Show your work. Box answers please. You are allowed a 3 x 5 notecard and a (non-graphing) calculator.

Problem 1 [10 pts] If possible, find values for A, B, C such that the solution set of $0 = Ax + By + Cz$ includes the points (1,1,1), (0,2,2) and (0,0,3).

$(1,1,1): \; 0 = A + B + C$

$(0,2,2): \; 0 = 2B + 2C$

$(0,0,3): \; 0 = 3C \quad \Rightarrow \boxed{C = 0} \quad \Rightarrow \quad B = -C = 0 \quad \therefore \boxed{B = 0}$

$\Rightarrow A = -B - C = 0 \quad \therefore \boxed{A = 0}$

Problem 2 [10 pts] Calculate $rref(A)$ for A given below.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow[r_3 + r_1]{r_2 - r_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{r_3 \leftrightarrow r_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2} r_2} \boxed{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} = rref(A)}$$

Problem 3 [5 pts] Find elementary matrices $E_1, E_2, \ldots, E_k$ such that $rref(A) = E_1 E_2 \cdots E_k A$.

$$rref(A) = E_{\frac{1}{2} r_2} E_{r_1 - r_2} E_{r_3 \leftrightarrow r_2} E_{r_3 + r_1} E_{r_2 - r_1} A$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{the order of these need not be unique... other answers possible!}} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Problem 4 [10pts] Suppose $x + 2y = a$ and $3x + 7y = b$ for a pair of given constants $a, b \in \mathbb{R}$. Write this system of equations as a matrix equation and solve it either by multiplication by inverse or Cramer's rule.

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$A^{-1} = \frac{1}{7-6}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7a - 2b \\ -3a + b \end{bmatrix}$$

$$\therefore \boxed{x = 7a - 2b,\ y = -3a + b}$$

$$x = \frac{\det\begin{bmatrix} a & 2 \\ b & 7 \end{bmatrix}}{\det(A)} = \frac{7a - 2b}{1}$$

$$y = \frac{\det\begin{bmatrix} 1 & a \\ 3 & b \end{bmatrix}}{\det A} = \frac{b - 3a}{1}$$

Problem 5 [10pts] Suppose $1/x^2 + 2/y^2 = a$ and $-2/x^2 - 4/y^2 = b$ for a pair of given constants $a, b \in \mathbb{R}$.

(a.) Make a subsitution to make this a linear system of equations.

(b.) solve the linear system and state any necessary conditions on a, b for solutions existing

(c.) undo the substitution and find all solutions.

(a.) Let $\mathbb{X}, \mathbb{Y}$ be defined as $\mathbb{X} = 1/x^2$, $\mathbb{Y} = 1/y^2$ then,

(b.)
$$\begin{aligned} \mathbb{X} + 2\mathbb{Y} &= a \\ -2\mathbb{X} - 4\mathbb{Y} &= b \end{aligned} \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & a \\ -2 & -4 & b \end{array}\right] \xrightarrow{r_2 + 2r_1} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 0 & b + 2a \end{array}\right]$$

Hence we need $\underline{b + 2a = 0}$ for consistency.

Moreover, $\mathbb{X} + 2\mathbb{Y} = a \Rightarrow \boxed{\mathbb{X} = a - 2\mathbb{Y},\ \mathbb{Y} \text{ free for } b + 2a = 0}$

(c.) $\mathbb{X} = 1/x^2$ and $\mathbb{Y} = 1/y^2$
mean $\mathbb{X}, \mathbb{Y} > 0$ are only allowable values.
While algebra of (b.) allows $\mathbb{Y}$ free we find $\mathbb{Y} > 0$ is needed to make the substitution reverse. Thus,

$$\frac{1}{x^2} = a - \frac{2}{y^2} \quad \text{for} \quad \frac{1}{y^2} \in (0, \infty) \text{ with } b + 2a = 0.$$

Note if $1/y^2 \in (0, \infty) \Rightarrow y \in \mathbb{R} - \{0\}$.

Thus, the sol$^{\underline{n}}$ is simply

$$\boxed{\frac{1}{x^2} + \frac{2}{y^2} = a \text{ for } y \in \mathbb{R} - \{0\} \text{ given } b = -2a}$$

Problem 6 [10pts] You are given $A = \overbrace{\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}}^{2\times 2}$ and $B = \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}}^{4\times 2}$. Calculate the matrix products given below if it is possible. However, if the requested calculation is not-defined then say "dne" for your answer.

(a.) $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \\ 3 & 4 \end{bmatrix}}.$

(b.) AB d.n.e.

$(2\times 2)(4\times 2)$ dimension mis match.

(c.) $B^T B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}}.$

Problem 7 [10pts] Solve $Av = b$ for z by Cramer's Rule. We define z by $v = [x, y, z]^T$ and A, b by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ -2 & 3 & 1 \end{bmatrix} \qquad \& \qquad b = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

$$\det(A) = 1(0-9) - 1(0+6) + 1(0-0) = -15$$

$$z = \frac{\det\begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ -2 & 3 & 6 \end{bmatrix}}{\det(A)} = \frac{0}{-15} = \boxed{0 = z.}$$

Problem 8 [5pts] Assume A, B are invertible. If $C = AB^{-1}$ then is C invertible?

since A^{-1}, B^{-1} exist.

Sol^{n} ① $\det(C) = \det(AB^{-1}) = \det(A)\det(B^{-1}) = \dfrac{\det(A)}{\det(B)} \neq 0$

Hence $\det(C) \neq 0$ and we find C^{-1} exists

Sol^{n} ② $C^{-1} = (AB^{-1})^{-1} = (B^{-1})^{-1}A^{-1} = \boxed{BA^{-1}}$ (and $(AB^{-1})BA^{-1} = AB^{-1}BA^{-1}$
$= AIA^{-1}$
$= AA^{-1}$
$= I$ in case you doubt me.)

YES it is invertible, here's the formula

Problem 9 [5pts] Let $A = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & 0 & 6 \end{bmatrix}$ and suppose let $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & -3 & 3 \end{bmatrix}$. Calculate $det(B)$.

$\det(A) = 4 \cdot 5 \cdot 6 = 120$ (upper triangular)

$\det(AB) = 1 \cdot 2 \cdot 3 = 6$ (lower triangular)

$$\det(B) = \frac{\det(AB)}{\det(A)} = \frac{6}{120} = \boxed{\frac{1}{20}}$$

Problem 10 [5pts] Suppose that the matrix below is the augmented coefficient matrix for a linear system of equations $Ax = b$ where $x = [x_1, x_2, x_3, x_4, x_5]^T$,

$$rref[A|b] = rref\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/4 \end{array}\right].$$

Find the general solution of $Ax = b$ in parametric form.

Set $x_4 = t$, $x_5 = s$ then

$$\boxed{\left.\begin{array}{l} x_1 = -t \\ x_2 = t - \frac{1}{2}s \\ x_3 = \frac{1}{2}s + \frac{1}{4} \\ x_4 = t \\ x_5 = s \end{array}\right\} t, s \in \mathbb{R}}$$

$$\boxed{\begin{array}{l} x_1 = -x_4 \\ x_2 = x_4 - \frac{1}{2}x_5 \\ x_3 = \frac{1}{2}x_5 + \frac{1}{4} \\ x_4, x_5 \in \mathbb{R} \end{array}}$$

whichever is fine

Problem 11 [10pts] Suppose M is a 3×3 matrix which appears as a submatrix of A from the previous problem, In other words, $M = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 4 & 4 \end{bmatrix}$. Find M^{-1} and solve $M[x, y, z]^T = [a, b, c]^T$ for $[x, y, z]^T$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = M^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1/2 & 0 \\ 0 & -1/2 & 1/4 \end{bmatrix}}_{\text{from } rref(M|I) \text{ given in Problem 10 !}}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ -a + b/2 \\ -b/2 + c/4 \end{bmatrix}$$

$$\therefore \boxed{\begin{array}{l} x = a \\ y = -a + b/2 \\ z = -b/2 + c/4 \end{array}}$$

Problem 12 [5pts] Suppose $D = P^{-1}AQ$ and $det(Q) \neq 0$. Solve for A.

$$PD = PP^{-1}AQ$$

$$PDQ^{-1} = PP^{-1}AQQ^{-1} = A$$

$$\therefore \boxed{A = PDQ^{-1}}$$

Problem 13 [5pts] Suppose $\{v_1, v_2, v_3\}$ forms a right-handed triple with volume 8. Is $\{v_3, v_1, v_2\}$ is also a right-handed triple?

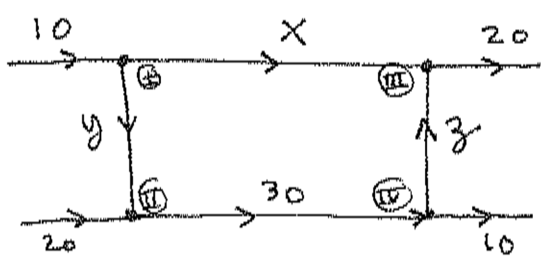
$$8 = \det[V_1|V_2|V_3] = -\det[V_1|V_3|V_2]$$

$$= -(-(\det[V_3|V_1|V_2]))$$

$$= \det[V_3|V_1|V_2]$$

Thus $\det[V_3|V_1|V_2] = 8 > 0$ hence $\{V_3, V_1, V_2\}$ is likewise a right-handed triple.

PROBLEM 14 (10pts)



I $10 = x + y$

II $20 + y = 30$

III $x + z = 20$

IV $z + 10 = 30$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 0 & 10 \\ 1 & 0 & 1 & 20 \\ 0 & 0 & 1 & 20 \end{array}\right] \xrightarrow{r_3 - r_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 0 & 10 \\ 0 & -1 & 1 & 10 \\ 0 & 0 & 1 & 20 \end{array}\right] \xrightarrow{r_3 + r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 20 \\ 0 & 0 & 1 & 20 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\begin{array}{l} x = 0 \\ y = 10 \\ z = 20 \end{array}}$$

PROBLEM 14