**Problem 1** [4pts] Suppose $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ and $v_1 = (-1, 1, 0)$ and $v_2 = (-1, 0, 1)$ and $v_3 = (1, 2, 3)$.

Show $v_1, v_2, v_3$ are eigenvectors of $A$ and identify the eigenvalue of each vector.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \therefore \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ has } \lambda_1 = 0.$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \therefore \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ has } \lambda_2 = 0.$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \therefore \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ has } \lambda_3 = 6.$$

**Problem 2** [5pts] Find the eigenvalues and a basis for each eigenspace of $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{pmatrix} = (\lambda-1)^2 - 25 = (\lambda-6)(\lambda+4)$$

$$\Rightarrow \underline{\lambda_1 = 6,\ \lambda_2 = -4}$$

(eigenvalues)

$\underline{\lambda_1 = 6}$ $\begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -5u + 5v = 0 \quad \therefore u = v \quad \Rightarrow \underline{\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

e-vector with e-value 6

$\underline{\lambda = -4}$ $\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 5u + 5v = 0 \quad \therefore u = -v \quad \Rightarrow \underline{\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

e-vector with e-value -4

Btw, $\begin{cases} W_{\lambda=6} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \\ W_{\lambda=-4} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \end{cases}$

**Problem 3** [1pts] Suppose $A$ is a $3 \times 3$ matrix such that there exist LI vectors $\{v_1, v_2, v_3\}$ such that $Av_1 = 3v_1$ and $Av_2 = 3v_2$ and $Av_3 = 8v_3$. If $T(v) = Av$ then find $[T]_{\beta,\beta}$ where $\beta = \{v_1, v_2, v_3\}$.

$$T(\underbrace{xV_1 + yV_2 + zV_3}) = xT(V_1) + yT(V_2) + zT(V_3) = 3xV_1 + 3yV_2 + 8zV_3$$

$$[T(v)]_\beta = (3x, 3y, 8z) \quad \text{whereas} \quad [v]_\beta = (x, y, z)$$

$$[T]_{\beta\beta} \text{ defined by } [T]_{\beta\beta}[v]_\beta = [T(v)]_\beta \Rightarrow \boxed{[T]_{\beta\beta} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}}$$