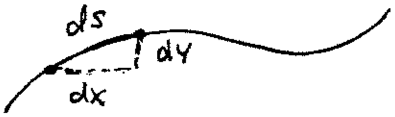
EXAMPLES OF ARCLENGTH

Heuristic :



$$ds^2 = dx^2 + dy^2$$
$$= \left(1 + (dy/dx)^2\right) dx^2$$
$$\Rightarrow \underline{ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

Hence to find the total arclength s from $x = a$ to $x = b$ we simple sum the little ds's which motivates

$$\boxed{s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx} - (*)$$

Remark: the argument above is not a "proof" and it can be established through more carefully constructed limiting procedures. The Heuristic above illustrates the infinitesimal method. Anyway, we wait for parametric curves to give general proof of the formula $s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ of which the formula $*$ is merely a special case.

1.) $y = x^{3/2}$ from $x = 1$ to $x = 4$. Calculate $\frac{dy}{dx} = \frac{3}{2}x^{1/2}$ thus $\left(\frac{dy}{dx}\right)^2 = \frac{9}{4}x$. Thus, using $*$, the arclength

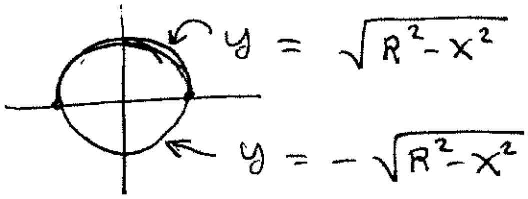
$$s = \int_1^4 \sqrt{1 + \frac{9x}{4}}\, dx \quad : \quad u = 1 + \frac{9x}{4},\ du = \frac{9\,dx}{4}$$
$$u(1) = 1 + 9/4,\ u(4) = 10$$

$$= \int_{1+9/4}^{10} \sqrt{u}\, \frac{4}{9}\, du$$

$$= \frac{4}{9} \cdot \frac{2}{3}\left[10^{3/2} - (1 + 9/4)^{3/2}\right]$$

$$= \boxed{\frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]} \approx 7.634$$

2.) $\underline{x^2+y^2=R^2 \text{ find arclength}}$,



$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{R^2-x^2}} = \frac{-x}{\sqrt{R^2-x^2}}$$

By symmetry we'll simply double the length of the top-half. Notice $(dy/dx)^2$ simplifies a bit,

$$S = 2\int_{-R}^{R} \sqrt{1+\frac{x^2}{R^2-x^2}}\,dx$$

common denominator

$$= 2\int_{-R}^{R} \sqrt{\frac{R^2-x^2+x^2}{R^2-x^2}}\,dx$$

$$= 2\int_{-R}^{R} \frac{R\,dx}{\sqrt{R^2-x^2}} \quad : \text{ we assume } R>0$$

$$= 4\int_{0}^{R} \underbrace{\frac{R\,dx}{\sqrt{R^2-x^2}}}_{\text{improper!}} \quad : \text{ ok, why do } \int_{-R}^{R} \text{ when symmetry allows this nice simplification.}$$

$$= 4\lim_{b\to R^-} \int_0^b \frac{R\,dx}{\sqrt{R^2-x^2}}$$

$$= 4R\lim_{b\to R^-}\left[\cancelto{\pi/2}{\sin^{-1}\left(\frac{b}{R}\right)} - \cancelto{0}{\sin^{-1}(0)}\right] = 4R\left(\frac{\pi}{2}\right) = \boxed{2\pi R}$$

$$** \int \underbrace{\frac{R\,dx}{\sqrt{R^2-x^2}}}_{} = \int \frac{R^2\cos\theta\,d\theta}{R\cos\theta} = R\theta + C = R\sin^{-1}\left(\frac{x}{R}\right)+C$$

$x = R\sin\theta$
$dx = R\cos\theta\,d\theta$
$R^2-x^2 = R^2\cos^2\theta$

<u>Remark</u>: with the parametric viewpoint this calculation becomes much much easier.

3.) $y = \sin x \, , \quad 0 \le x \le \pi$

$$S = \underbrace{\int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx}_{\text{Sometimes}} \underset{\substack{\uparrow \\ \text{numerical} \\ \text{integration}}}{\cong} 3.8202$$

Sometimes this is the answer. That's worthwhile since we have robust numerical methods to calculate <u>definite</u> integrals.

4.) $y = \ln(\cos x) \, , \quad 0 \le x \le \pi/3$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x$$

Thus we calculate arclength of given graph as follows,

$$S = \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx \quad : \quad \tan^2 x + 1 = \sec^2 x \text{ so}$$

$$= \int_0^{\pi/3} \sec(x)\, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/3}$$

$$= \ln\left|\frac{1}{\cos \pi/3} + \frac{\sin \pi/3}{\cos \pi/3}\right| - \cancelto{0}{\ln|1+0|}$$

$$= \ln|2 + \sqrt{3}|$$

$$= \boxed{\ln(2 + \sqrt{3}).}$$

5.) $y = \cosh x \, , \; 0 \leq x \leq \ln(3)$

$$\frac{dy}{dx} = \sinh x \implies \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\sinh^2 x} = \sqrt{\cosh^2 x} = \cosh x$$

Thus we calculate arclength by:

$$s = \int_0^{\ln(3)} \cosh x \, dx$$

$$= \sinh(\ln(3)) - \cancelto{0}{\sinh(0)}$$

$$= \frac{1}{2}\left(e^{\ln 3} - e^{-\ln 3}\right) \quad : \quad e^{-\ln 3} = e^{\ln 3^{-1}} = \frac{1}{3}$$

$$= \frac{1}{2}\left(3 - \frac{1}{3}\right)$$

$$= \boxed{4/3}$$

6.) $y = \frac{x^3}{3} + \frac{1}{4x} \quad 1 \leq x \leq 2$

$$\frac{dy}{dx} = x^2 - \frac{1}{4x^2} \implies \left(\frac{dy}{dx}\right)^2 = \left(x^2 - \frac{1}{4x^2}\right)^2 = x^4 - \frac{1}{2} + \frac{1}{4x^4}$$

Consequently, $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{x^4 + \frac{1}{2} + \frac{1}{4x^4}} = \sqrt{\left(x^2 + \frac{1}{2x^2}\right)^2} = x^2 + \frac{1}{2x^2}$.

Therefore,

$$s = \int_1^2 \left(x^2 + \frac{1}{2x^2}\right) dx$$

$$= \left(\frac{1}{3}x^3 - \frac{1}{2x}\right)\Big|_1^2$$

$$= \left(\frac{1}{3}8 - \frac{1}{4}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \boxed{31/12}$$