Problem 1 [5pts] Consider P_2 with basis $\{1, x, x^2\} = \beta$ and $\mathbb{R}^{2\times 2}$ with basis $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$.
Define $T : P_2 \to \mathbb{R}^{2\times 2}$ by

$$T(f(x)) = \begin{bmatrix} f(1) & f'(1) \\ f''(1) & f'''(1) \end{bmatrix}$$

Find $[T]_{\beta\gamma}$.

$$T(\underbrace{a+bx+cx^2}_{f}) = \left[\begin{array}{c|c} a+bx+cx^2 & b+2cx \\ \hline 2c & 0 \end{array}\right]\Bigg|_{x=1} = \left[\begin{array}{c|c} a+b+c & b+2c \\ \hline 2c & 0 \end{array}\right]$$

$$\Rightarrow \quad [T(f)]_\gamma = \begin{bmatrix} a+b+c \\ b+2c \\ 2c \\ 0 \end{bmatrix}$$

we need $[T(f)]_\gamma = [T]_{\beta\gamma}[f]_\beta$

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c \\ b+2c \\ 2c \\ 0 \end{bmatrix}}$$

$$\boxed{[T]_{\beta\gamma} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

Problem 2 [5pts] Let $\beta = \{(2,5), (-3,1)\}$. Find the coordinates of $v = (a,b)$ with respect to the the basis β. That is, find $[v]_\beta$.

$$[\beta] = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}$$

$$[\beta]^{-1} = \frac{1}{2+15}\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$$

$$\begin{aligned} [v]_\beta &= [\beta]^{-1} v \\ &= \frac{1}{17}\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \\ &= \boxed{\begin{bmatrix} \frac{1}{17}(a+3b) \\ \frac{1}{17}(-5a+2b) \end{bmatrix}} \end{aligned}$$