MATH 221 — TAKEHOME PART OF TEST 3 (DUE BEFORE THE TEST ON 12-4)

Show your work. Box answers please. If you use technology then explain how, do not just write answer. Thanks. You may work together, however, you must state who you worked with and you must write the answer in your own words in the end. Partial credit is given more generously to those who work alone. I would like the answers written on this page with the work attached. Thanks.

Problem 1 [10pts] Find the projection of (v, w, x, y, z) onto $W = span\{w_1, w_2, w_3\}$ where $w_1 = (1, 1, 1, 1, 1)$, $w_2 = (0, 0, 1, -1, 2)$ and $w_3 = (1, 1, 0, 0, 1)$ by the following methods:

1. apply the Gram-Schmidt procedure to the given basis for W and create and orthonormal basis with which you may form the projection formula (we usually did this)

$$\text{Proj}_W \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3v + 3w - x + y + z \\ 3v + 3w - x + y + z \\ -v - w + 5x + 2y + 2z \\ v + w + 2x + 5y - 2z \\ v + w + 2x - 2y + 5z \end{bmatrix}$$

2. apply the beatifully simple matrix formula for finding projections as I mentioned in the last lecture. This does not require Gram-Schmidt and you should use technology to compute the matrix inverses and products required for the formula.

See my sol^n for decimal approx. of the f-la for $\text{Proj}_W(v, w, x, y, z)$.

Problem 2 [10pts] Find the best approximation of $f(x) = e^x \sin(x)$ on $[-1, 1]$ in P_2. (use the Legendre polynomials we already normalized, you may use wolfram alpha etc... to peform the requisite integration, we assume $< f, g >= \int_{-1}^{1} f(x)g(x)dx$.

$$e^x \sin x \approx 0.273 + 0.7901x + 0.1765x^2$$

Problem 3 [10pts] Use the method of least squares and explicit matrix computation (which can be done with Matlab etc...) to find the equation of the plane which is closest to the points:

$$(1, 2, 3),\ (4, 5, 6),\ (1, 1, 1),\ (4, 8, 5),\ (0, 1, 1),\ (2, 2, 2).$$

$$z = 0.9383x + 0.1364y + 0.9616$$

Problem 4 [10pts] Suppose $dx/dt = x+y+z$ and $dy/dt = x-y+z$ and $dz/dt = x+2z$. If the solution is at $(1, 2, 3)$ at $t = 0$ then find x, y, z as functions of t. Your solution may either use the built-in Matlab commands or the general method of solution by eigenvectors and matrix exponential as presented in my notes. Of course you can use technology to compute the eigenvalues of the matrix in question, it is probably not pretty.

See sol^n

$$x(t) = -8.09\exp(2.81t) + 11.29\exp(0.53t) - 2.19\exp(-1.34t)$$
$$y(t) = -4.8\exp(2.81t) + 2.37\exp(0.53t) + 4.43\exp(-1.34t)$$
$$z(t) = 10.01\exp(2.81t) - 7.67\exp(0.53t) + 0.65\exp(-1.34t)$$

Takehome Test 3 Sol^{n} (applied linear alg.)

PROBLEM 1

$$W_1 = (1, 1, 1, 1, 1) \longrightarrow \underline{\underline{u_1 = \frac{1}{\sqrt{5}}\langle 1,1,1,1,1\rangle}}.$$

$$W_2 = (0, 0, +1, -1, 2)$$

$$W_3 = (1, 1, 0, 0, 1)$$

$$V_2 = W_2 - (W_2 \cdot u_1)u_1 = \langle 0,0,1,-1,2\rangle - \frac{2}{5}\langle 1,1,1,1,1\rangle$$

$$\Rightarrow V_2 = \langle -2/5, -2/5, 3/5, -7/5, 8/5\rangle$$

$$\Rightarrow V_2 = \frac{1}{5}\langle -2, -2, 3, -7, 8\rangle$$

$$\Rightarrow u_2 = \frac{1}{\sqrt{4+4+9+49+64}}\langle -2,-2,3,-7,8\rangle$$

$$\Rightarrow \underline{\underline{u_2 = \frac{1}{\sqrt{130}}\langle -2,-2,3,-7,8\rangle}}.$$

$$V_3 = W_3 - (W_3\cdot u_1)u_1 - (W_3\cdot u_2)u_2$$

$$= (1,1,0,0,1) - \frac{1}{5}(3)\langle 1,1,1,1,1\rangle - \frac{1}{130}\underbrace{(-2-2+8)}_{} \langle -2,-2,3,-7,8\rangle \quad \frac{-4}{130}$$

$$= \langle 1,1,0,0,1\rangle - \langle 3/5, 3/5, 3/5, 3/5, 3/5\rangle + \left\langle \frac{8}{130}, \frac{8}{130}, \frac{-12}{130}, \frac{28}{130}, \frac{-32}{130}\right\rangle$$

$$= \left\langle 1 - \frac{3}{5} + \frac{8}{130},\ 1 - \frac{3}{5} + \frac{8}{130},\ \frac{-3}{5} - \frac{12}{130},\ \frac{-3}{5} + \frac{28}{130},\ 1 - \frac{3}{5} - \frac{32}{130}\right\rangle$$

$$= \langle 6/13, 6/13, -9/13, -5/13, 2/13\rangle$$

$$= \frac{1}{13}\langle 6, 6, -9, -5, 2\rangle$$

Normalizing yields $\underline{\underline{u_3 = \frac{1}{\sqrt{182}}\langle 6, 6, -9, -5, 2\rangle}}.$

$$\text{Proj}_W \underbrace{\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix}}_{\vec{r}} = (u_1\cdot\vec{r})u_1 + (u_2\cdot\vec{r})u_2 + (u_3\cdot\vec{r})u_3 =$$

Problem 1.1 continued

$$\operatorname{Proj}_W(\vec{r}) = (u_1 \cdot \vec{r})u_1 + (u_2 \cdot \vec{r})u_2 + (u_3 \cdot \vec{r})u_3 \quad \text{where } \vec{r} = \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix}$$

$$= \frac{1}{5}(v+w+x+y+z)\langle 1,1,1,1,1\rangle +$$

$$+ \frac{1}{130}(-2v-2w+3x-7y+8z)\langle -2,-2,3,-7,8\rangle +$$

$$+ \frac{1}{182}(6v+6w-9x-5y+2z)\langle 6,6,-9,-5,2\rangle$$

$$= \begin{bmatrix} \frac{1}{5}(v+w+x+y+z) - \frac{2}{130}(-2v-2w+3x-7y+8z) + \frac{6}{182}(6v+\cdots+2z) \\ \frac{1}{5}(v+w+x+y+z) - \frac{2}{130}(-2v-2w+3x-7y+8z) + \frac{6}{182}(6v+6w-9x-5y+2z) \\ \frac{1}{5}(v+w+x+y+z) + \frac{3}{130}(-2v-2w+3x-7y+8z) - \frac{9}{182}(6v+6w-9x-5y+2z) \\ \frac{1}{5}(v+w+x+y+z) - \frac{7}{130}(-2v-2w+3x-7y+8z) - \frac{5}{182}(6v+6w-9x-5y+2z) \\ \frac{1}{5}(v+w+x+y+z) + \frac{8}{130}(-2v-2w+3x-7y+8z) + \frac{2}{182}(6v+6w-9x-5y+2z) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{7}v + \frac{3}{7}w - \frac{1}{7}x + \frac{1}{7}y + \frac{1}{7}z \\ \frac{3}{7}v + \frac{3}{7}w - \frac{1}{7}x + \frac{1}{7}y + \frac{1}{7}z \\ -\frac{1}{7}v - \frac{1}{7}w + \frac{5}{7}x + \frac{2}{7}y + \frac{2}{7}z \\ \frac{1}{7}v + \frac{1}{7}w + \frac{2}{7}x + \frac{5}{7}y - \frac{2}{7}z \\ \frac{1}{7}v + \frac{1}{7}w + \frac{2}{7}x - \frac{2}{7}y + \frac{5}{7}z \end{bmatrix}$$

$$\boxed{= \frac{1}{7}\begin{bmatrix} 3v + 3w - x + y + z \\ 3v + 3w - x + y + z \\ -v - w + 5x + 2y + 2z \\ v + w + 2x + 5y - 2z \\ v + w + 2x - 2y + 5z \end{bmatrix}}$$

Problem 1.2

Use Eq$^{=}$ (10) on pg. 372 to avoid Gram-Schmidt,

$$\mathrm{Proj}_W(\vec{r}) = A(A^TA)^{-1}A^T\vec{r} \quad \text{where} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Matlab calculates, to 4 decimals,

$$\mathrm{Proj}_W(\vec{r}) = \begin{bmatrix} 0.4286 & 0.4286 & -0.1429 & 0.1429 & 0.1429 \\ 0.4286 & 0.4286 & -0.1429 & 0.1429 & 0.1429 \\ -0.1429 & -0.1429 & 0.7143 & 0.2857 & 0.2857 \\ 0.1429 & 0.1429 & 0.2857 & 0.7143 & -0.2857 \\ 0.1429 & 0.1429 & 0.2857 & -0.2857 & 0.7143 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix}.$$

Multiply this gives to a good approximation the same result as we found in PROBLEM 1.1.

$\frac{3}{7} \approx 0.4286$, $\frac{1}{7} \approx 0.1429$, $\frac{5}{7} \approx 0.7143$ etc...

PROBLEM 2) $\mathrm{Proj}_{P_2}(f(x)) = \sum_{j=1}^{3} \langle f(x), u_j \rangle u_j$ where $\langle u_i, u_j \rangle = \delta_{ij}$

and we calculated in lecture (and on pg. 266 of my notes)

$$u_1 = \frac{1}{\sqrt{2}}(1)$$

$$u_2 = \sqrt{\frac{3}{2}}\, x$$

$$u_3 = \sqrt{\frac{8}{45}}\left(x^2 - \frac{1}{3}\right)$$

PROBLEM 2 continued: thanks to Wolfram Alpha!

$$\langle f(x), u_1 \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} e^x \sin x \, dx \cong 0.4692$$

$$\langle f(x), u_2 \rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}} \, x e^x \sin x \, dx = 0.9677$$

$$\langle f(x), u_3 \rangle = \int_{-1}^{1} \sqrt{\frac{8}{45}} \left(x^2 - \frac{1}{3}\right) e^x \sin x \, dx = 0.0744.$$

Hence,

$$e^x \sin x \approx (0.4692) \frac{1}{\sqrt{2}} + (0.9677) \sqrt{\frac{3}{2}} \, x + 0.0744 \sqrt{\frac{8}{45}} \left(x^2 - \frac{1}{3}\right)$$

$$\approx \boxed{0.3318 + \overset{1.185}{\cancel{0.7901}} x + \overset{0.0314}{\cancel{0.1765}} \left(x^2 - \frac{1}{3}\right)}.$$

$$\approx 0.3318 - \frac{1}{3}(0.1765) + 0.7901 x + 0.1765 x^2$$

$$= \boxed{\underset{0.3213}{\cancel{0.2730}} + \underset{1.185}{\cancel{0.7901}} x + \underset{0.0314}{\cancel{0.1765}} x^2}$$

PROBLEM 3 ~~A~~ ~~$ax + by + cz = d$~~

$\boxed{\text{PROBLEM 3}}$ Find plane $z = ax + by + c$ closest to the points

$$(1,2,3),\ (4,5,6),\ (1,1,1),\ (4,8,5),\ (0,1,1),\ (2,2,2)$$

We try to solve:

$$\begin{aligned} 3 &= a + 2b + c \\ 6 &= 4a + 5b + c \\ 1 &= a + b + c \\ 5 &= 4a + 8b + c \\ 1 &= b + c \\ 2 &= 2a + 2b + c \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 1 \\ 1 & 1 & 1 \\ 4 & 8 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{v}} = \underbrace{\begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \\ 1 \\ 2 \end{bmatrix}}_{\vec{b}}$$

The system $M\vec{v} = \vec{b}$ is inconsistent. We solve the normal eq$\underline{n}$'s $M^T M \vec{v} = M^T \vec{b}$ for the least-squares approx. sol$\underline{n}$ to the system $M\vec{v} = \vec{b}$.

$$\operatorname{rref}\left(M^T M \,\middle|\, M^T \vec{b}\right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.9383 \\ 0 & 1 & 0 & 0.1364 \\ 0 & 0 & 1 & 0.6916 \end{array}\right] \quad \text{by Matlab.}$$

$$\boxed{z = 0.9383x + 0.1364y + 0.6916}$$

PROBLEM 4

$$x' = x + y + z$$
$$y' = x - y + z$$
$$z' = x + 2z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We compute $[V, D] = \text{Eig}(A)$

$$V = \begin{bmatrix} -0.59 & -0.81 & 0.44 \\ -0.35 & -0.17 & -0.89 \\ -0.73 & 0.55 & -0.13 \end{bmatrix} \qquad D = \begin{bmatrix} 2.8136 & 0 & 0 \\ 0 & 0.5293 & 0 \\ 0 & 0 & -1.3429 \end{bmatrix}$$

Hence,
$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = C_1 \begin{bmatrix} -0.59 \\ -0.35 \\ -0.73 \end{bmatrix} e^{2.81t} + C_2 \begin{bmatrix} -0.81 \\ -0.17 \\ 0.55 \end{bmatrix} e^{0.53t} + C_3 \begin{bmatrix} 0.44 \\ -0.89 \\ -0.13 \end{bmatrix} e^{-1.34t}$$

$$\begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.59 & -0.81 & 0.44 \\ -0.35 & -0.17 & -0.89 \\ -0.73 & 0.55 & -0.13 \end{bmatrix}}_{T} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

Multiply by inverse T^{-1} $\Rightarrow$ $\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \cong \cancel{\begin{bmatrix} 13.72 \\ -13.94 \\ -4.98 \end{bmatrix}}$ $\begin{matrix} -3.4088 \\ 0.6849 \\ -1.0375 \end{matrix}$

$$x(t) = (13.72)(-0.59)e^{2.81t} - (13.94)(-0.81)e^{0.53t} - (4.98)(0.44)e^{-1.34t}$$

answer

$$\boxed{x(t) = -8.09e^{2.81t} + 11.29e^{0.53t} - 2.19e^{-1.34t}}$$

Likewise $\boxed{y(t) = -4.8e^{2.81t} + 2.37e^{0.53t} + 4.43e^{-1.34t}}$ & $\boxed{z(t) = 10.01e^{2.81t} - 7.67e^{0.53t} + 0.65e^{-1.3}}$

Repairing my answer for Problem 4:

$$x(t) = 2.01e^{2.81t} - 0.55e^{0.53t} - 0.46e^{-1.34t}$$

$$y(t) = 1.19e^{2.81t} - 0.12e^{0.53t} + 0.92e^{-1.34t}$$

$$z(t) = +2.49e^{2.81t} + 0.38e^{0.53t} + 0.13e^{-1.34t}$$