Problem 1 [2pts] Consider $W = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\}$. Is W a subspace of $\mathbb{R}^{2\times 2}$? If the answer is yes then find a basis for W and state the dimension of W.

Suppose $A \in W$.

$$A = A^T \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow A \in W \text{ has form } A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Observe $A = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Thus $A \in \text{span}\underbrace{\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}}_{\beta}$ and conversely it is clear $\text{span}(\beta) \subset W$ thus $W = \text{span}(\beta)$ which shows W is subspace. Furthermore

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = 0$$

and we find β is LI. Consequently β is a basis and we find $\dim(W) = 3$.

Problem 2 [2pts] Suppose $W = span\{v_1, v_2, v_3\} \subseteq \mathbb{R}^n$ for $n > 3$. Is W a subspace? If the answer is yes then state the possible dimensions of W.

Thm/ $\text{span}(S) \leq \mathbb{R}^n$.

We're given $W = \text{span}(S)$ thus $W \leq \mathbb{R}^n$.

$\dim(W) \in \{0, 1, 2, 3\}$.

($\dim W = 0$ if $v_1 = v_2 = v_3 = 0$, $\dim W = 1$ if $v_1 = kv_2 = lv_3 \neq$ etc..

Problem 3 [2pts] Let $\beta = \{1, x^2, x\}$ be the **ordered** basis for P_2. Find the coordinate vector of $f(x) = (x-3)^2$.

$$\begin{aligned} f(x) &= x^2 - 6x + 9 \\ &= 9 + x^2 - 6x \end{aligned} \Longrightarrow \boxed{[f(x)]_\beta = \begin{bmatrix} 9 \\ 1 \\ -6 \end{bmatrix}}$$

Problem 4 [4pts] For A given below:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

Find,

(a.) a basis for $Col(A)$

(b.) a basis for $Null(A)$; (basis for the solution set of $Ax = 0$)

(c.) solve $Ax = (1,1,1,1)$ if possible.

(d.) extend the basis of $Col(A)$ to obtain a basis for $\mathbb{R}^4$.

(a.) $\boxed{\beta = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}}$ gives basis for $Col(A)$.

(b.)
$x_1 = -x_2$
$x_3 = 3x_4$
$x_5 = 0$
x_2, x_4 free

$$x = \begin{bmatrix} -x_2 \\ x_2 \\ 3x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

form basis of $Null(A)$.

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

$Ax = (1,1,1,1)$ has sol$^{\underline{n}}$ when

(c.) $(1,1,1,1) \in Col(A)$ iff $\exists\, a, b, c$ such that

$$a\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\b\\c\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \Rightarrow \begin{matrix} a=1 \\ b=1 \\ b=1 \\ c=1 \end{matrix}$$

thus $Ax = (1,1,1)$ has sol$^{\underline{n}}$ $x_1 = 1$, $x_3 = 1$, $x_5 = 1$
where $x_2 = x_4 = 0$ (this is not a $\rightarrow\leftarrow$ to (b.)
notice we're solving $Ax = (1,1,1,1)$ not $Ax = 0$)
Thus $\boxed{x = (1, 0, 1, 0, 1)}$

(d.) In (c.) changing $(1,1,1,1)$ to $(1,1,0,1)$ makes $\underbrace{b=1, b=0}_{\text{inconsistent}}$
thus $\beta \cup \{(1,1,0,1)\}$ will work.
(many answers here.)