Meromorphic Function. $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Def^n/. A map $f : \underset{\text{open}}{D} \subseteq \mathbb{C} \to \bar{\mathbb{C}}$ is called meromorphic iff the following hold:

a). $S(f) = f^{-1}\{\infty\}$ is discrete in D.

b). $f_0 = f|_{D - S(f)} : D - S(f) \to \mathbb{C}$.

c) the points in $S(f)$ are poles of f_0.

Let $M(D)$ = set of meromorphic fncts on D.

Remark: $\mathcal{O}(D)$ is the set of analytic fncts on D.

If D is a domain (open & connected $\subseteq \mathbb{C}$).

then $fg \equiv 0$ for $f, g \in \mathcal{O}(D) \Rightarrow f \equiv 0$ or $g \equiv (0)$. ($\nexists$ zero divs in $\mathcal{O}(D)$)

Note: $\mathcal{O}(D)$ is ring, commutative, with identity $\Rightarrow \mathcal{O}(D)$ is an integral domain

if $\mathcal{O}(D)$ is an integral domain $\Rightarrow D$ is connected.

Remark: $\mathcal{O}(D)$ form subring of $M(D)$

Prop: $f \in M(D) \Rightarrow \exists\, g, h \in \mathcal{O}(D)$ with $g \not\equiv 0$. s.t. $f = \frac{h}{g}$

Def^n/. $f : \overset{\text{open}}{D} \subseteq \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is meromorphic (m.m).

a). f is mm in $D \cap \mathbb{C}$.

b). if $\infty \in D$ then, $\hat{f}(z) = f(1/z)$ is m.m. in $\hat{D} = \{z \in \mathbb{C} \mid \frac{1}{z} \in D\}$.

eg/ $P(z) = a_n z^n + \cdots + a_1 z + a_0$, $a_n \neq 0$, $a_j \in \mathbb{C}$.

$\hat{P}(\frac{1}{z}) = \underbrace{a_n(\frac{1}{z})^n + \cdots + a_1(\frac{1}{z})}_{\text{principle part of Laurent expan.}} + a_0 \Longleftarrow$ pole of order n.

ex) $h(z)=\exp(\frac{1}{z})$ $\hat{h}(z) = \exp(\frac{1}{\frac{1}{z}}) = \exp(z)$. ∞ is regular pts.

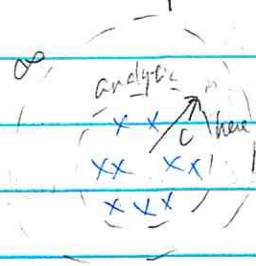
ex) $g(z) = \exp(z)$

$\hat{g}(z) = \exp(\frac{1}{z}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\frac{1}{z})^n$ ← essentially singular at zero.

⇒ g is essentially sing at ∞.

A.6. P158 Prop: $M(\bar{\mathbb{C}})$ = rational fncts on $\mathbb{C}$.

proof: Let $f \in M(\bar{\mathbb{C}})$ ⇒ f is analytic on $\underbrace{\{z \in \mathbb{C} \mid |z| > c\}}_{\text{puncture disk at } \infty}$ for some $c > 0$.



thus f has finitely many poles.

Hence the principle part of f at s: $h_s(\frac{1}{z-s})$ where h_s is poly.

$\frac{1}{0} = \infty$

Notice $g(z) = f(z) - \sum_{\substack{s \in \mathbb{C} \\ f(s)=\infty}} h_s(\frac{1}{z-s})$

is analytic on all of $\mathbb{C}$ ⇒ $g(z)$ = poly.

Remark: $\underbrace{f(z) = g(z) + \sum_{\substack{s \in \mathbb{C} \\ f(s)=\infty}} h_s(\frac{1}{z-s})}_{\text{partial fract. decomp. of } f(z)}$.

Pns. churchill also look up the fritag.

$\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} Res(z_j)\, \underbrace{\chi(z_j)}_{\text{winding \#}}$

$Res(f; z_1) = 3$
$Res(f; z_2) = 1$
$Res(f; z_3) = -7$

$\int_C f(z)dz = 2\pi i(1 \cdot 3 + 2 \cdot 1 + 2(-7))$

$\chi_1 = -1$ (cw)
$\chi_2 = 2$ (ccw)
$\chi_3 = 2$.