Show your work. Box answers please. You are allowed a 3 x 5 notecard and a (non-graphing) calculator. In this test the phrase "find the eigenvalues and eigenvectors" is meant to indicate you should find a basis for eigenspace of A. In the complex eigenvalue case, it suffices to find a particular complex e-vector. Supposing A is $n \times n$, in the non-diagonal case it may suffice to find less than n vectors. You can earn bonus points by (1.) finding a Jordan basis for the non-diagonalizable example (2.) showing A with complex e-values is similar to a matrix which is the product of a dilation and a rotation.

Problem 1 [20pts] Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix} = (\lambda+1)(\lambda-3) \quad \therefore \quad \underline{\lambda_1 = -1, \ \lambda_2 = 3}.$$

(not surprising)

$\underline{\lambda_1 = -1}$ $\quad (A+I)\vec{u}_1 = 0$

$$\begin{bmatrix} 4 & 0 \\ 8 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} u = 0 \\ v \text{ free} \end{matrix} \Rightarrow \boxed{\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}.$$

$\underline{\lambda_2 = 3}$ $\quad (A - 3I)\vec{u}_2$

$$\begin{bmatrix} 0 & 0 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 8u - 4v = 0 \\ \to v = 2u \\ \text{let } u = 1 \end{matrix} \Rightarrow \boxed{\vec{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

Problem 2 [20pts] Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$

$$\begin{matrix} 2 & 2 \\ -1 & 2 \end{matrix}$$

$$(\lambda-2)^2+2$$

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 2 \\ -1 & 5-\lambda \end{pmatrix}$$
$$= (\lambda-2)(\lambda-5)+2$$
$$= \lambda^2 - 7\lambda + 10 + 2$$
$$= \lambda^2 - 7\lambda + 12$$
$$= (\lambda-3)(\lambda-4) \Rightarrow \underline{\lambda_1 = 3,\ \lambda_2 = 4}.$$

$$(A-3I)\vec{u}_1 = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \underline{\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}.$$

$$-u+2v=0$$
$$u=2v$$

$$(A-4I)\vec{u}_2 = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \underline{\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$

$$-u+v=0$$

$A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}$ (optional choice)

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 2 \\ -1 & 2-\lambda \end{pmatrix} = (\lambda-2)^2+2 = 0$$
$$\Rightarrow \underline{\lambda = 2 \pm i\sqrt{2}}.$$

$$\left(A-(2+i\sqrt{2})I\right)\vec{u} = \begin{bmatrix} -i\sqrt{2} & 2 \\ -1 & -i\sqrt{2} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u = -i\sqrt{2}v \quad \Rightarrow \quad \underline{\vec{u} = \begin{bmatrix} -i\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}}$$
$$v=1$$

Normalizing $\vec{u} = \dfrac{1}{\sqrt{1+(\sqrt{2})^2}}\begin{bmatrix} -i\sqrt{2} \\ 1 \end{bmatrix} = \dfrac{1}{\sqrt{3}}\begin{bmatrix} -i\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{3} \end{bmatrix} + i\begin{bmatrix} -\sqrt{2}/\sqrt{3} \\ 0 \end{bmatrix}$

Let $P = \begin{bmatrix} 0 & -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{3} & 0 \end{bmatrix} \Rightarrow P^{-1} = \dfrac{3}{\sqrt{2}}\begin{bmatrix} 0 & \sqrt{2}/\sqrt{3} \\ -1/\sqrt{3} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{3} \\ -\sqrt{3}/\sqrt{2} & 0 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} 0 & \sqrt{3} \\ -\sqrt{3}/\sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{3} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{3} \\ -\sqrt{3}/\sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} 2/\sqrt{3} & -2\sqrt{2}/\sqrt{3} \\ 2/\sqrt{3} & \sqrt{2}/\sqrt{3} \end{bmatrix}$$

Problem 2 optional continued,

$$P^{-1}AP = \begin{bmatrix} 0 & \sqrt{3} \\ -\sqrt{3}/\sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} 2/\sqrt{3} & -2\sqrt{2}/\sqrt{3} \\ 2/\sqrt{3} & \sqrt{2}/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \sqrt{2} \\ -2/\sqrt{2} & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}$$

$$= \sqrt{6}\begin{bmatrix} 2/\sqrt{6} & \sqrt{2}/\sqrt{6} \\ -\sqrt{2}/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{6} \end{bmatrix}}_{\text{dilation by } \sqrt{6}.}\underbrace{\begin{bmatrix} 2/\sqrt{6} & \sqrt{2}/\sqrt{6} \\ -\sqrt{2}/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}}_{\text{rotation}}$$

Problem 3 [20pts] Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

$$\det \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} = -\lambda \det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} - 2 \det \begin{bmatrix} 1 & 2-\lambda \\ 1 & 0 \end{bmatrix}$$

$$= -\lambda(\lambda-2)(\lambda-3) - 2(\lambda-2)$$

$$= [-\lambda(\lambda-3) - 2](\lambda-2)$$

$$= (2-\lambda)[\lambda^2 - 3\lambda + 2]$$

$$= (2-\lambda)(\lambda-1)(\lambda-2) \Rightarrow \underline{\lambda_1 = 1,\ \underbrace{\lambda_2 = 2}^{\text{repeated}}.}$$

$$(A - I)\vec{u}_1 = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\hookrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$u = 0, \quad v = w.$

$$\underline{\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}.$$

$$(A - 2I)\vec{u}_2 = \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow v$ free & $u = -w$

$$\therefore \vec{u}_2 = \begin{bmatrix} u \\ v \\ -u \end{bmatrix} = u \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + v \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{\vec{u}_{2,3} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \leftrightarrow \boxed{\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = W_{\lambda=2}}$$

Problem 4 [20pts] Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$

lower Δ matrix can read e-values by inspection.

Clearly e-values of A are $\underline{\lambda_1 = 1, \ \lambda_2 = 2, 2}$.

$$(A - I)\vec{u}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$u + v = 0 \Rightarrow \underline{v = -u}$.

$-3u + 5v + w = 0 \quad w = 3u - 5v = \underline{8u}$.

Set $u = 1$, $\quad \underline{\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}}$.

(You can check: $A\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$ as req'd.)

$$(A - 2I)\vec{u}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} u = 0 \\ -3u + 5v = 0 \end{array}$$

$\Rightarrow v = \frac{3}{5}u = 0$.

w free.

Hence $\underline{\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$.

(Bonus ↓)

__Note__: A __not__ diagonalizable because $\nexists$ eigenbasis for A as the geometric multiplicity of $\lambda = 2$ is one whereas the algebraic multiplicity of $\lambda = 2$ is two.

Seek $\vec{u}_3$ such that $(A - 2I)\vec{u}_3 = \vec{u}_2$

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} u = 0 \\ -3u + 5v = 1 \end{array} \Rightarrow \underline{v = 1/5}.$$

w is free, set $w = 0$ to find $\underline{\vec{u}_3 = (0, 1/5, 0)}$.

Setting $P = [\vec{u}_1 | \vec{u}_2 | \vec{u}_3]$ we'll find $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. (Jordan form)

Problem 5 [15pts] Determine the type of conic section is given by the equation $5x^2 - 4xy + 8y^2 = 36$ by finding the eigenvalues and eigenvectors of the associated quadratic form. Sketch the graph.

$$5x^2 - 4xy + 8y^2 = (x, y)\underbrace{\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix}$$

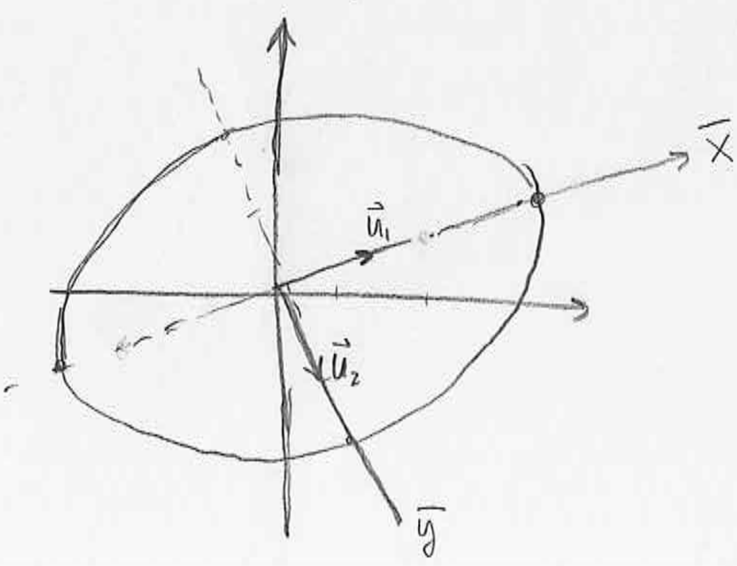
$$\det\begin{pmatrix} 5-\lambda & -2 \\ -2 & 8-\lambda \end{pmatrix} = (\lambda-5)(\lambda-8) - 4$$
$$= \lambda^2 - 13\lambda + 40 - 4$$
$$= \lambda^2 - 13\lambda + 36$$
$$= (\lambda - 4)(\lambda - 9) \quad \therefore \underline{\lambda_1 = 4, \ \lambda_2 = 9}.$$

$$(A - 4I)\vec{u}_1 = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} u - 2v = 0 \\ \therefore u = 2v \end{array}$$

$$\vec{u}_1 = k\begin{bmatrix} 2 \\ 1 \end{bmatrix} \xrightarrow{\text{normalizing}} \underline{\vec{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}},$$

$$(A - 9I)\vec{u}_2 = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -2u - v = 0 \\ \therefore v = -2u \end{array}$$

$$\vec{u}_2 = k\begin{bmatrix} 1 \\ -2 \end{bmatrix} \xrightarrow{\text{normalizing}} \underline{\vec{u}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}}.$$



$$Q(\bar{x}, \bar{y}) = \underbrace{4\bar{x}^2 + 9\bar{y}^2 = 36}_{\text{ellipse}}.$$

$$\frac{\bar{x}^2}{9} + \frac{\bar{y}^2}{4} = 1$$

Problem 6 [5pts] Suppose $\{u_1, u_2, u_3\}$ forms and orthonormal set of vectors in $\mathbb{R}^3$. Furthermore, suppose A is a matrix such that $A = 3u_1u_1^T + 7u_2u_2^T - u_3u_3^T$.

(a.) find the eigenvalues of A

(b.) find the minimum and maximum values of $Q(v) = v^TAv$ for $v \in S_2 = \{v \in \mathbb{R}^3 \mid v^Tv = 1\}$ (S_2 is the unit-sphere)

(c.) calculate $\det(A)$

(d.) calculate $\text{trace}(A)$

(e.) is $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ similar to A?

a.) $\underline{\lambda_1 = 3, \; \lambda_2 = 7, \; \lambda_3 = -1}$.

b.) $\underline{Q_{min} = -1}$ $\quad$ $\underline{Q_{max} = 7}$.

c.) $\det(A) = \lambda_1\lambda_2\lambda_3 = \boxed{-21}$

d.) $\text{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3 = 3 + 7 - 1 = \boxed{9.}$

e.) $\det(B) = 0 \neq -21$

Hence $\nexists$ P such that $B = P^{-1}AP$

since $\det(B) = \det(P^{-1}AP)$

$= \det(P^{-1})\det(A)\det(P)$

$= \det(A)$.

(which is not true here).