MATH 221 | TEST 2

Show your work. Box answers please. You are allowed a 3 x 5 notecard and a (non-graphing) calculator.

Problem 1 [16pts] A matrix A and its reduced row echelon form are given below:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 1 & 2 & 0 & 3 & 5 \\ -1 & 0 & 1 & -1 & -3 \\ 3 & 2 & 1 & 5 & 11 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a.) Find basis for Col(A) and calculate the rank of A.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \Rightarrow \underline{\text{rank}(A) = 3}.$$

(b.) Find basis for Row(A).

$$\{ [1,0,0,1,3], [0,1,0,1,1], [0,0,1,0,0] \}$$

(c.) Find basis for Null(A) and calculate the nullity of A.

$$Ax=0 \Rightarrow \begin{cases} x_1 = -x_4 - 3x_5 \\ x_2 = -x_4 - x_5 \\ x_3 = 0 \\ x_4 = x_4 \\ x_5 = x_5 \end{cases} \Rightarrow x = x_4 \underbrace{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{V_1} + x_5 \underbrace{\begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{V_2} \Rightarrow \boxed{\{V_1, V_2\} \text{ is basis for Null}(A) \text{ and } \nu(A) = 2}$$

(d.) Find the solution set for $Av = b$ where A was given in the first problem and $b = \underbrace{(1,1,-1,3)}_{Col_1(A)} + 2\underbrace{(2,2,0,2)}_{Col_2(A)} + 3\underbrace{(0,0,1,1)}_{Col_3(A)}$.

$$A \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} = Col_1(A) + 2\,Col_2(A) + 3\,Col_3(A) = b$$

Hence $(1,2,3,0,0)$ is a sol$^{\underline{n}}$ of $Av = b$ and in general we find by the Th$^{\underline{m}}$ from notes,

$$\boxed{\text{Sol}^{\underline{n}} \text{ Set} = \{ (1,2,3,0,0) + c_1 V_1 + c_2 V_2 \mid c_1, c_2 \in \mathbb{R} \}}$$

Problem 2 [10pts] Let $\gamma = \{(3,4), (-2,6)\}$ find the coordinates of $v = (a,b)$ with respect to the γ-basis for $\mathbb{R}^2$.

$$[\gamma]^{-1} = \begin{bmatrix} 3 & -2 \\ 4 & 6 \end{bmatrix}^{-1} = \frac{1}{18+8}\begin{bmatrix} 6 & 2 \\ -4 & 3 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 6 & 2 \\ -4 & 3 \end{bmatrix}$$

$$[V]_\gamma = [\gamma]^{-1} V = \frac{1}{26}\begin{bmatrix} 6 & 2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 6a+2b \\ -4a+3b \end{bmatrix}$$

$$\boxed{[V]_\gamma = \left(\tfrac{1}{26}(6a+2b), \tfrac{1}{26}(-4a+3b) \right)}$$

Problem 3 [10pts] Find a condition on a, b, c for which $\{x^2+3, x+2x, ax^2+bx+c\}$ forms a LI set of polynomial functions.

Use $\beta = \{1, x, x^2\}$ hence

$$[x^2+3]_\beta = (3, 0, 1)$$
$$[x+2x]_\beta = [3x]_\beta = (0, 3, 0)$$
$$[ax^2+bx+c]_\beta = (c, b, a)$$

LI of sets is preserved by coordinate map hence we may check for LI of the coordinate vectors in $\mathbb{R}^3$. But, 3-vectors v_1, v_2, v_3 in $\mathbb{R}^3$ are LI iff $\det[v_1|v_2|v_3] \neq 0$.

$$\det\begin{bmatrix} 3 & 0 & c \\ 0 & 3 & b \\ 1 & 0 & a \end{bmatrix} = 3(3a) + c(-3) = \boxed{9a - 3c \neq 0}$$

Problem 4 [10pts] Let $A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Let $S = \{A_1, A_2, A_3, A_4\}$ and find $[v]_S$ given that $v = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$.

Find V_1, V_2, V_3, V_4 such that $V = V_1A_1 + V_2A_2 + V_3A_3 + V_4A_4$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} = V_1\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + V_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + V_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + V_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We find $-1 = V_3$ and $3 = V_4$ from 2^{nd} row components.
From 1^{st} row we read off,

$$(1,1): \quad 2 = -V_1 + V_2 \xrightarrow{(+)} 2 = 2V_2 \Rightarrow V_2 = 1$$
$$(1,2): \quad 0 = V_1 + V_2 \xrightarrow{(-)} 2 = -2V_1 \Rightarrow V_1 = -1$$

$$\therefore \boxed{[V]_S = (-1, 1, -1, 3)}$$

Problem 5 [10pts] Suppose $T(x,y,z) = (x-2y+2z,\ 2x+y+z,\ x+y)$ is the formula for $T : \mathbb{R}^3 \to \mathbb{R}^3$. Find the standard matrix $[T]$.

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \therefore \quad \boxed{[T] = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$

Problem 6 [16pts] For T defined in Problem 3, calculate $T^{-1}(a,b,c)$.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 5 & -3 & -2 & 1 & 0 \\ 0 & 3 & -2 & -1 & 0 & 1 \end{array}\right] \xrightarrow[5r_3]{3r_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 15 & -9 & -6 & 3 & 0 \\ 0 & 15 & -10 & -5 & 0 & 5 \end{array}\right]$$

$$\xrightarrow{r_3 - r_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 15 & -9 & -6 & 3 & 0 \\ 0 & 0 & -1 & 1 & -3 & 5 \end{array}\right] \xrightarrow[r_2 - 9r_3]{r_1 + 2r_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 3 & -6 & 10 \\ 0 & 15 & 0 & -15 & 30 & -45 \\ 0 & 0 & -1 & 1 & -3 & 5 \end{array}\right]$$

$$\xrightarrow[\frac{1}{5}r_2]{3r_1} \left[\begin{array}{ccc|ccc} 3 & -6 & 0 & 9 & -18 & 30 \\ 0 & 3 & 0 & -3 & 6 & -9 \\ 0 & 0 & -1 & 1 & -3 & 5 \end{array}\right] \xrightarrow{r_1 + 2r_2} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 3 & -6 & 12 \\ 0 & 3 & 0 & -3 & 6 & -9 \\ 0 & 0 & -1 & 1 & -3 & 5 \end{array}\right]$$

$$\xrightarrow[\substack{1/3\, r_2 \\ -r_3}]{1/3\, r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & -1 & 3 & -5 \end{array}\right] \qquad \text{check: } [T][T]^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 & 4 \\ -1 & 2 & -3 \\ -1 & 3 & -5 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

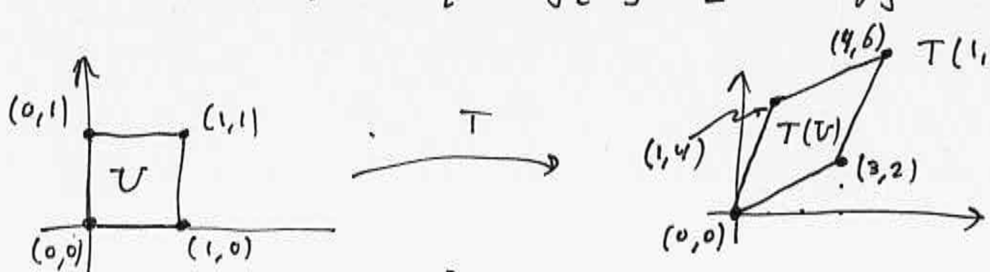
$$\underbrace{\begin{matrix} 1 & -2 & 4 \\ -1 & 2 & -3 \\ -1 & 3 & -5 \end{matrix}}_{[T]^{-1}}$$

$$T^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ -1 & 2 & -3 \\ -1 & 3 & -5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \boxed{(a - 2b + 4c,\ -a + 2b - 3c,\ -a + 3b - 5c)}$$

Problem 7 [10pts] Suppose $T(1,0) = (3,2)$ and $T(0,1) = (1,4)$. Assume T is linear and find a formula for $T : \mathbb{R}^2 \to \mathbb{R}^2$. Graph the image of the unit-square under T.

$$[T] = [T(e_1) \,|\, T(e_2)] = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$T(x,y) = [T]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + y \\ 2x + 4y \end{bmatrix} = \boxed{(3x+y,\ 2x+4y)}$$



$$T(1,1) = T(1,0) + T(0,1) = (3,2) + (1,4) = (4,6)$$

$$U = [0,1] \times [0,1].$$

Problem 8 [10pts] Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and define $T(B) = AB$ for each $B \in \mathbb{R}^{2\times 2}$.

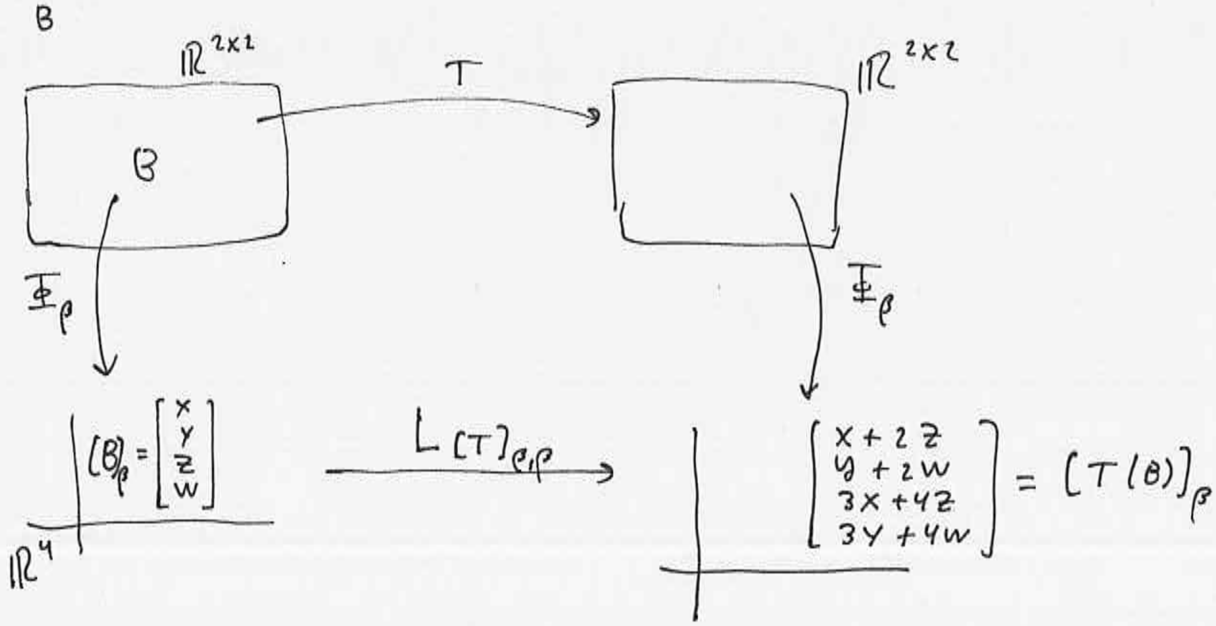
(a.) Show T is a linear transformation.

(b.) Let $\mathbb{R}^{2\times 2}$ have its standard basis of matrix units $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Find $[T]_{\beta,\beta}$. Hint: study $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and draw a picture as we did for similar problems.

(a.) Let $B_1, B_2 \in \mathbb{R}^{2\times 2}$ and $c \in \mathbb{R}$.

$$\begin{aligned} T(B_1 + cB_2) &= A(B_1 + cB_2) \\ &= AB_1 + cAB_2 \\ &= T(B_1) + cT(B_2) \implies T \text{ is linear.} \end{aligned}$$

(b.) $$T\underbrace{\begin{pmatrix} x & y \\ z & w \end{pmatrix}}_{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \left[\begin{array}{c|c} x + 2z & y + 2w \\ \hline 3x + 4z & 3y + 4w \end{array}\right] = T(B)$$



$$\left[\begin{array}{c|c} & \\ \hline & \end{array}\right]\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x + 2z \\ y + 2w \\ 3x + 4z \\ 3y + 4w \end{bmatrix}$$

$$\implies \boxed{[T]_{\beta,\beta} = \left[\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ \hline 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{array}\right]}$$

$\curvearrowright X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ per our custom. this is 4×1

Problem 9 [4pts] Suppose $Ax = b$ has solutions $x_1 = (1,2,3,4)$ and $x_2 = (1,1,1,1)$ and $x_3 = (0,0,2,2)$.

(a.) If A is $m \times n$ then what m, n are reasonable given the data of this problem.

(b.) What is the smallest possible dimension for $Null(A)$?

(c.) If $T(v) = Av$ then is T injective? Explain.

(d.) Is it possible that T defined in (c.) is surjective? Explain.

(a.) Ax is product of $m \times n$ with (4×1) hence we need $\boxed{n=4}$ no condition on n is seen as of now, however we may see something as we study b-d.

(b.) $A(X_2 - X_1) = AX_2 - AX_1 = b - b \therefore X_2 - X_1 \in Null(A)$. Set $V_{21} = X_2 - X_1$. Likewise $V_{31} = X_3 - X_1$ and $V_{32} = X_3 - X_2$ are null-vectors. Calculate,

$$V_{21} = (0,-1,-2,-2) \qquad V_{31} = (-1,-2,-1,-2) \qquad V_{32} = (-1,-1,1,1)$$

$$\text{rref}\begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \\ 1 & -2 & -2 \end{bmatrix} = \ ? \overset{(\text{see } ✡.)}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

thus $\{V_{21}, V_{31}, V_{32}\}$ is LI $\Rightarrow \nu(A) \geq 3$.

(c.) No. $T(X_1) = T(X_2) = b$ but $X_1 \neq X_2 \therefore T$ not 1-1.

(d.) For example, $b = 0$ and choose A with $m = 3$. We'll study how this is systematically accomplished in a week or three. Many correct answers here.

Problem 10 [10pts] Given that $\text{rref}\begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find a basis for

CCP

$$W = \text{span}\{(1,1,1), (2,2,2), (-1,0,1)\}.$$

Is $(4,8,12) \in W$?

BASIS By CCP is $\boxed{\underbrace{\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{pivot columns.}}}$

Note : $(4,8,12) = 4(1,2,3) = 4\left(2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) \therefore (4,8,12) \in W$ ✓.

✡ $$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \\ 1 & -2 & -2 \end{bmatrix} \xrightarrow[\substack{r_3 + r_1 \\ r_4 + r_1}]{r_2 - r_1} \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow[\substack{r_3 - 2r_2 \\ r_4 - 3r_2}]{r_1 - r_2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $\{V_{21}, V_{31}, V_{32}\}$ is LI