Problem 1 [4pts] Suppose $w_1 = (1,2,3,0)$ and $w_2 = (0,-3,2,0)$ and $w_3 = (1,1,1,1)$. Find an orthonormal basis for $W = \text{span}\{w_1, w_2, w_3\}$. Find the projection of (a,b,c,d) onto the subspace W; that is, calculate $\text{Proj}_W(a,b,c,d)$.

Let $\underline{u_1 = \frac{1}{\sqrt{14}}\langle 1,2,3,0\rangle}$ note $u_1 \cdot u_1 = 1$.

Let $V_2 = W_2 - (W_2 \cdot u_1)u_1 = (0,-3,2,0) - \frac{1}{14}(0)u_1 \Rightarrow \underline{u_2 = \frac{1}{\sqrt{13}}\langle 0,-3,2,0\rangle}$.

Let
$$V_3 = W_3 - (W_3 \cdot u_1)u_1 - (W_3 \cdot u_2)u_2$$
$$= (1,1,1,1) - \tfrac{1}{14}(1+2+3)\langle 1,2,3,0\rangle - \tfrac{1}{13}(-3+2)\langle 0,-3,2,0\rangle$$
$$= \langle 1,1,1,1\rangle - \tfrac{3}{7}\langle 1,2,3,0\rangle + \tfrac{1}{13}\langle 0,-3,2,0\rangle$$
$$= \langle 1 - 3/7,\ 1 - 6/7 - 3/13,\ 1 - 9/7 + 2/13,\ 1\rangle$$
$$= \langle 4/7,\ -8/91,\ -12/91,\ 1\rangle$$
$$\Rightarrow \underline{u_3 = \sqrt{\tfrac{91}{123}}\langle 4/7,\ -8/91,\ -12/91,\ 1\rangle}.$$

Thus,
$$\text{Proj}_W(a,b,c,d) = ((a,b,c,d)\cdot u_1)u_1 + [(a,b,c,d)\cdot u_2]u_2 + [(a,b,c,d)\cdot u_3]u_3$$
$$= \tfrac{1}{14}(a+2b+3c)\langle 1,2,3,0\rangle + \tfrac{1}{13}(-3b+2c)\langle 0,-3,2,0\rangle + \tfrac{91}{123}\left(\tfrac{4a}{7} - \tfrac{8b}{91} - \tfrac{12c}{91} + d\right)\langle \tfrac{4}{7}, \tfrac{-8}{91}, \tfrac{-12}{91}, 1\rangle.$$

Problem 2 [3pts] Find the least square fit line which best approximates the data set $(1,2),(2,4),(3,7)$ $(4,1),(5,10),(-1,0)$. (do this by solving appropriate normal equations via technology ~~or hand-to-hand combat)~~

$$= \boxed{\begin{bmatrix} \frac{1}{14}a + \frac{2}{14}b + \frac{3}{14}c + \frac{4}{7}\left(\frac{91}{123}\right)\left(\frac{4a}{7} - \frac{8b}{91} - \frac{12c}{91} + d\right) \\ \frac{2}{14}a + \frac{4}{14}b + \frac{6}{14}c - \frac{3}{13}(-3b+2c) - \frac{8}{91}\left(\frac{91}{123}\right)\left(\frac{4a}{7} - \frac{8b}{91} - \frac{12c}{91} + d\right) \\ \frac{6}{14}a + \frac{6b}{14} + \frac{9}{14}c + \frac{2}{13}(-3b+2c) - \frac{12}{91}\left(\frac{91}{123}\right)\left(\frac{4a}{7} - \frac{8b}{91} - \frac{12c}{91} + d\right) \\ \frac{91}{123}\left(\frac{4a}{7} - \frac{8b}{91} - \frac{12c}{91} + d\right) \end{bmatrix}}$$

Alternatively, and computationally easier, let $A = [w_1|w_2|w_3]$
$$\text{Proj}_W(v) = A(A^TA)^{-1}A^Tv$$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 2 & -3 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 2 & -3 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$
← can calculate via Matlab.

Problem 4 [2pt] Let $w_1 = E_{11}, w_2 = E_{12} + E_{21}, w_3 = E_{22} + E_{11}$. Use the gram-schmidt algorithm to find an orthonormal basis for $W = \text{span}\{w_1, w_2, w_3\}$ given that we use the inner product $< A, B >= \text{Trace}(AB^T)$.

$$\langle W_1, W_1 \rangle = \text{Trace}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1.$$

$$\langle W_2, W_1 \rangle = \text{Tr}\left((E_{12}+E_{21})E_{11}\right) = \text{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{Tr}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0.$$

$$\langle W_2, W_2 \rangle = \text{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \text{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2.$$

$$\langle W_3, W_3 \rangle = \text{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2$$

$$\langle W_2, W_3 \rangle = \text{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{Tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0.$$

$$\langle W_1, W_3 \rangle = \text{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1.$$

Set $u_1 = W_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ already normalized.

Set
$$\begin{aligned} V_2 &= W_2 - \langle W_2, u_1 \rangle u_1 \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 0 u_1 \\ &= \underline{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \Rightarrow \underline{u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}. \end{aligned}$$

Next,
$$\begin{aligned} V_3 &= W_3 - \langle W_3, u_1 \rangle u_1 - \langle W_3, u_2 \rangle u_2 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \cancelto{0}{\text{Trace}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)} u_2 \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{note } \langle V_3, V_3 \rangle = 1 \end{aligned}$$

$$\boxed{\text{Thus } \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ forms } \langle , \rangle\text{-orthonormal basis for } W.}$$