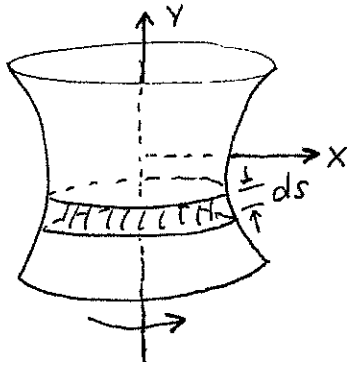
# AREA OF SURFACE OF REVOLUTION EXAMPLES

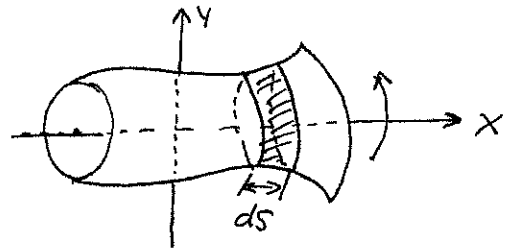



$$dA = 2\pi x \, ds$$

- radius of band is $x$
- revolving $x = g(y)$ about $y$-axis for $y_0 \le y \le y_1$
- $ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$

Add-up all the $dA$,

$$A = \int_{y_0}^{y_1} 2\pi g(y) \sqrt{\left(\frac{dg}{dy}\right)^2 + 1}\, dy$$




$$dA = 2\pi y \, ds$$

- radius of band is $y$
- revolving $y = f(x)$ about $x$-axis for $x_0 \le x \le x_1$
- $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

Add-up all the $dA$,

$$A = \int_{x_0}^{x_1} 2\pi f(x) \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$

Remark: the idea of finding $dA$ for an infinitesimal band then using integration to calculate $A = \int dA$ is another example of the infinitesimal method. This heuristic principle is a short-hand for writing a finite approximation model for the application then passing to the limit to identify the finite $\Sigma$ transmogrifying to $\int_a^b$

not a word.

1.) Find surface area of $y=\sqrt{R^2-x^2}$, $0 \le x \le R$ revolved about x-axis.

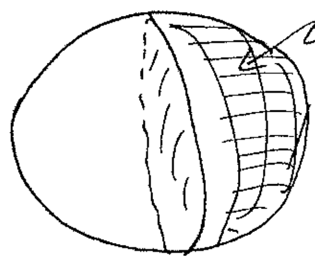
$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{R^2-x^2}} = \frac{-x}{\sqrt{R^2-x^2}} \quad \therefore \quad 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{x^2}{R^2-x^2} = \frac{R^2}{R^2-x^2}$$

Thus,

$$A = \int_0^R 2\pi\sqrt{R^2-x^2}\sqrt{\frac{R^2}{R^2-x^2}}\,dx$$

$$= 2\pi\int_0^R R\,dx$$

$$= \boxed{2\pi R^2}$$



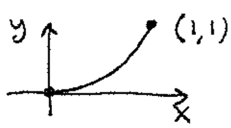

Remark: we find $4\pi R^2$ is surface area of sphere of radius R.

2.) Find surface area of $y=x^2$ $0 \le x \le 1$
a.) about x-axis b.) about y-axis
Just set-up the integrals.

a.) $y=x^2 \therefore \frac{dy}{dx}=2x$ and $dS=\sqrt{1+4x^2}\,dx$

thus $\boxed{A = \int_0^1 2\pi x^2\sqrt{1+4x^2}\,dx.}$

b.) Notice




$y=1^2=1$ where $x=1$ so $0 \le y \le 1$ for this curve (generally the x and y bounds will not match!)

$y=x^2 \Rightarrow x=\pm\sqrt{y}$ but $x \ge 0 \therefore \underline{x=\sqrt{y}}$.

observe $\frac{dx}{dy} = \frac{1}{2\sqrt{y}} \Longrightarrow 1+\left(\frac{dx}{dy}\right)^2 = 1+\frac{1}{2y}$. Thus,

$$\boxed{A = \int_0^1 2\pi\sqrt{y}\sqrt{1+\frac{1}{2y}}\,dy.}$$

3.) $y = \sqrt{1+e^x}$, $0 \le x \le 1$ revolved around x-axis, find surface area.

$$\frac{dy}{dx} = \frac{e^x}{2\sqrt{1+e^x}} \quad \therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{e^{2x}}{4(1+e^x)}$$

$$A = \int_0^1 2\pi \sqrt{1+e^x}\sqrt{1+\frac{e^{2x}}{4(1+e^x)}}\,dx$$

$$= \int_0^1 2\pi \sqrt{(1+e^x)\left(1+\frac{e^{2x}}{4(1+e^x)}\right)}\,dx$$

$$= \int_0^1 2\pi \sqrt{1+e^x+\frac{1}{4}e^{2x}}\,dx$$

$$= \int_0^1 2\pi \sqrt{\frac{1}{4}(e^{2x}+4e^x+4)}\,dx$$ : notice $e^{2x} = (e^x)^2$ and we see

$$= \int_0^1 \pi \sqrt{(e^x+2)^2}\,dx$$

$$= \pi \int_0^1 (e^x+2)\,dx$$

$$= \pi(e^1 + 2 - e^0)$$

$$= \boxed{\pi(e+1)}$$

4.) $y = \frac{1}{3}x^{3/2}$, $0 \le x \le 12$ find S.A. of surface formed by rotating given graph about y-axis.

We need to find $x$ as function of $y$, solve $y = \frac{1}{3}x^{3/2}$ for $x$: $3y = x^{3/2} \Rightarrow x = (3y)^{2/3} \Rightarrow \frac{dx}{dy} = 3^{2/3}\cdot\frac{2}{3}y^{-1/3} = \frac{2}{\sqrt[3]{3y}}$

when $x=0$ get $y = \frac{1}{3}(0)^{3/2} = 0$, when $x = 12$ get $y = \frac{12^{3/2}}{3} = 8\sqrt{3}$

$$A = \int_0^{8\sqrt{3}} 2\pi(3y)^{2/3}\sqrt{\frac{4}{3^{2/3}}\frac{1}{y^{2/3}}+1}\,dy$$

Remark: I used a C.A.S. to integrate here.

$$= 2\pi(3)^{2/3}\int_0^{8\sqrt{3}}\sqrt{(y^{2/3})^2\left[\frac{4}{3^{2/3}}\frac{1}{y^{2/3}}+1\right]}\,dy$$

$$= 2\pi\sqrt[3]{9}\int_0^{8\sqrt{3}}\sqrt{y^{4/3}+y^{2/3}\cdot\frac{4}{3^{2/3}}}\,dy \doteq 2\pi\cancel{\sqrt[3]{9}}\left[\frac{1856}{15\cancel{(3)^{2/3}}}\right] = \boxed{\frac{3712\pi}{15}}$$

$\approx 777.44.$

4.) an approach at the integration directly,

$$\int \sqrt{y^{4/3} + \frac{4}{3^{2/3}} y^{2/3}}\, dy =$$

$$= \int \sqrt{y^{2/3}\left[y^{2/3} + 4/3^{2/3}\right]}\, dy$$

$$= \int y^{1/3} \sqrt{\underbrace{y^{2/3} + 4/3^{2/3}}_{w}\,]}\, dy$$

$$dw = \frac{2}{3} y^{-1/3}\, dy$$

$$dy = \frac{3}{2} y^{1/3}\, dw$$

$$= \int y^{1/3} \sqrt{w}\, \frac{3}{2} y^{1/3}\, dw$$

$$= \frac{3}{2}\int y^{2/3} \sqrt{w}\, dw$$

$$w = y^{2/3} + 4/3^{2/3}$$

$$y^{2/3} = w - 4/3^{2/3}$$

$$= \frac{3}{2}\int \left(w - \frac{4}{3^{2/3}}\right)\sqrt{w}\, dw$$

$$= \int \left(\frac{3}{2} w^{3/2} - 2\sqrt[3]{3}\sqrt{w}\right) dw$$

$$= \frac{3}{2}\cdot\frac{2}{5} w^{5/2} - 2\sqrt[3]{3}\cdot\frac{2}{3} w^{3/2} + C$$

$$= \frac{3}{5}\left(y^{2/3} + \frac{4}{3^{2/3}}\right)^{5/2} - \frac{4}{3^{2/3}}\left(y^{2/3} + \frac{4}{3^{2/3}}\right)^{3/2} + C$$

(there is likely an easier way, but this is a path)