Problem 1 [4pts] Find the eigenvalues and a basis for each eigenspace of $A = \begin{bmatrix} 4 & -5 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{bmatrix} 4-\lambda & -5 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix} = (\lambda+1)^2(4-\lambda) = 0 \therefore \underline{\lambda_1 = \lambda_2 = -1, \ \lambda_3 = 4}.$$

$\underline{\lambda = -1}$

$$(A + I)\vec{u}_1 = \begin{bmatrix} 5 & -5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \left.\begin{matrix} 5u - 5v = 0 \\ w \text{ free} \end{matrix}\right\} \to \vec{u}_1 = \begin{bmatrix} u \\ u \\ w \end{bmatrix} = u\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

basis for $\lambda = -1$ e-space.

Thus $\underline{W_{\lambda=-1} = \text{span}\{(1,1,0), (0,0,1)\}}$.

$\underline{\lambda = 4}$

$$(A - 4I)\vec{u}_3 = \begin{bmatrix} 0 & -5 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \left.\begin{matrix} -5v = 0 \\ u \text{ free} \\ -5w = 0 \end{matrix}\right\} \to \vec{u}_3 = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} = u\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

basis for e-space with $\lambda = 4$.

Thus $\underline{W_{\lambda=4} = \text{span}\{(1,0,0)\}}$.

Problem 2 [5pts] Find the eigenvalues and one e-vector of $A = \begin{bmatrix} 5 & 51 \\ -3 & 11 \end{bmatrix}$.

$$0 = \det(A - \lambda I) = \det\begin{bmatrix} 5-\lambda & 51 \\ -3 & 11-\lambda \end{bmatrix} = (\lambda - 5)(\lambda - 11) + 153$$
$$= \lambda^2 - 16\lambda + 208$$
$$= (\lambda - 8)^2 + 144 \Rightarrow \boxed{\lambda = 8 \pm 12i}.$$

$\underline{\lambda = 8 + 12i}$

$$(A - (8+12i)I)\vec{u}_1 = \begin{bmatrix} -3-12i & 51 \\ -3 & 3-12i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to -3u = (12i - 3)v$$
$$\underline{u = (1-4i)v}.$$

$$\vec{u}_1 = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1-4i \\ 1 \end{bmatrix}$$ thus

$\boxed{\vec{u}_1 = (1-4i, 1) \text{ is complex e-vector with e-value } \lambda = 8 + 12i}$