k = \pm 10$ gives no sol$^{\underline{n}}$ for this problem)

Problem 3 **(a.)** Calculate the volume of the solid spanned by the vectors $v_1 = e_1 + e_2 + e_3$, $v_2 = 2e_2 + e_3$ and $v_3 = 3e_3$ where $e_1, e_2, e_3 \in \mathbb{R}^3$ are the usual standard basis.
(b.) is $\{v_1, v_2, v_3\}$ a right handed triple?

a.) $\det[V_1|V_2|V_3] = \det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix} = 1(2)(3) = \boxed{6.}$

b.) yes. $\det[V_1|V_2|V_3] > 0$ hence $\{V_1, V_2, V_3\}$ forms right-handed triple.

Problem 4 Solve $Av = b$ for t by Cramer's Rule. We define t by $v = [x, y, z, t]^T$ and A, b by

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 5 \\ -2 & 3 & 1 & 0 \\ 0 & 3 & 3 & -4 \end{bmatrix} \qquad \& \qquad b = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det(A) = 3\det\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 3 & -4 \end{bmatrix} + 5\det\begin{bmatrix} 1 & 1 & 1 \\ -2 & 3 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

$$= 3[-4-8-6] + 5(9-3-1(\cancel{-6})+1(\cancel{-6}))$$

$$= 3[-18] + 5[6]$$

$$= -54 + 30$$

$$= \boxed{-24}$$

$$\det\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} = -3\det\begin{bmatrix} 0 & 3 & 0 \\ -2 & 3 & 1 \\ 0 & 3 & 3 \end{bmatrix} = -3(-3(-6)) = \underline{-54}$$

$$\therefore \boxed{t = \frac{-54}{-24} = \frac{54}{24}} = \boxed{\frac{9}{4}}$$

Problem 5 Given that

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix} \qquad \& \qquad B = [2e_2 + 3e_3 + 4e_4 + 5e_5 | 3e_3 + 4e_4 + 5e_5 | 4e_4 + 5e_5 | 5e_5 | 0]$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 5 & 0 \end{bmatrix}, \quad B^T = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find

clear because row of zeros or repeated or for (c.) diagonal.

(a.) $\det(A) = \boxed{0}$

(b.) $\det(B) = \boxed{0}$

(c.) $\det(A - B) = \boxed{5! = 120}$

(d.) $\det(A - B^T) =$ ugly.

$$A - B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$A - B^T = \begin{bmatrix} 1 & -1 & -2 & -3 & -4 \\ 2 & 2 & -1 & -2 & -3 \\ 3 & 3 & 3 & -1 & -2 \\ 4 & 4 & 4 & 4 & -1 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

$$\underbrace{\det(A - B^T)}_{\text{obtained via 14 row-ops, I checked with the linked website to the coursepage which shows steps...}} = 600 \text{ as it happens.}$$