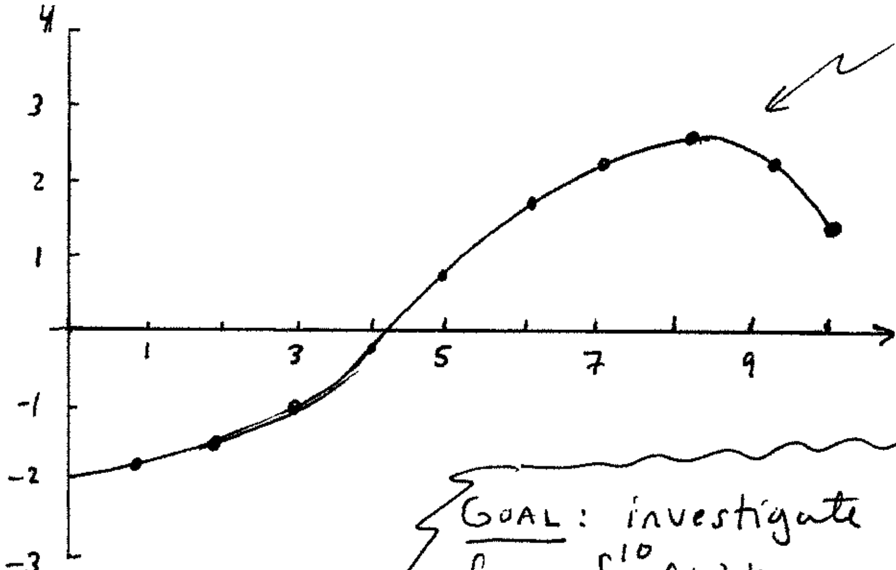
# APPROXIMATE INTEGRATION:

- THE LEFT, RIGHT AND MIDPOINT RULES AS WELL AS TRAPEZOID AND SIMPSONS RULE GIVE US METHODS TO CALCULATE $\int_a^b f(x)dx$ when no antiderivative formula for $f(x)$ is easily found. Comparing $L_N$, $R_N$, $M_N$, $T_N$, $S_N$ requires basic graphical intuition and/or graphing with calculus ideas we learned in CALCULUS I.

1.)




GOAL: investigate $L_5$, $R_5$, $M_5$ for $\int_0^{10} f(x)dx$.

Identify, $a = 0$
$b = 10$

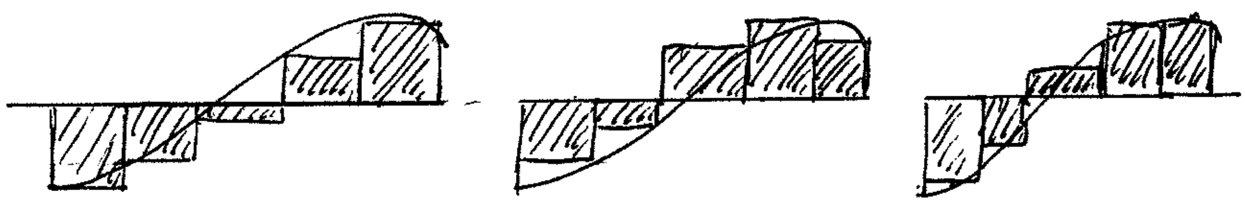
$$\Delta x = \frac{10-0}{5} = 2 \Rightarrow x_0 = 0,\ x_1 = 2,\ x_2 = 4,\ x_3 = 6$$
$$x_4 = 8,\ x_5 = 10$$

$$L_5 = \sum_{i=0}^{4} f(x_i)\Delta x = \big(f(0) + f(2) + f(4) + f(6) + f(8)\big)(2)$$
$$\cong (-2 - 1.5 - 0.3 + 1.7 + 3.4)(2)$$
$$\cong \boxed{2.6}$$

$$R_5 = \sum_{i=1}^{5} f(x_i)\Delta x = \big(f(2) + f(4) + f(6) + f(8) + \overbrace{f(10)}^{1.5}\big)(2) = \boxed{9.6}$$

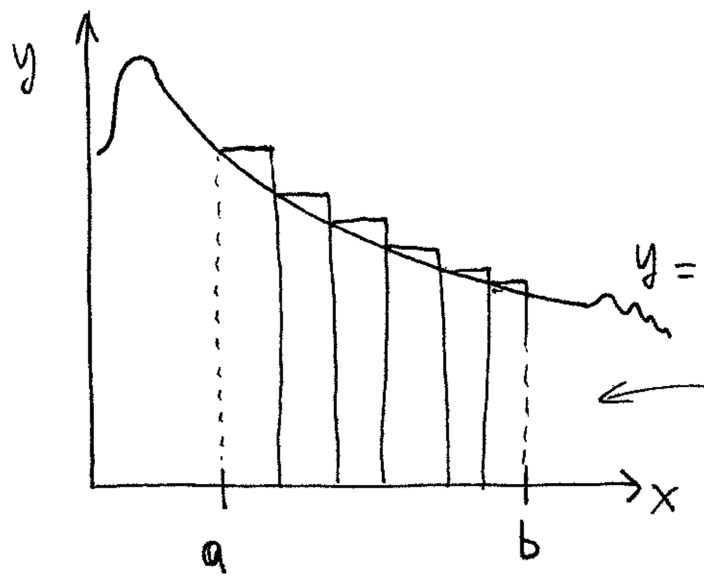
$$M_5 = \sum_{i=1}^{5} f(\bar{x}_i)\Delta x = \big(f(1) + f(3) + f(5) + f(7) + f(9)\big)(2) \cong \boxed{4.8}$$

Remark: we found $L_5 \cong 2.6$, $R_5 \cong 9.6$ and $M_5 \cong 4.8$ of these $M_5$ gives the best approximation. We find $L_5$ under-estimates $\int_0^{10} f(x)dx$ whereas $R_5$ over-estimates $\int_0^{10} f(x)dx$. You should be able to see why that's true by picturing $L_5$, $R_5$ and $M_5$



$L_5$ $R_5$ $M_5$

2.)




Observe $f(x)$ is decreasing on $[a,b]$

sketch of $L_6$ for $\int_a^b f(x)dx$

we find $R_n < \int_a^b f(x)dx < L_n$ by graphical intuition.

Remark: for $n \geq 10$ you should certainly use technology!

3.) Consider $\mathcal{I} = \int_0^2 \sin(x^2)\,dx.$

$\Delta x = \frac{2-0}{4} = 0.5$

We'll examine $n=4$ approximations

| | |
|---|---|
| $x_0 = 0$ | $\bar{x}_1 = 0.25$ |
| $x_1 = 0.5$ | $\bar{x}_2 = 0.75$ |
| $x_2 = 1$ | $\bar{x}_3 = 1.25$ |
| $x_3 = 1.5$ | $\bar{x}_4 = 1.75$ |
| $x_4 = 2$ | |

LEFT | $L_4 = \left(f(0) + f(0.5) + f(1) + f(1.5)\right)(0.5)$

$= \left[\sin(0) + \sin\left((0.5)^2\right) + \sin(1) + \sin(2.25)\right](0.5)$

$\cong \underline{0.9335}.$

RIGHT | $R_4 = \left[\sin(0.5)^2 + \sin(1) + \sin(2.25) + \sin(4)\right](0.5) \cong \underline{0.5551}.$

TRAPEZOID | $T_4 = \frac{L_4 + R_4}{2} \cong \underline{0.7443}.$

MIDPOINT | $M_4 = \left(f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4)\right)\Delta x$

$= \left(\sin(0.25^2) + \sin(0.75^2) + \sin(1.25^2) + \sin(1.75^2)\right)(0.5)$

$\cong \underline{0.8374}.$ (notice almost as good as $S_4$ here)

SIMPSONS' | $S_4 = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$

$= \frac{0.5}{3}\left[\sin(0) + 4\sin(0.5^2) + 2\sin(1) + 4\sin(2.25) + \sin(4)\right]$

$\cong \underline{0.8380}.$

It's not obvious, but $f(x) = \sin(x^2)$ has $|f^{(4)}(x)| \le \underbrace{1.7}_{K}$ on $[0,2]$.

The Error Bound for Simpson's Rule says

$$|E_{S_4}| \le \frac{K(b-a)^5}{180 \cdot 4^4} = \frac{(1.7)(2)^5}{(180)(4)^4} \cong 0.0012$$

so we can be sure $\underline{\int_0^2 \sin(x^2)\,dx = 0.8380 \pm 0.0012}.$