2/6/2

"Curvature & Homology" by Samuel Goldberg
pg 64 — More on Harmonic functions, diff, co-diff., Laplacian, ...

There are 4 fields of interest:
$\vec{E}, \vec{D}, \vec{H}, \vec{B}$ (vector field defined on Minikowski space M)
identify $\mathbb{R}^4 = M$,
$(x^0, x^1, x^2, x^3) \leftarrow$ global coordinates on $\mathbb{R}^4$

$$\eta\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = \eta_{\mu\nu} \qquad (\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad x^0 = ct \ \& \ c = 1$$

$$\vec{E}(t, x^1, x^2, x^3) = E^i(t, x_1, x_2, x_3) \frac{\partial}{\partial x^i}$$

note: $i,j,k \in \{1,2,3\}$
$\mu, \nu \in \{0,1,2,3\}$

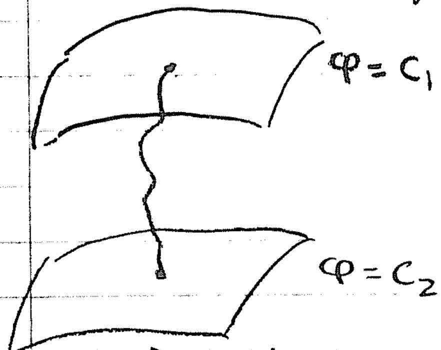
Similarly for $\vec{D}, \vec{H}, \vec{B}$.

(y^0, y^1, y^2, y^3) is inertial if $y^\mu = \Lambda^\mu_\nu x^\nu$ $\quad \eta\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\right) = \eta_{\mu,\nu}$

$x \circ y^{-1} = \Lambda$ is a Lorentz matrix (note: Lookup holonomic charts)

The field $\vec{E}$ is called the <u>electric field</u>
?<u>Generally</u>?, $\vec{E} = -\nabla\varphi$ (scalar potential)

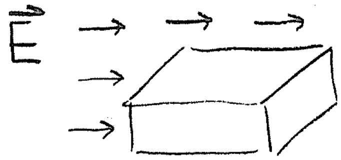
$$\vec{E}(t,x,y,z) = -\frac{\partial\varphi}{\partial x}(t,..) - \frac{\partial\varphi}{\partial y}(t,..) - \frac{\partial\varphi}{\partial z}(t,..)$$



voltage drop $\int_C \vec{E}\, d\vec{\sigma}$
Line integrals are natural
$\vec{E}$ is measured in volts/meter

$\vec{E} = E^1\partial_1 + E^2\partial_2 + E^3\partial_3$ we write $E = E_i\, dx^i$
where $E_i(t, x^1, x^2, x^3) = E^i(t, x^1, x^2, x^3)$

The field $\vec{D}$ is called a displacement field (defined in the presence of matter)



$\ominus \to \oplus$

$\vec{P}$ polarization

permativity

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ is measured in coulombs/meters^2

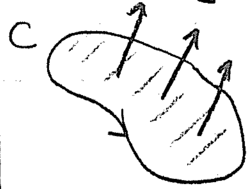
$\int_S \vec{D} \cdot d\vec{A}$ free charge enclosed by S (S is a closed surface) (like a sphere)

$\vec{E}$ is associated with total charge.
$\vec{D}$ is assoc. w/ free charge.
$\vec{E}$ polar vector (Changes sign when (x^1, x^2, x^3) changes to $(-x^1, -x^2, -x^3)$)
$\vec{D}$ axial vector (Doesn't change when (x^1, x^2, x^3) goes to $(-x^1, -x^2, -x^3)$)

$$D = \tfrac{1}{2} \epsilon_{ijk} D^i (dx^j \wedge dx^k)$$

One defines a magnetization vector $\vec{M}$ analogous to $\vec{P}$

$$I_M = \oint_C M d\vec{\sigma} \quad \text{(current induced by mag.)}$$



$$\vec{B} = \vec{B}_0 + \mu_0 \vec{M} \qquad \oint_C \vec{B}_0 d\vec{\sigma} = \mu_0 I$$

$$\oint_C \left(\frac{\vec{B} - \mu_0 \vec{M}}{\mu_0} \right) d\vec{\sigma} = I \leftarrow \text{current through the surface}$$

so $$\vec{H} \equiv \frac{\vec{B} - \mu_0 \vec{M}}{\mu_0}$$

$\vec{H}$ magnetic field strength
$\vec{M}$ magnetization vector
$\vec{B}$ magentic induction

For some materials
$\vec{B} = \mu \vec{H}$
$\vec{D} = \epsilon \vec{E}$

Maxwell's Equations

$$\underbrace{\nabla \cdot \vec{B}}_{Div(B)} = 0 \qquad \nabla \vec{D} = \rho$$

$$\underbrace{\nabla \times \vec{H}}_{Curl(H)} - \frac{\partial \vec{D}}{\partial t} = \vec{j} \qquad (\underbrace{\nabla \times \vec{E}}_{Curl(E)}) + \frac{\partial \vec{B}}{\partial t} = 0$$

Extra Assumptions (media dep.)
$\vec{B} = \mu \vec{H}$ and $\vec{D} = \epsilon \vec{E}$

$$\vec{\mathfrak{X}} = \mathfrak{X}^i \frac{\partial}{\partial x^i} \quad \vec{\Upsilon} = \Upsilon^i \frac{\partial}{\partial x^i}$$

$$\mathfrak{X} = \mathfrak{X}_i \, dx^i \quad \Upsilon = \frac{1}{2} \epsilon_{ijk} \Upsilon^i (dx^j \wedge dx^k)$$
(where $\mathfrak{X}^i = \mathfrak{X}_i$)

$$* \quad d\mathfrak{X} = \frac{\partial \mathfrak{X}_i}{\partial t}(dt \wedge dx^i) + Curl(\vec{\mathfrak{X}})^1 (dy \wedge dz) + \ldots + Curl(\vec{\mathfrak{X}})^3 dx \wedge dy$$

$$d\Upsilon = d\left[\Upsilon^1 dy \wedge dz + \Upsilon^2 dz \wedge dx + \Upsilon^3 dx \wedge dy\right]$$

$$= \frac{\partial \Upsilon^1}{\partial t} dt \wedge dy \wedge dz + \ldots + \frac{\partial \Upsilon^3}{\partial t} dt \wedge dx \wedge dy + Div(\vec{\Upsilon}) dx \wedge dy \wedge dz$$

$$* \quad d\Upsilon = dt \wedge \left(\frac{\partial \Upsilon^1}{\partial t} dy \wedge dz + \frac{\partial \Upsilon^2}{\partial t} dz \wedge dx + \frac{\partial \Upsilon^3}{\partial t} dx \wedge dy\right) + Div(\Upsilon) dx \wedge dy \wedge dz$$

2/8/2

Repair last time $\quad M \equiv \mathbb{R}^4 \equiv \mathbb{R}\times\mathbb{R}^3$

$\vec{X} = X^i \partial_i$ let $X = \eta_{ij} X^i dx^j$

$\vec{Y} = Y^i \partial_i$ let $Y = \frac{1}{2}\epsilon_{ijk} Y^i dx^j \wedge dx^k$

$\vec{X}$ and $\vec{Y}$ are time-varying vector fields on $\mathbb{R}^3$

$$d^3 X = -\left[(\mathrm{Curl}(\vec{X}))^1 dy\wedge dz + \ldots\right]$$

$$d^4 X = -\sum \partial_t X^i dt\wedge dx^i + d^3 X = dt\wedge\frac{\partial X}{\partial t} + d^3 X$$

$$d^3 Y = \mathrm{Div}(\vec{Y})\, dx\wedge dy\wedge dz$$

$$d^4 Y = dt\wedge\frac{\partial Y}{\partial t} + d^3 Y$$

$$E = E_1 dx + E_2 dy + E_3 dz = -E^1 dx - E^2 dy - E^3 dz$$

$$B = \tfrac{1}{2}\epsilon_{ijk} B^i dx^j\wedge dx^k,\ H = H_i dx^i,\ D = \tfrac{1}{2}\epsilon_{ijk} D^i dx^j\wedge dx^k$$

where $\vec{E}, \vec{B}, \vec{H}$, and $\vec{D}$ are our v.f.'s

$$(\nabla\times\vec{H}) - \frac{\partial\vec{D}}{\partial t} = (j_1, j_2, j_3) = \vec{j} \qquad \mathrm{Div}(\vec{D}) = \rho$$

$$d^4 D = dt\wedge\frac{\partial D}{\partial t} + \mathrm{Div}(\vec{D})\, dx\wedge dy\wedge dz$$

$$dt\wedge d^4 D = \overbrace{\cancel{dt\wedge dt}}^{0}\wedge\frac{\partial D}{\partial t} + \mathrm{Div}(\vec{D})\, dt\wedge dx\wedge dy\wedge dz$$
$$= \rho\, dt\wedge dx\wedge dy\wedge dz$$

$$d^4(-D + dt\wedge H) = d^4 D + d^4(dt\wedge H) = dt\wedge\frac{\partial D}{\partial t} + \mathrm{Div}(\vec{D})\, dx\wedge dy\wedge dz$$
$$+ \overbrace{\cancel{d^4(dt)}}^{0}\wedge H - dt\wedge d^4 H = dt\wedge\left[\left(\frac{-\partial D}{\partial t}\right) + \underline{d^3(-H)}\right] - \mathrm{Div}(\vec{D})\, dx\wedge dy\wedge dz$$

$$d^4(dt\wedge H - D) = dt\wedge(-j_1 dy\wedge dz - j_2 dz\wedge dx - j_3 dx\wedge dy) + \rho\, dx\wedge dy\wedge dz$$

$\boxed{\text{Def}}$ $G = D - dt \wedge H$ (a 2-form) then we have

$$dG = \rho\, dx \wedge dy \wedge dz - j_1 dt \wedge dy \wedge dz - j_2 dt \wedge dz \wedge dx - j_3 dt \wedge dx \wedge dy$$

"The current 3-form." denote by J $\Rightarrow dG = J$

$\rightarrow$ $*dG = *J = \rho\, dt + j_1 dx + j_2 dy + j_3 dz$ (2 of maxwell's Eqs.)

$dG = J$ is equivalent to $\mathrm{Div}(\vec{D}) = \rho$ and $\mathrm{Curl}(\vec{H}) - \frac{\partial \vec{D}}{\partial t} = \vec{j}$

(we went one way, but the other way goes too)

$\mathrm{Div}(\vec{B}) = 0$ $\quad \mathrm{Curl}(\vec{E}) + \frac{\partial \vec{B}}{\partial t} = 0$

$\boxed{\text{Def}}$ $F = B - dt \wedge E$ then we get $dF = 0$

$dF = 0$ is equivalent to $\mathrm{Div}(\vec{B}) = 0$ and $\mathrm{Curl}(\vec{E}) + \frac{\partial \vec{B}}{\partial t} = 0$

dual in 3-dim.

Constit. Eq. $\vec{B} = \mu \vec{H} \longleftrightarrow B = \mu * H$

$\vec{D} = \epsilon \vec{E} \longleftrightarrow D = \epsilon * E$

We can get

$$G = (\text{const.}) * F$$

thus

$$\left[\begin{array}{c} dF = 0 \\ d((\text{const}) * F) = J \end{array}\right]$$

Poincaré Lemma $\Rightarrow$ for $dF = 0$ $\exists A \ni dA = F$

So in the end we get

$$d * dA = J$$

a system of DE's to solve (any A will then satisfies $dF = 0$)

2/11/2

$\alpha(v) = g(g^{\#}\alpha, v)$

Minkowski Space: $(\mathbb{R}^4)$ $(*^4)$ 4-dual

$*(dx^2 \wedge dx^3) = dx^0 \wedge dx^1$ $*(dx^0 \wedge dx^1) = dx^3 \wedge dx^2$

$*(dx^3 \wedge dx^1) = dx^0 \wedge dx^2$ $*(dx^0 \wedge dx^2) = dx^1 \wedge dx^3$

$*(dx^1 \wedge dx^2) = dx^0 \wedge dx^3$ $*(dx^0 \wedge dx^3) = dx^2 \wedge dx^1$

Euclidean: $(\mathbb{R}^3)$ $(*^3)$ 3-dual

$*dx^1 = dx^2 \wedge dx^3$

$*dx^2 = dx^3 \wedge dx^1$

$*dx^3 = dx^1 \wedge dx^2$

$$*^3 E = E_1 dx^2 \wedge dx^3 + E_2 dx^3 \wedge dx^1 + E_3 dx^1 \wedge dx^2$$

$$= E_1 *^4(dx^1 \wedge dx^0) + E_2 *^4(dx^2 \wedge dx^0) + E_3 *^4(dx^3 \wedge dx^0)$$

$$= *^4(E \wedge dx^0) = -*^4(dt \wedge E)$$

$$(*^3 B) \wedge dt = B^1 dx^1 \wedge dt + B^2 dx^2 \wedge dt + B^3 dx^3 \wedge dt$$

$$= B^1 *^4(dx^3 \wedge dx^2) + B^2 *^4(dx^3 \wedge dx^1) + B^3 *^4(dx^2 \wedge dx^1)$$

$$= -*^4\left(\tfrac{1}{2}\epsilon_{ijk} B^i dx^j \wedge dx^k\right) = -*^4(B)$$

$$\therefore\ *^3 E = -*^4(dt \wedge E) \quad \& \quad dt \wedge (*^3 B) = *^4(B)$$

$$F = B + dt \wedge E, \quad G = D - dt \wedge H$$

$$*^4 F = *^4 B + *^4(dt \wedge E) = dt \wedge (*^3 B) - *^3 E$$

with $c=1$ we get (for some materials) $\mu\epsilon = 1$

$$B = -\mu *^3 H \qquad D = -\epsilon *^3 E$$

$$*^4 F = \mu(dt \wedge (-H)) + \cancel{\tfrac{1}{\epsilon}}^{\,\mu} D = \mu(D - (dt\wedge H))$$

$$\Rightarrow *^4 F = \mu G$$

<u>Our Equations are</u>: $dF = 0$, $dG = J$, $*^4 F = \mu G$

$$\boxed{\text{Maxwell's Eq's} \quad dF = 0, \quad d(*^4 F) = \mu J}$$

$$(F_{\mu\nu}) = \begin{array}{c} \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{bmatrix} \end{array} \begin{array}{c} t \\ x \\ y \\ z \end{array}$$

$dx\wedge dy$'s coeff ✓

$dy\wedge dz$'s coeff ✓

$$(G_{\mu\nu}) = \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{bmatrix} 0 & H^1 & H^2 & H^3 \\ -H^1 & 0 & D^3 & -D^2 \\ -H^2 & -D^3 & 0 & D^1 \\ -H^3 & D^2 & -D^1 & 0 \end{bmatrix} \end{array}$$

Thm M a manifold w/ metric g. If $\{e_i\}$ is a g-orthonormal basis of T_pM for some $p \in M$ then $\{e^{i_1}\wedge \dots \wedge e^{i_k} \mid 1 \le i_1 < \dots < i_k \le n\}$ is an orthonormal basis of $\Lambda^k T_pM^*$ relative to $\tilde{g}$. The index of $\tilde{g}$ is the # of inc. sequences $i_1 < \dots < i_k$ for which $g(e_{i_1}, e_{i_1}) \dots g(e_{i_k}, e_{i_k}) = -1$ $\left(\rightarrow \prod_{j=1}^{k} g(e_{i_j}, e_{i_j}) = -1\right)$

Thm $L: V \rightarrow V$ an isometry. $*(L^*\beta) = L^*(*\beta)$ for forms β

<u>In Exercises</u>: $L \cdot \beta = L^* \beta$

Handout:
Bleecker pg 18 – 22

2/13/2

$\boxed{\text{Thm}}$ The index of $\tilde{g}$ is the numeber of $i_1 < \dots < i_k \ni$
$g_{i_1 i_1} \dots g_{i_k i_k} = -1$ and $\{e^{i_1} \wedge \dots \wedge e^{i_k}\}$ are o.n.

$\boxed{\text{Def}}$ V and W are vector spaces with metrics g & h and volumes μ_g & μ_h resp. then a linear map $L: V \to W$ is an <u>isometry</u> iff L is invertible and $h(L(v), L(w)) = g(v, w)$

In this case you can show $\tilde{g}(L^*\alpha, L^*\beta) = \tilde{h}(\alpha, \beta)\ \forall \alpha, \beta \in \Lambda^p W^*$

We say L is <u>orientation preserving</u> iff $L^*\mu_h = \mu_g$

$\boxed{\text{Thm}}$ Let V, W be vector spaces with metrics g, h and volumes μ_g, μ_h. Let $L: V \to W$ be an orientation pres. isometry, then
$$*(L^*\beta) = L^*(*\beta)$$

<u>proof</u>
Observe that for $\beta \in \Lambda^k W^*$, $L^*\beta \in \Lambda^k V^*$, $*(L^*\beta) \in \Lambda^{n-k} V^*$
moreover $*\beta \in \Lambda^{n-k} W^*$ and $L^*(*\beta) \in \Lambda^{n-k} V^*$ (so they're the right size)

$$\tilde{g}(\alpha, L^*(*\beta))\mu_g = \tilde{g}(L^*((L^{-1})^*\alpha), L^*(*\beta))\mu_g$$
$$= \tilde{h}((L^{-1})^*\alpha, *\beta)(L^*\mu_h) = L^*(\tilde{h}((L^{-1})^*\alpha, *\beta)\mu_h)$$
$$= L^*[(L^{-1})^*\alpha \wedge (**\beta)] = L^*(L^{-1})^*\alpha \wedge L^*(**\beta) = \alpha \wedge L^*(**\beta)$$
$$= \alpha \wedge (-1)^{k(n-k)+s+1} ** L^*((-1)^{k(n-k)+s+1}\beta)$$
$$= \tilde{g}(\alpha, \cancel{(-1)^{2(\dots)}} *(L^*\beta))\ \forall \alpha \in \Lambda^k V^*$$
$$\Rightarrow L^*(*\beta) = *(L^*\beta) \ //$$

Isometry $\Rightarrow$

$\{e_i\}$ o.n. $g(e_i, e_j) = h(L(e_i), L(e_j))$ signature is same

$\boxed{\text{Cor}}$ M is a manifold, g is a metric on M, μ_g a compat. volume.
If $\varphi : M \to M$ is an orientation preserving isometry then
$$\varphi^*(*\beta) = *(\varphi^*\beta) \quad \forall \beta \in \Omega^k M$$
(φ or. pres. isom. $\Leftrightarrow$ $d_p\varphi : T_pM \to T_{\varphi(p)}M$ or. pres. isom. $\forall p \in M$)

Ex: $M = \mathbb{R}^4$ Minkowski $g_p = \eta$ (identify $T_pM \equiv T_0M \equiv \mathbb{R}^4$)
$\eta(v,w) = v^0w^0 - v^1w^1 - v^2w^2 - v^3w^3$.

A Lorentz transformation is a linear isometry $\Lambda : \mathbb{R}^4 \to \mathbb{R}^4$
(call it proper if also orientation pres.)

$$\Lambda(e_i) = \Lambda_i^j e_j \quad \eta(\Lambda_i^j e_j, \Lambda_k^\ell e_\ell) = \Lambda_i^j \Lambda_k^\ell \eta_{j\ell} = \eta(e_i, e_k) =$$
$$\Lambda_i^j \Lambda_k^\ell \eta_{j\ell} = \eta_{ik} \quad \dots \Leftrightarrow \boxed{\Lambda^t \eta \Lambda = \eta}$$

$\boxed{\text{Thm}}$ The vacuum Maxwell equations
$$dF = 0, \ dG = 0, \ *F = \mu G$$
are invariant under the action of the Lorentz group.

proof
$$d(\Lambda \cdot F) = d((\Lambda^{-1})^* F) = (\Lambda^{-1})^* \cancel{dF}^{\,0} = 0 \ \checkmark$$
$$d(\Lambda \cdot G) = (\Lambda^{-1})^* \cancel{dG}^{\,0} = 0 \ \checkmark$$
$$*(\Lambda \cdot F) = *((\Lambda^{-1})^* F) = (\Lambda^{-1})^*(*F) = (\Lambda^{-1})^*(\mu G) =$$
$$\mu (\Lambda^{-1})^* G = \mu(\Lambda \cdot G) \ \checkmark$$
$$(L \cdot \alpha(v_1, v_2) = \alpha(L^{-1}(v_1), L^{-1}(v_2)) = (L^{-1})^* \alpha(v_1, v_2))$$

Notice $*F = \mu G \Rightarrow d\mu G = 0$ ie $d*F = 0$
$\longleftrightarrow dF = 0$ & $\delta F = j$
Poincaré Lemma $\Rightarrow \exists A \ni dA = F \Rightarrow \delta dA = j$
$\delta \bar{A} = 0$, $\Delta \bar{A} = j \Rightarrow \delta d\bar{A} = j \Rightarrow \dots$
$\bar{A} = A + df$ $\quad\dots$
$\leftarrow$ Gauge Freedom

2/15/2

$\boxed{\text{Def}}$ A <u>Lie group</u> is a manifold G with group structure $\ni$
$$\mu : G \times G \to G \qquad iv : G \to G \qquad \mu(g_1, g_2) = g_1 \cdot g_2 \quad iv(g) = g^{-1}$$
are smooth.

Fix a Lie group G.

For $g \in G$, $L_g : G \to G$ def by $L_g(x) = gx$ (left mult.)

<u>note</u> L_g is smooth $\quad G \overset{i}{\hookrightarrow} \{g\} \times G \xrightarrow{\mu} G$

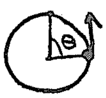
Similarly, define R_g (right mult.)

$\sigma_g : G \to G$ def by $\sigma_g(x) = gxg^{-1}$ $\quad$ ($\sigma_g = R_{g^{-1}} \cdot L_g$ hence smooth)

$\ell : T_eG \to \mathfrak{X}(G)$ (= vector fields on G)

$A \in T_eG \quad \ell(A) = \ell_A \in \mathfrak{X}(G) \quad \ell_A(g) = d_e L_g(A)$

$d_e L_g : T_eG \to T_gG$

<u>Ex:</u>  $\quad \theta : S^1 \to \mathbb{R}$ (coordinate is just the angle)
$$d_e L_\varphi \left(\frac{\partial}{\partial \theta}\Big|_e \right) = \frac{\partial}{\partial \theta}\Big|_\varphi \quad \Rightarrow \quad \ell_{\frac{\partial}{\partial \theta}|_e}(\varphi) = \frac{\partial}{\partial \theta}\Big|_\varphi$$

<u>Remark</u>: $d_x L_g(\ell_A(x)) = \ell_A(L_g(x)) = \ell_A(gx)$

<u>proof</u>
$$d_x L_g(\ell_A(x)) = d_x L_g(d_e L_x(A)) = d_e L_g \circ L_x(A) =$$
$$d_e L_{gx}(A) = \ell_A(gx) \; //$$

Let $\mathfrak{X}_{inv}(G) = \{ X \in \mathfrak{X}(G) \mid d_x L_g(X(x)) = X(gx) \}$

called these "Left invariant" vector fields

<u>note</u>: $\ell_A \in \mathfrak{X}_{inv}(G) \quad \forall A \in T_eG$

$\rightarrow \; \ell : T_eG \to \mathfrak{X}_{inv}(G)$ want to show 1-1/onto & linear

$$\ell(\alpha A_1 + A_2)(g) = d_e L_g(\alpha A_1 + A_2) = \alpha\, d_e L_g(A_1) + d_e L_g(A_2)$$
$$= \alpha\, \ell(A_1)(g) + \ell(A_2)(g)$$ (basically b/c differential is lin.)

Sp $\ell(A) = 0 \Rightarrow d_e L_g(A) = 0 \ \forall g \in G \Rightarrow$
$$d_g L_{g^{-1}}(d_e L_g(A)) = d_e L_{g^{-1}g}(A) = d_e L_e(A) = A = 0 \checkmark$$

We show ℓ is onto: Let $\mathfrak{X} \in \mathcal{X}_{inv}(G)$ Let $A = \mathfrak{X}(e)$
$\ell_A(g) = d_e L_g(A) = d_e L_g(\mathfrak{X}(e)) = \mathfrak{X}(ge) = \mathfrak{X}(g) \ \forall g \in G$
$\Rightarrow \ell_A = \mathfrak{X}$ //

ℓ is an isomorphism $\checkmark$

<u>Claim</u>: $\mathcal{X}_{inv}(G)$ is a Lie Algebra (a subalg. of $\mathcal{X}(G)$)

Lie derivative $L_x(y) = [x, y]$ (thus $L_x[y,z] = [L_x y, z] + [y, L_x z]$)

$[\mathfrak{X}, \mathfrak{Y}]^i = \mathfrak{X}(\mathfrak{Y}^i) - \mathfrak{Y}(\mathfrak{X}^i)$ $(= d\mathfrak{Y}^i(\mathfrak{X}) - d\mathfrak{X}^i(\mathfrak{Y}))$ where (U, x) a chart $[\mathfrak{X}, \mathfrak{Y}] = [\mathfrak{X}, \mathfrak{Y}]^i \partial_i$ (or $[\mathfrak{X}, \mathfrak{Y}](f) = \mathfrak{X}(\mathfrak{Y}(f)) - \mathfrak{Y}(\mathfrak{X}(f))$) then $\mathcal{X}(G)$ w/ $[,]$ is a Lie Alg. (skip proof)

Let $\varphi: M \to N$, let $\mathfrak{X}, \mathfrak{Y}$ be v.f.'s on M and N resp. then we say $\mathfrak{X}$ is <u>φ-related</u> to $\mathfrak{Y}$ iff $d_p\varphi(\mathfrak{X}(p)) = \mathfrak{Y}(\varphi(p)) \ \forall p$

$\boxed{\text{Thm}}$ If $\varphi: M \to N$, $\mathfrak{X} \overset{\varphi}{\sim} \mathfrak{Y}$ and $Z \overset{\varphi}{\sim} W$ then $[X, Z] \overset{\varphi}{\sim} [Y, W]$

<u>Note</u>: $\ell_A \overset{L_g}{\sim} \ell_A \Rightarrow [\ell_A, \ell_B] \overset{L_g}{\sim} [\ell_A, \ell_B]$

and ℓ_A left inv. $\Leftrightarrow \ell_A \overset{L_g}{\sim} \ell_A$ $\therefore$ $[,]$ closed in $\mathcal{X}_{inv}(G)$
hence $\mathcal{X}_{inv}(G)$ is a sub.alg.

HMWK #4

2/18/2

Maurer-Cartan Structure equations

$\beta^i(g) \in T^*_g G$

$\beta^i(e) = e^i \qquad d\beta^i = -\frac{1}{2} f^i_{jk}(d\beta^j \wedge d\beta^k)$

$\beta^i = \ell e^i, \quad X_i = \ell e_i \quad X_i(e) = e_i$

$[X_j, X_k] = f^i_{jk} X_i$ $\qquad \{X_i\}$ basis for Lie Alg. & f^i_{jk} are structure consts.

$\beta \in \Lambda^k G$

$$d\beta(X_1, \dots, X_{k+1}) = \sum (-1)^{i+1} X_i(\beta(X_1, \dots, \hat{X}_i, \dots, X_{k+1})) + \sum (-1)^{i+j} \beta([X_i, X_j], X, \dots)$$

$$d\beta(X, Y) = X(\beta(Y)) - Y(\beta(X)) - \beta([X, Y])$$

Cartan's Magic Formula $\mathcal{L}_X \beta = i_X d + d\, i_X$

(1) We defined $\ell : T_e G \to \mathfrak{X}_{inv}(G)$ vector space

$\ell(A) \equiv \ell_A, \quad \ell_A(g) = d_e L_g(A)$ We prove ℓ is an isomorphism

(2) $\ell_A \overset{L_g}{\sim} \ell_A \quad \ell_B \overset{L_g}{\sim} \ell_B \Rightarrow [\ell_A, \ell_B] \overset{L_g}{\sim} [\ell_A, \ell_B]$

$\left[\begin{matrix} X \overset{\varphi}{\sim} Y \\ Z \overset{\varphi}{\sim} W \end{matrix} \Rightarrow [X, Z] \overset{\varphi}{\sim} [Y, W] \quad \begin{matrix} X, Z \in \mathfrak{X}(M) \\ Y, W \in \mathfrak{X}(N) \end{matrix} \quad \varphi : M \to N \text{ smooth.} \right]$

$d_g L_h(\ell_c(g)) = \ell_c(L_h(g)) = \ell_c(hg) \quad \therefore \ell_c \overset{L_h}{\sim} \ell_c$

$X \overset{\varphi}{\sim} Y \Leftrightarrow d_p\varphi(X(p)) = Y(\varphi(p))$

(3) $\ell_A \overset{L_g}{\sim} \ell_A \; \forall g \Leftrightarrow \ell_A \in \mathfrak{X}_{inv}(G) \Rightarrow [\ell_A, \ell_B] \in \mathfrak{X}_{inv}(G) \; \forall \ell_A, \ell_B \in \mathfrak{X}_{inv}(G)$

(4) Since $\ell : T_e G \to \mathfrak{X}_{inv}(G)$ is onto, we can write ℓ_A, ℓ_B $\forall$ any elements of $\mathfrak{X}_{inv}(G)$ Define $[A, B] = \ell^{-1}([\ell_A, \ell_B])$

$\hookrightarrow$ Thus we induce a Lie Algebra on $T_e G$ and,

$\ell([A, B]) = \ell(\ell^{-1}[\ell_A, \ell_B]) = [\ell_A, \ell_B] \Rightarrow \ell$ is a Lie Algebra isomorphism.

<u>Summary</u>: $T_e G \cong \mathfrak{X}_{inv}(G)$ (as Lie Algebras)

$\mathfrak{X}_{inv}(G) \subseteq \mathfrak{X}(G)$

finite dim'l
Real vector space

infinite dim'l
Real vectorspace
But finite dim'l $\mathcal{F}(G)$-module

(5) If $\{e_i\}$ is a basis of T_eG then $[\ell e_j, \ell e_k] = f^i_{jk} \ell e_i$ for some $f^i_{jk} \in \mathbb{R}$.

proof $\{e_i\}$ a basis of $T_eG \Rightarrow \{\ell e_i\}$ a basis of $\mathfrak{X}_{inv}(G)$
and $[\ell e_j, \ell e_k] \in \mathfrak{X}_{inv}(G)$

Note:

$$[f\mathbb{X}, g\mathbb{Y}] = \mathcal{L}_{f\mathbb{X}}(g\mathbb{Y}) = L_{f\mathbb{X}}(g)\mathbb{Y} + g L_{f\mathbb{X}}(\mathbb{Y})$$

$$= f\mathbb{X}(g)\mathbb{Y} + g[f\mathbb{X}, \mathbb{Y}] = \ldots = f\mathbb{X}(g)\mathbb{Y} - g\mathbb{Y}(f)\mathbb{X} + fg[\mathbb{X}, \mathbb{Y}]$$

so if f, g constant you get $[f\mathbb{X}, g\mathbb{Y}] = fg[\mathbb{X}, \mathbb{Y}]$

(6) $\forall A \in T_eG$, ℓ_A is complete. This means that if γ is a solution of ℓ_A through the identity then γ is defined $\forall$ time.

$$\gamma'(t) = \ell_A(\gamma(t)) \qquad \gamma(0) = e$$

2/20/2

(6) For each $A \in T_eG$, l_A is <u>complete</u>.
($ie\ \exists!\ \gamma : (-\infty,\infty) \to G$, $\gamma(0) = e$, $\gamma'(t) = l_A(\gamma(t))$)

$$\gamma'(t) = \sum \frac{d}{dt}(x^i \circ \gamma)(t) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)} \qquad l_A(p) = l_A^i(p)\frac{\partial}{\partial x^i}\Big|_p$$

$$\gamma'(t) = l_A(\gamma(t)) \iff \boxed{\frac{d}{dt}(x^i \circ \gamma)(t) = l_A^i(\gamma(t))} \ \forall \text{ chart}$$

<u>Flow</u>: $X \in \mathfrak{X}(M)$ then φ_t is a flow for X ($\varphi_t : M \to M$) $t \in I_p$ open int about zero

$$\frac{d}{dt}(\varphi_t(p)) = X(\varphi_t(p)) \ \& \ \varphi_t(p)\Big|_{t=0} = p$$

$\hookrightarrow \varphi_t(\varphi_s(p)) = \varphi_{t+s}(p)$

If $\tilde{\varphi}(t,p) = \varphi_t(p)$ then $\tilde{\varphi} : \mathcal{D} \subseteq \mathbb{R} \times M \to M$ smooth (open maximal.)

$gl(n) = \{A \mid A \text{ } n\times n \text{ matrices}\}$ $\quad Gl(n) = \{A \in gl(n) \mid \det(A) \neq 0\}$

- $\det : gl(n) \to \mathbb{R}$ (smooth because its a polynomial in the entries of the matrices)

$\Rightarrow Gl(n) = \det^{-1}(\mathbb{R} - \{0\})$ open in $gl(n)$ (det cont. / $\mathbb{R}-\{0\}$ open)

- $\exp : gl(n) \to Gl(n)$ $\quad (\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots)$

<u>note</u>: exp does converge $\forall A$, and $\det(\exp(A)) = e^{tr(A)} \neq 0$

$$\exp(A+B) = \exp(A)\exp(B) \text{ if } [A,B] = 0$$

$$t \longmapsto \exp(tA) \quad \& \quad s+t \longmapsto \exp((s+t)A) = \exp(sA)\exp(tA)$$

$\therefore$ homomorphism. $\quad \frac{d}{dt}[\exp(tA)] = \exp(tA)A \ (= A\exp(tA))$

$$\varphi(t,A) = \exp(tA)$$

then $\frac{d}{dt}\varphi(t,A) = A\exp(tA) = A\varphi(t,A)$ also $\varphi(0,A) = I$

$\therefore \gamma_A(t) = \exp(tA)$, $\gamma_A(0) = I$, and $\gamma_A'(0) = A$
and $\gamma_A(1) = \exp(A)$

proof of (6) l_A is complete

Sketch

$\exists\, \gamma(-a,b) \to G \ni \gamma(0) = e$ (by uniqueness existance)
$\gamma'(t) = l_A(\gamma(t))$ let $(-a,b)$ be maximal
$\$$ b is finite. Let $s > b/2$, $\gamma_1(t) = \gamma(s+t)$, $\gamma_2(t) = \gamma(s)\gamma(t)$
$\gamma_1'(t) = \gamma'(s+t) = l_A(\gamma(s+t)) = l_A(\gamma_1(t))$
$\gamma_2'(t) = l_A(\gamma_2(t))$ $\gamma_1(0) = \gamma(s)$ $\gamma_2(0) = \gamma(s)$
$\Rightarrow \gamma_1 = \gamma_2$ but γ_2 is defined on $(-a,b)$ and γ_1 is defined on $(-a,b) \Rightarrow \gamma_1(s+s)$ is defined but $2s > b$ $\rightarrow\leftarrow$
$\therefore b$ infinite likewise a infinite

Then we show $\psi_t(g) = g\gamma_A(t)$ is the flow //

Handout: pgs 1–7
pg 1 starts Let $gl(n) = \ldots$

2/22/2

Def Let B be an algebra with norm $\|\cdot\|$, if B is complete w/t respect to $\|\cdot\|$ and $\|ab\| \leq \|a\|\|b\|$ (also $\exists e \in B$ a mult. id and $\|e\| = 1$) then B is a <u>Banach Algebra</u>

flow of a vector field: (for a complete v.f.)

$$\mathbb{R} \xrightarrow{\varphi} \text{Diff}\, M \ni \varphi(t,p) = \varphi_t(p)$$

$$\frac{d}{dt}[\varphi_t(p)] = X(\varphi_t(p)) \qquad \varphi_t(p)\Big|_{t=0} = p$$

φ is a hom, and $\varphi_0 = id_M$

Def X a vector field then $\mathcal{D} \subseteq \mathbb{R} \times M$ (open) $\tilde{\varphi} : D \to M$ smooth

$$\ni \frac{d}{dt}[\tilde{\varphi}(t,p)] = X(\tilde{\varphi}(t,p)) \qquad \tilde{\varphi}(0,p) = p$$

then we call $\tilde{\varphi}$ a <u>flow</u> of X

Thm On a smooth vector field $\exists$ a flow $\tilde{\varphi}$ with maximal domain $\mathcal{D} \subseteq \mathbb{R} \times M$

We say X is <u>complete</u> if $\mathcal{D} = \mathbb{R} \times M$ ($\varphi_t(p)$ defined $\forall t \in \mathbb{R}, p \in M$)

l_A is <u>complete</u>

<u>proof</u> $\psi_A(t,g) = g\gamma_A(t)$, $\gamma_A'(t) = l_A(\gamma(t))$, $\gamma_A(0) = e$

now γ def. $\forall t \in \mathbb{R} \Rightarrow \psi_A$ defined on $\mathbb{R} \times G$, smooth because γ is smooth & mult. is smooth ✓

Now, $\frac{d}{dt}(\psi_A(t,g)) = \frac{d}{dt}(g\gamma_A(t)) = \frac{d}{dt}[L_g \gamma_A(t)]$

$$= d_{\gamma_A(t)} L_g(\gamma_A'(t)) = d_{\gamma_A(t)} L_g(l_A(\gamma_A(t))) = l_A(L_g(\gamma_A(t)))$$

$$= l_A(g\gamma_A(t)) = l_A(\psi(t,g)) \quad \text{also } \psi(0,g) = g\gamma_A(0) = ge = g \checkmark$$

flow on ℓ_A

(7) Define $\exp: T_eG \to G$ by $\exp(A) = \psi_A(1,e)$ $(= \gamma_A(1))$

Claim: $\exp(sA) = \gamma_A(s)$

proof

Isom.

$$\frac{d}{dt}(\psi_{sA}(t,e)) = \frac{d}{dt}(\gamma_{sA}(t)) = \ell_{sA}(\gamma_{sA}(t)) = s\,\ell_A(\gamma_{sA}(t))$$

$$\frac{d}{dt}(\gamma_A(st)) = \gamma_A'(st)\cdot s = s\,\ell_A(\gamma_A(st))$$

now $\gamma_A(s\cdot 0) = \gamma_A(0) = e$ and $\gamma_{sA}(0) = e \Rightarrow$

$\gamma_{sA}(t) = \gamma_A(st)$ (by uniqueness & existence)

$\therefore \gamma_{sA}(1) = \gamma_A(s) \Rightarrow \exp(sA) = \gamma_A(s)$ //

(8) exp is a local diffeomorphism at 0.

proof

$$\gamma_A(s) = \exp(sA) \Rightarrow \gamma_A'(s) = d_{sA}(\exp)(A)$$

$\Rightarrow \gamma_A'(0) = d_0\exp(A)$ but $\gamma_A'(0) = \ell_A(\gamma_A(0)) = \ell_A(e) = A$

$\therefore A = d_0\exp(A) \Rightarrow d_0\exp = I_{T_eG} \Rightarrow d_0\exp$ is invertible

by inverse function thm $\exists\, \mathcal{U}$ open about 0 in $T_eG \ni$

$\hookrightarrow$ $\exp|_{\mathcal{U}}$ is a diffeomorphism onto an open subset of G

$\therefore$ We can use exp as a chart about 0

2/25/2

(9) Identify $\mathfrak{X}_{inv}(Gl(n))$:

$A \in Gl(n) \subseteq gl(n)$ $\quad L_A : gl(n) \longrightarrow gl(n)$ is a linear map,

$\Rightarrow d_p L_A : T_p gl(n) \longrightarrow T_{Ap} gl(n)$ $\quad$ If $T_x \mathbb{R}^n \equiv \mathbb{R}^n$ then $d_p L_A = L_A$

$U^{open} \subseteq \mathbb{R}^n$ then identify $T_p U = T_p \mathbb{R}^n$ so because $Gl(n)$ open in $gl(n)$ we can identify $T_x Gl(n) = T_x gl(n)$

Define a chart: $x^{ij}(A) = A^{ij}$ (i-j th component of A)

$T_I Gl(n) = \{ A^{ij} \partial/\partial x^{ij}|_I \mid A \in gl(n) \} \equiv gl(n)$

$A \in T_I Gl(n)$ $\left(\Rightarrow A = A^{ij} \partial/\partial x^{ij}|_I\right)$

For $B \in Gl(n)$, $l_A(B) \in T_B Gl(n)$ $\quad l_A(B) = d_I L_B(A) = L_B A = BA$

$l_A(B) = (BA)^{ij} \partial/\partial x^{ij}|_B$ $\quad l_A(B) = \sum_{i,j,k} B^{ik} A^{kj} \partial/\partial x^{ij}|_B$

$l_A(B) = A^{kj} x^{ik}(B) \partial/\partial x^{ij}|_B$

$\therefore l_A = A^{kj} x^{ik} \partial/\partial x^{ij}$ $\quad \mathfrak{X}_{inv}(Gl(n)) = \{ A^{kj} x^{ik} \partial/\partial x^{ij} \mid A \in gl(n) \}$

(10) $[l_{A_1}, l_{A_2}]_{Lie} = l_{[A_1,A_2]_{matrix}}$ $(= l_{A_1A_2 - A_2A_1})$ $\quad A_1, A_2 \in T_I Gl(n) = gl(n)$

$[l_{A_1}, l_{A_2}]^{pq} = \left[A_1^{kj} x^{ik} \partial/\partial x^{ij}, A_2^{kj} x^{ik} \partial/\partial x^{ij} \right]^{pq}$

$= A_1^{kj} x^{ik} \partial/\partial x^{ij} \left(A_2^{kq} x^{pk} \right) - A_2^{kj} x^{ik} \partial/\partial x^{ij} \left(A_1^{kq} x^{pk} \right)$

$= l_{A_1}^{ij} A_2^{kq} \delta^{ip} \delta^{jk} - l_{A_2}^{ij} A_1^{kq} \delta^{ip} \delta^{jk} = l_{A_1}^{pj} A_2^{jq} - l_{A_2}^{pj} A_1^{jq}$

$= A_1^{lk} A_2^{kq} x^{pl} - A_2^{lk} A_1^{kq} x^{pl} = [A_1, A_2]^{lq} x^{pl} = l^{pq}_{[A_1,A_2]}$

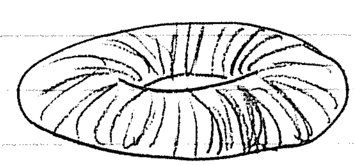
Lie Subgroups

H is a Lie subgroup of a Lie group G iff
(1) H is a subgroup of G.
(2) H is a submanifold of G (strong sense)
- $H \subseteq G$, H has relative topology, $i: H \hookrightarrow G$ is smooth.

Ex: Dense Wind on the Torus (not a sub. man.)

$G = U(1) \times U(1)$, $U(1) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$

$H = \{(e^{i\theta}, e^{i\alpha\theta}) \mid \theta \in \mathbb{R}, \alpha \text{ irrational}\}$



H is a Lie group,

$$\left\{ \begin{array}{l} \mu_H|_{H\times H} \ (= \mu \circ (i_H \times i_H)) \text{ hence smooth} \\ i_V|_H \ (= i_V \circ i_H) \text{ hence smooth} \end{array} \right\}$$

$L_h^G \circ i_H = L_h^H$ (ie Left mult. in H is Left mult in G restr. to H)

$d_e L_h^H = d_e L_h^G \circ d_e i_H = d_e L_h^G|_{T_eH}$ $\quad (d_e i_H : T_eH \to T_eG)$

$h \in H,\ A \in T_eH \quad \ell_A^H(h) = d_e L_h^H(A) = d_e L_h^G(A) = \ell_A^G(h)$

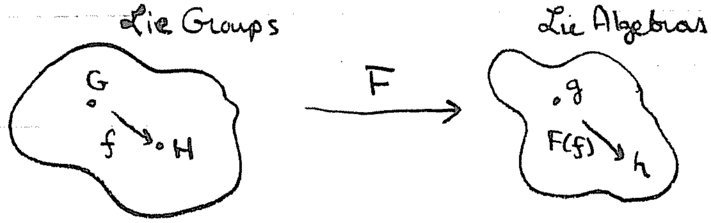
- $\exp^H(A) = \exp^G(A) \ \forall A \in T_eH$. $\quad \exp^G|_{T_eH} = \exp^H$

Ex: $O(n)$ $\quad \lambda \in I \subseteq \mathbb{R}$, $B(\lambda)^T B(\lambda) = I$ $\quad B(\lambda)$ a smooth curve in $O(n)$

$\ni B(0) = I$ then $B(\lambda)^T \frac{\partial B}{\partial \lambda}(\lambda) + \left(\frac{\partial B}{\partial \lambda}(\lambda)\right)^T B(\lambda) = 0$

$$\Rightarrow I \frac{\partial B}{\partial \lambda}(0) + \left(\frac{\partial B}{\partial \lambda}(0)\right)^T I = 0 \Rightarrow o(n) = \{A \in gl(n) \mid A + A^T = 0\}$$

2/27/2



The functor F is onto but not 1-1
1-1 If we restrict to simply connected Lie Grps.

and If G simply con. $F(G) = \mathfrak{g}$ then if $F(\overline{G}) = \mathfrak{g}$ also
$\exists$ a discrete group $D \ni G/D = \overline{G}$

---

(10) Let G and H be Lie groups. $\mathfrak{g} = T_eG$, $\mathfrak{h} = T_eH$ the corresponding Lie Algebras.
If $F: G \to H$ is a Lie Group homomorphism.
then $d_eF = F_*: \mathfrak{g} \to \mathfrak{h}$ is a Lie Algebra hom.

[$\boxed{\text{Def}}$ $F: G \to H$ is a <u>Lie Group hom.</u> iff F is smooth and F is a group hom.]

- In particular, $i: H \hookrightarrow G$ is a Lie grp. hom, (1-1) $\Rightarrow$ $i_*: \mathfrak{h} \to \mathfrak{g}$ is a Lie alg. hom. (also 1-1) $\therefore$ $\mathfrak{h}$ is a subalgebra $\mathfrak{g}$.

<u>proof</u>

$F: G \to H$ smooth, grp. hom.

<u>Claim</u>: $\ell_A^G \overset{F}{\sim} \ell^H_{d_eF(A)}$ $A \in T_eG$

$\boxed{X \overset{\varphi}{\sim} Y \text{ iff } d_p\varphi(X_p) = Y_{\varphi(p)}}$

$$d_gF(\ell_A^G(g)) = d_gF(d_eL_g^G(A)) = d_e(F \circ L_g^G)(A)$$

[but F is a hom.

$$(F \circ L_g^G)(h) = F(L_g^G(h)) = F(gh) = F(g)F(h) = L^H_{F(g)}(F(h))$$

$$\Rightarrow F \circ L_g^G = L^H_{F(g)} \circ F$$]

$$= d_e(L^H_{F(g)} \circ F)(A) = d_{F(g)}L^H_{F(g)}(d_eF(A)) = \ell^H_{d_eF(A)}(F(g))$$

$$\therefore \ell_A^G \overset{F}{\sim} \ell^H_{d_eF(A)} \Rightarrow [\ell_A^G, \ell_B^G] \overset{F}{\sim} [\ell^H_{d_eF(A)}, \ell^H_{d_eF(B)}]$$

$\therefore\ d_eF([l_A^G, l_B^G](e)) = [l^H_{d_eF(A)}(e), l^H_{d_eF(B)}(e)]$

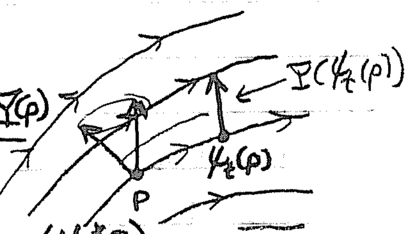
$\Rightarrow d_eF([A,B]) = [d_eF(A), d_eF(B)]$ $\therefore$ d_eF pres. brackets.

$\boxed{\text{Def}}$ X, Y vector fields on a manifold M define

$$(L_X Y)(p) = \frac{d}{dt}\left[d_{\psi_t(p)}\psi_{-t}(Y(\psi_t(p)))\right]\Big|_{t=0}$$

where ψ_t is the flow of X

(*Lie derivative* of a vector field)



Also, $L_X Y(p) = \lim_{t\to 0} \frac{d_{\psi_t(p)}\psi_{-t}(Y(\psi_t(p))) - Y(p)}{t}$

for,

β a 1-form on M, $L_X\beta(p) = \lim_{t\to 0}\frac{(\psi_t^*\beta)_p - \beta_p}{t}$

V be a finite dimensional vector space

$gl(V) = \{\varphi : V \to V \mid \varphi \text{ linear}\}$

$Gl(V) = \{\varphi \in gl(V) \mid \varphi \text{ invertible}\}$

Let $V, \|\cdot\|$ be a Banach Space, then $Gl(V) \overset{\text{open}}{\subseteq} gl(V)$

Let G be a Lie group. $\sigma : G \to \mathrm{Hom}(G,G)$ (Lie Hom)

$\sigma(g)(h) = ghg^{-1}$, $\sigma(g) \in \mathrm{Hom}(G,G)$ [b/c $\sigma(g)(x_1x_2) =$
$gx_1x_2g^{-1} = gx_1g^{-1}gx_2g^{-1} = \sigma(g)(x_1)\sigma(g)(x_2)$ and σ smooth
because mult. & inv. are smooth.]

σ is a hom. [b/c $\sigma(g_1g_2)(h) = g_1g_2h(g_1g_2)^{-1} = g_1(g_2hg_2^{-1})g_1^{-1}$
$= \sigma(g_1)\circ\sigma(g_2)(h)$]

$\sigma : G \to \mathrm{Hom}(G,G)$ is a hom. (not 1-1 since $Z(G) \to$ identity)

now $\sigma_g^{-1} = \sigma_{g^{-1}} \Rightarrow \sigma(g)$ is 1-1

$\mathrm{Ad} : G \to Gl(\mathfrak{g})$, $\mathrm{Ad}(g) = d_e\sigma_g$

$d_e\sigma_{g^{-1}}\, d_e\sigma_g = d_e\sigma_g^{-1} d_e\sigma_g = d_e\, id_G = id_{\mathfrak{g}}$ $\therefore$ $\mathrm{Ad}(g) \in Gl(\mathfrak{g})$

$(\mathrm{Ad}(g)\circ\mathrm{Ad}(g^{-1}) = id)$

$\mathrm{Ad} : G \to Gl(\mathfrak{g})$ is a Lie Hom.

$\mathrm{ad} : \mathfrak{g} \to T_e Gl(\mathfrak{g}) = T_e gl(\mathfrak{g})$ $\mathrm{ad}(x) = d_e(\mathrm{Ad})(x)$

3/1/2

$$\beta^i(X) = \beta^i(\lambda^j X_j) = \lambda^i \Rightarrow \text{smooth since } \beta^i \text{ \& } X \text{ are.}$$

$$\therefore \forall X \in \mathcal{X}(G)\ \exists f^i \in \mathcal{F}(G) \ni X = f^i X_i \quad (X_i \in \mathcal{X}_{inv}(G))$$

$$\det e^B = e^{\operatorname{tr} B}$$

$$SU(n) = \{A \in gl(n) \mid A^\dagger A = I,\ \det A = 1\}$$

$$u(n) = \{B \in gl(n, \mathbb{C}) \mid B^\dagger = -B\}$$

$$1 = \det e^B = e^{\operatorname{tr} B} \Rightarrow \operatorname{tr} B = 0 \quad \therefore su(n) = \{B \mid B^\dagger = -B,\ \operatorname{tr} B = 0\}$$

X, Y vector fields on M,

$$(L_X Y)(p) = \frac{d}{dt}\left[d_{\psi_t(p)} \psi_{-t}(Y(\psi_t(p)))\right]\Big|_{t=0}$$

where ψ_t is the flow of X

$\boxed{\text{Lemma}}$ $L_X Y = [X, Y]$ proof (maybe later)

$\boxed{\text{Thm}}$ $A, B \in \mathfrak{g} = T_e G$

$$ad(A)(B) = \frac{\partial^2}{\partial s \partial t}\left[\exp(tA)\exp(sB)\exp(-tA)\right]\Big|_{s=t=0} = [A, B]$$

proof

Let ψ_t be the flow of l_A, note $\psi_t(g) = g\gamma_A(t) = g\exp(tA)$

$$[A, B] \overset{\text{Def}}{=} [l_A, l_B](e) \overset{\text{Lemma}}{=} L_{(l_A)} l_B(e) = \frac{d}{dt}\left[d_{\psi_t(e)} \psi_{-t}(l_B(\psi_t(e)))\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[d_{\psi_t(e)} \psi_{-t}(d_e L_{\psi_t(e)}(B))\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[\cancel{d_e \psi_{-t} \circ L_{\psi_t(e)}(B)}\right]\Big|_{t=0}$$

$\underline{\text{use}}$ $\frac{d}{ds}[g\exp(sB)] = \frac{d}{ds}[L_g(\exp(sB))] = dL_g\left(\frac{d}{ds}(\exp sB)\right|$
$= dL_g(B)$

$$= \frac{d}{dt}\left[d_{\psi_t(e)}\psi_{-t}\left(\frac{d}{ds}[\psi_t(e)\exp(sB)]\Big|_{s=0}\right)\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\frac{d}{ds}\left[\psi_{-t}\left(\psi_t(e)\exp(sB)\right)\right]\Big|_{s=t=0}$$

$$= \frac{\partial^2}{\partial t\partial s}\left[\left(\psi_t(e)\exp(sB)\exp(-tA)\right)\right]\Big|_{s=t=0}$$

$$= \underline{\frac{\partial^2}{\partial t\partial s}\left[\cancel{e}\cdot\exp(tA)\exp(sB)\exp(-tA)\right]\Big|_{s=t=0}} \checkmark$$

$$= \frac{d}{dt}\left[\frac{d}{ds}\left[\sigma_{\exp(tA)}(\exp(sB))\right]\right]\Big|_{s=t=0}$$

$$= \frac{d}{dt}\left[d_I\sigma_{\exp(tA)}\left(\frac{d}{ds}(\exp(sB))\Big|_{s=0}\right)\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[Ad(\exp(tA))(B)\right]\Big|_{t=0}$$

$$= d(Ad)\left[\frac{d}{dt}(\exp(tA))\Big|_{t=0}\right](B)$$

$$= d(Ad)(A)(B) = ad(A)(B) \,/\!/$$

$$Ad: \underset{\substack{\underline{\text{think}}\\ Gl(n)}}{G} \longrightarrow \underset{\substack{\underline{\text{think}}\\ Gl(n^2)}}{Gl(\mathfrak{g})}$$

HMWK we use Θ (Mayer-Cartan form)

$$M \xrightarrow{g} G \subseteq Gl(n)$$

$$g^*\Theta = g^{-1}dg$$

$$\overline{A} = g^{-1}Ag + \boxed{g^{-1}dg}$$ inhomogenus terms

Gauge Trans.

if Abelian $\overset{\shortparallel}{g^{-1}gA} + e^{-i\theta}de^{i\theta} = A + d\Theta$

3/4/2

Left Invariant Vector Fields are Smooth

note:
$$Y(f)(x) = Y_x(f) = df(Y_x)$$

Let $f: G \to \mathbb{R}$ be a smooth function and $A \in T_eG$

$$l_A(f)(g) = l_A(g)(f) = d_eL_g(A)(f) = d_gf\, d_eL_g(A)$$
$$= d_e f\circ L_g(A)$$

Is $g \mapsto d_e(f\circ L_g)(A)$ smooth?

$$f\circ L_g(x) = f(\mu(g,x)).$$
$$f\circ\mu(x^{-1}\times x^{-1}) \quad \frac{\partial[f\circ\mu(x^{-1}\times x^{-1})]}{\partial u^i}(v,0) \quad \text{smooth ✓}$$

$\ldots \Rightarrow g \mapsto f\circ L_g$ smooth $\Rightarrow \ldots$

Fiber Bundles

$\boxed{\text{Def}}$ A fiber bundle is a mapping $\pi: E \to M$ where E & M are manifolds $\ni$

(1) π is smooth & surjective

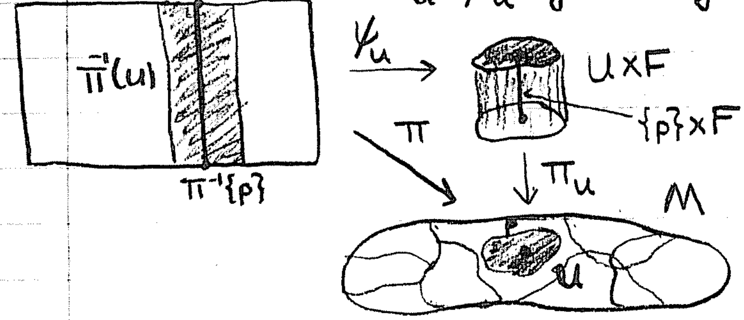
(2) $\exists$ a manifold F called the fiber of π and an open cover $\mathcal{U}$ of M along with a corresponding family of maps $\psi_U: \pi^{-1}(U) \to U\times F$ $(U\in\mathcal{U})$ $\ni$

(a) ψ_U is a diffeomorphism

(b) if $\pi_U: U\times F \to U$ is the projection then

$\pi_U\circ\psi_U(y) = \pi(y) \quad \forall\, y\in\pi^{-1}(U)$

E



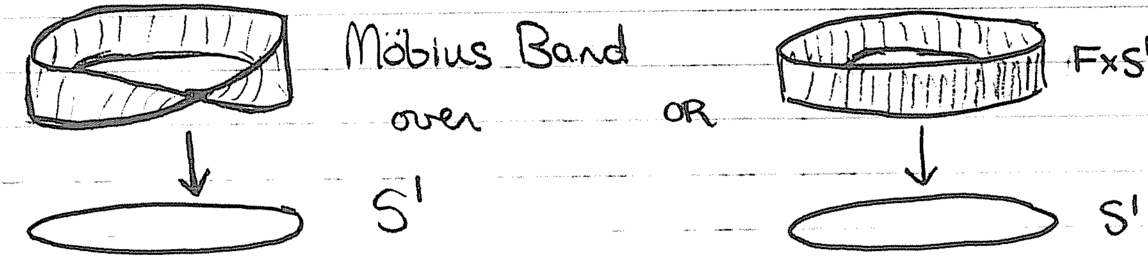
$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\psi_U"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "///"] & U\times F \arrow[dl, "\pi_U"] \\
& U &
\end{tikzcd}$$

$$\boxed{\psi_U|_{\pi^{-1}\{p\}} = \{p\}\times F}$$

Now because $\psi_u|_{\pi^{-1}(p)} = \{p\} \times F \Rightarrow \pi^{-1}\{p\}$ is diffeomorphic to $\{p\} \times F$ which is diffeomorphic to F

where $F = (-1,1)$

Ex:



also $TM, T^*M, \Lambda^p M$ use $\mathbb{R}^n, (\mathbb{R}^n)^*, (\Lambda^p \mathbb{R}^n)^*$ as fibers

note: ψ_u are called locally trivializing maps

M base manifold
E bundle space

$TM = \{(p,v) \mid p \in M, v \in T_pM\}$
(U, x) chart of M then define (TU, dx) a chart on TM by $TU = \{(p,v) \mid p \in U, v \in T_pU (\cong T_pM)\}$

$$dx(p,v) = (x(p), d_p x(v)) \; (= (x^1(p), \ldots, x^n(p), dx^1_p(v), \ldots, dx^n_p(v)))$$

$(\Rightarrow \dim(TM) = 2n$ if $\dim(M) = n)$

note: $dx : TU \underset{(\text{onto})}{\twoheadrightarrow} x(U) \times \mathbb{R}^n$

This gives TM a manifold structure.

now use $\psi_u = (x^{-1} \times id) \circ dx : TU \rightarrow U \times \mathbb{R}^n \; (F = \mathbb{R}^n)$

$\pi : TM \rightarrow M \quad \pi(p,v) = p$

$\pi^{-1}(U) = \{(p,v) \mid \pi(p,v) \in U\} = TU$

$\Rightarrow TM$ is a fiber bunder over M w/t fiber $\mathbb{R}^n$

3/6/2

HW: $\Theta_g(V) = \beta^i_g(V)\,\mathfrak{X}_i$ where $g \in G$, $V \in T_gG$

$\{e_i\}$ basis for T_eG, $\mathfrak{X}_i = l_{e_i}$ & $\beta^i = l_{e^i}$ where $\{e^i\}$ is the dual of $\{e_i\}$.

Thus $\Theta_g(V)$ is a vector in the Lie algebra $\mathfrak{X}_{inv}(G)$

$\Theta_g(\mathfrak{X}(g)) = \lambda^i(g)\,\mathfrak{X}_i$ for <u>any</u> $\mathfrak{X} \in \mathfrak{X}(G)$

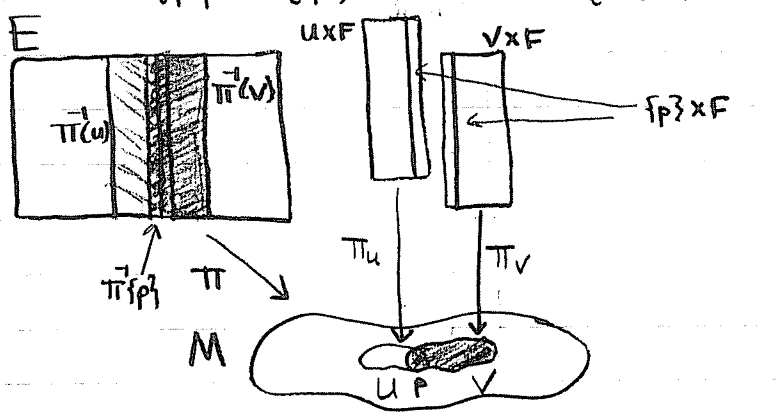
$\therefore\ \Theta(\mathfrak{X}) : G \to \mathfrak{X}_{inv}(G) \quad g \mapsto \lambda^i(g)\,\mathfrak{X}_i$

$E \xrightarrow{\pi} M$ (smooth) $\quad U \subseteq M$ (open)

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi_u"] \arrow[dr, "\pi"'] & {} \arrow[d, phantom, "\cong"] & U \times F \arrow[dl, "\pi_u"] \\ & M & \end{tikzcd}$$

Where $\{U\}$ is an open cover of the base space M.
and $\forall\ U$ in the cover $\exists$ a diffeomorphism $\psi_u : \pi^{-1}(U) \to U \times F$
E is called the total space and F is the fiber.

<u>note:</u> $\pi^{-1}\{p\} \cong \{p\} \times F \cong F \quad (\psi_u(\pi^{-1}\{p\}) = \{p\} \times F = \pi_u^{-1}\{p\})$



This is the most primative fiber bundle
later we put structure on the fiber
- F is a vector space
- F is a Lie group.

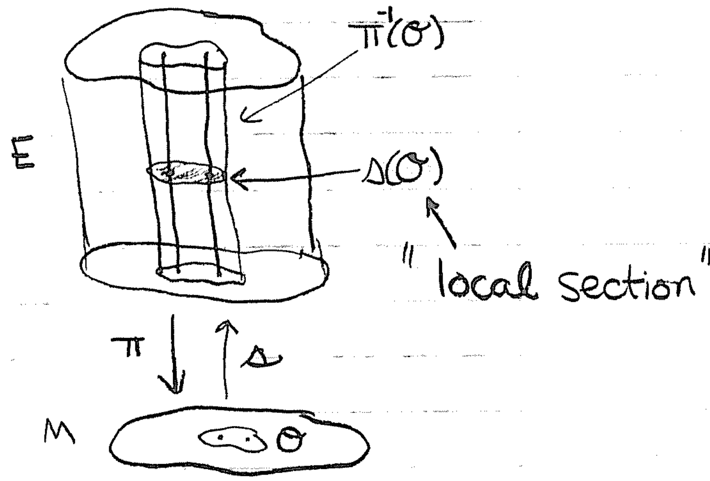
ψ_u, ψ_v are local trivializing maps

If ψ_U is a local trivializing map, we can define $s_U : U \to \pi^{-1}(U)$ by $s_U(x) = \psi_U^{-1}(x, f_0)$ for some fixed f_0 in the fiber, we call s_U a <u>gauge</u>

$$\pi(s_U(x)) = \pi_U \circ \psi_U(s_U(x)) = \pi_U(x, f_0) = x$$

$s_U : U \to E$, $s_U(U)$ is a cross-section.
s_U intersects each fiber in 1 and only 1 point (and does this smoothly)

$\boxed{\text{Def}}$ A function $s : \mathcal{O} \to E$ (smooth) where $\mathcal{O}$ is open in M $\ni \pi \circ s(x) = x \ \forall x \in \mathcal{O}$ then s is called a <u>local section</u> of π



$\boxed{\text{Def}}$ $E_1 \xrightarrow{\pi_1} M_1$, F_1
$E_2 \xrightarrow{\pi_2} M_2$, F_2
A <u>bundle isomorphism</u> from (E_1, π_1, F_1) to (E_2, π_2, F_2) is a pair of maps $(\Phi, \varphi) \ni$

$\Phi : E_1 \to E_2$
$\varphi : M_1 \to M_2 \ni$
Φ, φ are diffeomorphisms $\ni$
(ie $\pi_2 \circ \Phi = \varphi \circ \pi_1$)

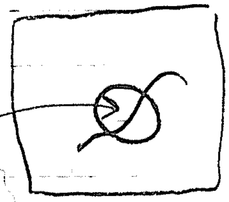
$$\begin{tikzcd} E_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "///"] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "\varphi"] & M_2 \end{tikzcd}$$

for Gauge Theory use $M_1 = M_2$ $\varphi = id_{M_1}$
General Relativity φ diffeomorphism.

Handout:
"Assume that M is a...

3/8/2

<u>HW</u>: $z(u,f) = (x(u), y(f) - y(\varphi(u)))$
$\bar{z}(u,f) = (\bar{x}(u), \bar{y}(f) - \bar{y}(\varphi(u)))$
$s(u) = (u, \varphi(u))$



want on graph map to be $x \to (x,0)$

(also let $\dim M = m$)
M a manifold $\mathcal{F}M = \{(p, \{e_i\}) \mid \{e_i\}$ is a basis for $T_pM\}$
(frame bundle)
In general, $TM = \bigcup_p T_pM \neq M \times \mathbb{R}^m$ (equality if M is a Lie group & other special structures)
<u>Ex</u>: $TS^2 \neq S^2 \times \mathbb{R}^2$ $\therefore \nexists$ a Lie group structure for S^2 ✓

For each chart (U,x) of M define:
$\mathcal{F}U = \{(p,\{e_i\}) \mid p \in U, \{e_i\}$ a basis for $T_pM (= T_pU)\}$
and $\mathcal{F}_x : \mathcal{F}U \to x(U) \times Gl(m) \underset{\text{open}}{\subseteq} x(U) \times gl(m)$ $(= x(U) \times \mathbb{R}^{m^2})$

$$\mathcal{F}_x(p, \{e_i\}) = (x(p), (d_px^j(e_i)))$$ matrix

Consider $A = (A^j_i)$, $A^j_i = d_px^j(e_i)$ then A is nonsingular (because $\partial_i|_p$ & e_i are bases for T_pM ($e_i = A^j_i \partial_j|_p$) A is a change of basis matrix) $\Rightarrow A \in Gl(m)$

$$\mathcal{F}_x^{-1}(a, A) = (x^{-1}(a), \{A^i_j \tfrac{\partial}{\partial x^i}\big|_{x^{-1}(a)}\})$$ $(\mathcal{F}_x \circ \mathcal{F}_x^{-1} = id, \mathcal{F}_x^{-1} \circ \mathcal{F}_x = id ✓)$
$\Rightarrow \mathcal{F}_x$ is a chart

Consider, (U,x) & (V,y) charts on M

smooth ✓

$$[\mathcal{F}_y \circ \mathcal{F}_x^{-1}](a, A) = \mathcal{F}_y(x^{-1}(a), A^i_j \tfrac{\partial}{\partial x^i}\big|_{x^{-1}(a)})$$
$$= (y \circ x^{-1}(a), (dy^k(A^i_j \tfrac{\partial}{\partial x^i}))) = (y \circ x^{-1}(a), (\tfrac{\partial y^k}{\partial x^i} A^i_j))$$
$\therefore$ smooth on overlap hence is an atlas.

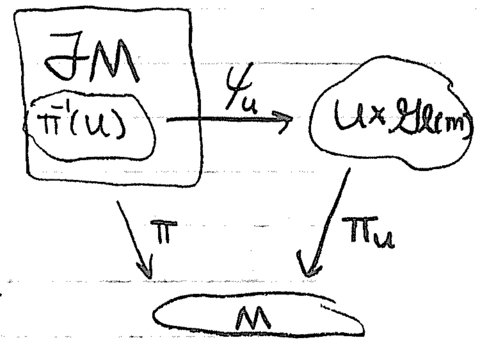
$(\dim(M) = m)$

Claim $\mathcal{F}M \xrightarrow{\pi} M$, $\pi(p, \{e_i\}) = p$ is a fiber bundle with fiber $Gl(m)$

Given a chart (U, x) of M define $\psi_u : \mathcal{F}U = \pi^{-1}(U) \to U \times Gl(m)$

$$\psi_u(p, \{e_i\}) = (p, (d_p x^k(e_i)))$$

now,
$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\psi_u"] \arrow[d, "\mathcal{F}x"'] \arrow[dr, phantom, "/\!/"] & U \times Gl(m) \arrow[d, "x \times id"] \\ x(U) \times Gl(m) \arrow[r, "id"] & x(U) \times Gl(m) \end{tikzcd}$$



$\therefore \psi_u = (x^{-1} \times id) \circ \mathcal{F}_x \Rightarrow$ smooth ✓

Let M be a manifold with metric g of index k (k -1's on diag)
let $p = n - k$ $\therefore \exists$ a basis $\{e_i\}$ of $T_q M \ni$
$g_q(e_i, e_j) = \pm \delta_{ij}$ ordered $(g_q(e_i, e_j)) = G_{ij} = \begin{Bmatrix} 0 & i \neq j \\ 1 & 1 \leq i = j \leq p \\ -1 & p+1 \leq i = j \leq m \end{Bmatrix}$

Define $\mathcal{O}_g M$ (orthonormal frame bundle over M with metric g)

$$\mathcal{O}_g M = \{ (q, \{e_i\}) \mid g_q(e_i, e_j) = G_{ij} \}$$

define again $\pi : \mathcal{O}_g M \to M$ $\pi(q, \{e_i\}) = q$

$\boxed{\text{Thm}}$ If M is a manifold and g is a metric on M with index k (again $p = m - k$) then $\forall q_0 \in M$ $\exists$ open set $\mathcal{U}$ about q_0 and vectors fields $\{X_i \mid 1 \leq i \leq m\} \ni g_q(X_i(q), X_j(q)) = G_{ij}$ $\forall 1 \leq i, j \leq m$ $\forall q \in \mathcal{U}$.

Let $\mathcal{O}(p, k) = \{ A \in Gl(m) \mid A^T G A = G \}$

By the Thm $\exists$ an open cover $\mathcal{U}$ of $M \ni \forall U \in \mathcal{U}$ $\exists$ $\{X_i^U\}$ vector fields $\ni g_q(X_i^U(q), X_j^U(q)) = G_{ij}$ $\forall q \in U$.

X_U^j dual to X_j^U

$$\psi_u : \pi^{-1}(U) \to U \times \mathcal{O}(p,k) \ni \psi_u(q, \{e_i\}) = (q, (X_U^j(q) e_i))$$

Handout: Chapter 2
pg 39 – 42
"Constructing Manifolds"

3/18/2

Lorentz, time pres. pos. determinant.

$$\mathcal{O}(p,k) = \{A \in \mathcal{GL}(\overset{p+k}{\overset{\shortparallel}{n}}) \mid A^T G A = G\} \quad (= \boxed{\mathcal{L}_+^\uparrow} \oplus \mathcal{L}_-^\uparrow \oplus \mathcal{L}_+^\downarrow \oplus \mathcal{L}_-^\downarrow)$$

$$\mathcal{O}_g M = \{(m,\{e_j\}) \mid m\in M,\ g_m(e_i,e_j) = G_{ij}\}$$

$\pi : \mathcal{O}_g M \to M$ defined by $\pi(m,\{e_j\}) = m$

By the $\boxed{\text{Thm}}$ $\exists\ \mathcal{U}$ open cover of $M \ni \forall\, U \in \mathcal{U}\ \exists\ \{\mathfrak{z}_i^U\}_{i=1}^{\dim M} \subseteq \mathfrak{X}M$
$\ni g_m(\mathfrak{z}_i^U(m), \mathfrak{z}_j^U(m)) = G_{ij}\ \forall\, m \in U$
We can assume $U \in \mathcal{U}$ is a chart domain

ji^{th} comp. of a matrix

$$\psi_U : \pi^{-1}U \to U \times \mathcal{O}(p,k) \ni \psi_U(m,\{e_i\}) = (m, [\mathfrak{z}_U^j(m)(e_i)])$$

$\Lambda = (\lambda_i^j)$ where $\lambda_i^j = \mathfrak{z}_U^j(m)(e_i)$ (note: $\mathfrak{z}_U^j$ is dual of $\mathfrak{z}_i^U$)

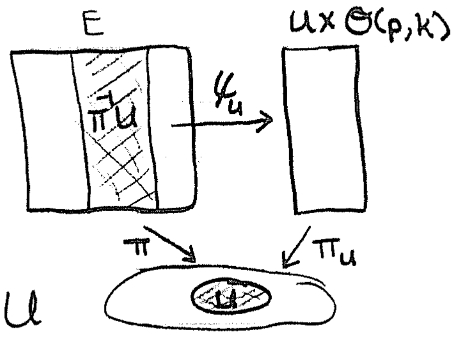
then
$$\mathfrak{z}_\ell^U(m)\lambda_i^j = \mathfrak{z}_\ell^U(m)\mathfrak{z}_U^j(m)(e_i) = \delta_\ell^j e_i \Rightarrow e_i = \mathfrak{z}_j^U(m)\lambda_i^j$$

$\Lambda \in \mathcal{GL}(n)$ because it is a change of basis matrix
$$G_{ij} = g_m(e_i,e_j) = \lambda_i^k \lambda_j^\ell\, g_m(\mathfrak{z}_k^U(m), \mathfrak{z}_\ell^U(m)) = \lambda_i^k \lambda_j^\ell G_{k\ell}$$
$$\therefore G^i_j = \lambda^{T\,i}{}_k G^k_\ell \lambda^\ell_j \Rightarrow G = \Lambda^T G \Lambda \Rightarrow \Lambda \in \mathcal{O}(p,k)$$

hence $\psi_U(m,\{e_i\}) \in U \times \mathcal{O}(p,k)$

o Now we need a differentiable structure and that $\{\psi_U\}$ are smooth with respect to that structure – better than that – diffeomorphisms

Examine, $\psi_U : \pi^{-1}(U) \to U \times \mathcal{O}(p,k)$
$\psi_V : \pi^{-1}(V) \to V \times \mathcal{O}(p,k)$



assume $U \cap V \neq \emptyset$
Let $\{\mathfrak{z}_i^U\}$ $\{\mathfrak{z}_j^V\}$ be G-orthog. fields on U and V respectively

Define a mapping $T : U \cap V \longrightarrow O(p,k) \ni$

$T(q) = A_q \qquad \zeta_i^V(q) = (A_q)_i^j \, \zeta_j^U(q)$ as before $A_q \in O(p,k)$

T is called a <u>transition function</u>

basis <u>at</u> q still g-orthog.

Observe that $\psi_V^{-1}(q, \Lambda) = (q, \{\Lambda_j^i \zeta_i^V(q)\})$

$$\psi_U \circ \psi_V^{-1}(q, \Lambda) = (q, \zeta_U^k(q)(\Lambda_j^i \zeta_i^V(q))) = (q, \zeta_U^k(q) \Lambda_j^i \, T(q)_i^l \zeta_l^U(q))$$

$$= (q, \Lambda_j^i T(q)_i^l \overset{\delta_l^k}{\cancel{\zeta_U^k \zeta_l^U}}) = (q, \Lambda_j^i T(q)_i^k) = (q, T(q)\Lambda)$$

$\hookrightarrow \psi_U \circ \psi_V^{-1}(q, \Lambda) = (q, T_{UV}(q)\Lambda)$ where $T_{UV} : U \cap V \longrightarrow O(p,k)$

$\rightarrow$ We will show $T_{WU} \circ T_{UV} = T_{WV}$ this is called a Cocycle condition for Chech-Cohomology.

...

3/20/2

$$(=\{A\in \mathfrak{gl}(n) \mid A^T G A = G\})$$

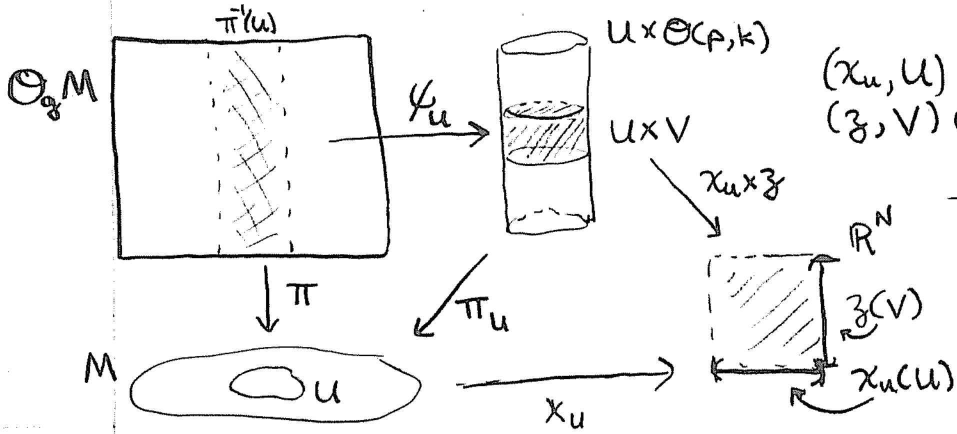
$$\psi_u : O_g M \longrightarrow U \times O(p,k)$$

$$\gamma_i^V(q) = T_{uv}(q)^j_i\, \gamma_j^U(q) \qquad T_{uv} : U\cap V \longrightarrow O(p,k) \text{ transition function}$$

$$(\psi_u \circ \psi_v^{-1})(m,\Lambda) = (m, T_{uv}(m)\Lambda) \qquad m\in U\cap V,\ \Lambda\in O(p,k)$$

Let $\mathcal{Q}$ be an atlas on $O(p,k)$

$\{(x_u\times z)\circ\psi_u \mid x_u \text{ is a chart of } M \text{ on } U,\ z\in\mathcal{Q}\}$



(x_u, U) chart on M

(z, V) chart on $O(p,k)$

$$\left([(x_u\times z_1)\circ\psi_u]\circ[(x_v\times z_2)\circ\psi_v]^{-1}\right)(s,t)$$

$$= (x_u\times z_1)\circ(\psi_u\circ\psi_v^{-1})\circ(x_v^{-1}\times z_2^{-1})(s,t)$$

$$= (x_u\times z_1)(\psi_u\circ\psi_v^{-1})(x_v^{-1}(s), z_2^{-1}(t)) = (x_u\times z_1)(x_v^{-1}(s), T_{uv}(x_v^{-1}(s))\, z_2^{-1}(t))$$

$$= \Big(\underbrace{x_u\circ x_v^{-1}(s)}_{\text{smooth}},\ z_1(\underbrace{T_{uv}(x_v^{-1}(s))\, z_2^{-1}(t)}_{\text{smooth b/c composite of smooth maps + smooth multiplication.}})\Big)$$

$\Rightarrow \{(x_u\times z)\circ\psi_u \mid x_u;\ z\in\mathcal{Q}\}$ is an atlas on $O_g M$

ψ_u is smooth, $q\in\pi^{-1}(U)$, $\psi_u(q)\in O(p,k)$ choose z a chart $\ni$ $\mathrm{proj}_{O(p,k)}(\psi_u(q))\in$ domain of z.

$\mathrm{dom}(z) = V_z$

$U_z = \psi_u^{-1}(U \times V_z)$ now U_z is a chart domain for $O_g M$

$$\begin{tikzcd} U_z \arrow[r, "\psi_u"] \arrow[d, "(x_u \times z)\circ\psi_u"'] & U \times V_z \arrow[dl, "x_u \times z"] \\ x_u(U) \times z(V_z) & \end{tikzcd}$$

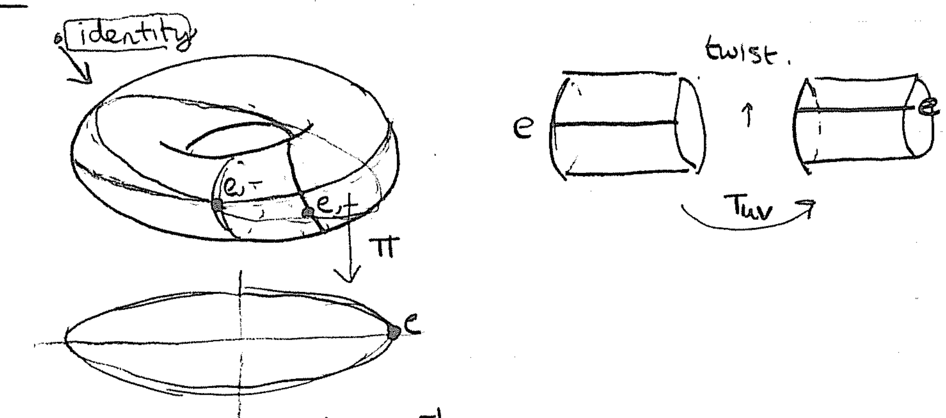
$(x_u \times z)\circ\psi_u \circ [(x_u \times z)\circ \psi_u]^{-1} = id$ is smooth.

$\Rightarrow$ $\underline{\psi_u \text{ is smooth}}$ ✓

$[(x_u \times z)\circ \psi_u] \circ \psi_u^{-1} \circ (x_u \times z)^{-1} = id$ is smooth

$\Rightarrow$ $\underline{\psi_u^{-1} \text{ is smooth}}$ ✓

$\therefore$ ψ_u are diffeomorphisms.



<u>Remark</u>: If $\psi_u : \pi^{-1}(U) \longrightarrow U \times F$

$\psi_v : \pi^{-1}(V) \longrightarrow V \times F$ are local trivializing maps of a fiber bundle E over M with fiber F, $\pi : E \to M$.

and $U \cap V \neq \emptyset$ then $\exists$ a smooth mapping

$$\varphi_{uv} : U \cap V \longrightarrow \mathrm{Diff}(F) \ni$$

$$(\psi_u \circ \psi_v^{-1})(m, f) = (m, \varphi_{uv}(m)(f))$$

- To say φ_{uv} is <u>smooth</u> means that $\tilde{\varphi}_{uv}$ defined by $\tilde{\varphi}_{uv}(m,f) \equiv \varphi(m)(f)$ is smooth ($\tilde{\varphi}_{uv} : U \times F \to F$)

(don't want to mess with a manifold structure on $\mathrm{Diff}(F)$ b/c ∞-dim'l)

If U, V, W are domains of local trivializing maps ψ_u, ψ_v, ψ_w and $m \in U \cap V \cap W (\neq \emptyset)$ then

$\varphi_{wu}(m) \circ \varphi_{uv}(m) = \varphi_{wv}(m)$ (co-cycle condition)

3/22/2

$\pi : E \to M$ a fiber bundle with fiber F

$\psi_u : \pi^{-1}(U) \to U \times F$

$\psi_v : \pi^{-1}(V) \to V \times F \quad U \cap V \neq \emptyset$

$$(\psi_u \circ \psi_v^{-1})(m,f) = (m, \tilde{\varphi}_{uv}(m,f))$$

$$\tilde{\varphi}_{uv}(m,f) = (\pi_F \circ \psi_u \circ \psi_v^{-1})(m,f) \qquad \left[\begin{array}{c} \pi_F : (U \cap V) \times F \to F \\ \pi_F(u,f) = f \end{array} \right]$$

transition function

$\varphi_{uv} : U \cap V \to \text{Maps}(F,F)$

$\varphi_{uv}(u)(f) = \tilde{\varphi}(u,f)$

Now $\tilde{\varphi}_{vu}(m,f) = (\pi_F \circ \psi_v \circ \psi_u^{-1})(m,f)$.

Let U, V, W be open sets with ψ_u, ψ_v, ψ_w local trivializing maps. also $U \cap V \cap W \neq \emptyset$

$$\psi_w \circ \psi_v^{-1} = \psi_w \circ \psi_u^{-1} \circ \psi_u \circ \psi_v^{-1}$$

$$(m, \varphi_{wv}(m)(f)) = \psi_w \circ \psi_v^{-1}(m,f) = \psi_w \circ \psi_u^{-1}(m, \varphi_{uv}(m)(f))$$
$$= (m, \varphi_{wu}(m)(\varphi_{uv}(m)(f)))$$

$\Rightarrow \varphi_{wv}(m)(f) = \varphi_{wu}(m) \circ \varphi_{uv}(m)(f)$

$\therefore \varphi_{wv}(m) = \varphi_{wu}(m) \circ \varphi_{uv}(m)$ (cocycle condition)

$\Rightarrow \cancel{\varphi_{uu}}^{\,id}(m) = \varphi_{uv}(m) \circ \varphi_{vu}(m) \quad \Rightarrow \varphi_{uv}(m)^{-1} = \varphi_{vu}(m)$

hence $\varphi_{uv}(m) \in \text{Diff}(F,F)$

Define $(\varphi_{uv}\varphi_{vw})(m) \equiv \varphi_{uv}(m) \circ \varphi_{vw}(m) \quad \forall\, m \in U \cap V \cap W$

$\Rightarrow$ we write $\varphi_{uv}\varphi_{vw} = \varphi_{uw}$

$\boxed{\text{Thm}}$ Given M, F manifolds and an open cover $\mathcal{U}$ of M with maps $\varphi_{uv} : U \cap V \longrightarrow \mathrm{Diff}(F,F)$ $\forall\, U, V \in \mathcal{U} \ni U \cap V \neq \emptyset$ $\ni U \cap V \cap W \neq \emptyset$ then $\varphi_{wv} = \varphi_{wu}\varphi_{uv}$
Then $\exists$ a fiber bundle $\pi : E \to M$ with fiber F and local trivializing maps $\psi_u : \pi^{-1}(U) \longrightarrow U \times F$, $\psi_v : \pi^{-1}(V) \longrightarrow V \times F \ni$
$(\psi_u \circ \psi_v^{-1})(m, f) = (m, \varphi_{uv}(m)(f))$ $\forall\, m \in U \cap V$, $f \in F$

"proof"
Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ and write $\varphi_{\alpha\beta}$ for $\varphi_{U_\alpha U_\beta}$
Let S be the disjoint union of $\{U_\alpha \times F\}_{\alpha \in A}$
$S = \{(\alpha, u, f) \mid \alpha \in A, u \in U_\alpha, f \in F\}$ $\quad S_\alpha = \{(\alpha, u, f) \mid u \in U_\alpha, f \in F\}$
Define an atlas $\mathcal{A}_\alpha$ on S_α (just use atlas on U_α & F) ...
then let $\mathcal{A} = \bigcup_{\alpha \in A} \mathcal{A}_\alpha$ so we have an atlas on S.

Define $\sim$ on S by $(\alpha, x, f_1) \sim (\beta, y, f_2) \iff x = y$ and $f_2 = \varphi_{\beta\alpha}(x, f_1)$
this is an equiv. relation b/c $x = x$, $f_1 = \underbrace{\varphi_{\alpha\alpha}(x)}_{\text{Id}}(f_1)$ ✓ reflexive.
and $(\alpha, x, f_1) \sim (\beta, y, f_2) \Rightarrow x = y$ and $f_2 = \varphi_{\beta\alpha}(f_1)$
$\Rightarrow \varphi_{\alpha\beta}(f_2) = f_1$ $(\varphi_{\alpha\beta}^{-1} = \varphi_{\beta\alpha})$ $\therefore (\beta, y, f_2) \sim (\alpha, x, f_1)$ ✓ symm.
trans. also ...
Let $E = \{[\alpha, x, f] \mid (\alpha, x, f) \in S\}$ (equivalence classes)
define $\tilde{\pi} : S \to M$ by $\tilde{\pi}(\alpha, x, f) = x$ is smooth
(since a proj.) $\tilde{\pi}^{-1}(U_\alpha) = S_\alpha$ $\quad \tilde{\pi}|_{[\alpha, x, f]} = \{x\}$ (const.)

$\therefore \pi : E \to M$ def. by $\pi[\alpha, x, f] = x$ is well def.

...

may sketch more on Monday...
(looks much like our example $\mathcal{O}_g M$, $\mathcal{O}(p,k)$ ...)

3/25/2 <u>finishing sketch of proof</u>

$\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ open cover of M and

$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \text{Diff}(F)$

We built

$S = \bigcup_{\alpha\in A} S_\alpha$ where $S_\alpha = \{(\alpha, x, f) \mid x \in U_\alpha, f \in F\}$

then $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$ and $S_\alpha \overset{\text{diff}}{\cong} U_\alpha \times F$

$\mathcal{Q}_\alpha$ is an atlas on S_α (derived from $U_\alpha \times F$)

we get $\mathcal{Q} = \bigcup_{\alpha\in A} \mathcal{Q}_\alpha$

$\tilde{\pi} : S \longrightarrow M$ $\tilde{\pi}(\alpha, x, f) = x$ $\tilde{\pi}\big|_{S_\alpha} : \{\alpha\} \times U_\alpha \times F \longrightarrow U_\alpha$

is smooth $\Rightarrow \tilde{\pi}$ smooth

$(\alpha, x, f) \sim (\beta, y, f') \iff x = y$ and $f' = \varphi_{\beta\alpha}(y)(f)$

equiv. relation

<u>note:</u> $\tilde{\pi}[\alpha, x, f] = \{x\}$ define $\pi : \overset{=E}{S/\sim} \longrightarrow M$ by

$\pi([\alpha, x, f]) = x$ ✓

$\pi^{-1}(U_\alpha) = \{[\beta, y, f] \mid y \in U_\alpha\}$

$\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$

$\psi_\alpha([\beta, y, f]) = \psi_\alpha([\alpha, y, \varphi_{\alpha\beta}(f)]) = (y, \varphi_{\alpha\beta}(y)(f))$

$\mathfrak{P}$ $(\beta, y, f) \sim (\gamma, z, f') \Rightarrow \ldots$ well defined ✓

$\psi_\beta \circ \psi_\alpha^{-1}(x, f) = \psi_\beta([\alpha, x, f]) = \psi_\beta[\beta, x, \varphi_{\beta\alpha}(x)(f)]$

$= (x, \varphi_{\beta\alpha}(x)(f))$

$\Rightarrow \psi_\beta \circ \psi_\alpha^{-1}$ is smooth ✓

Let $\mathcal{Q}_E = \{(x_\alpha \times z) \circ \psi_\alpha \mid x_\alpha$ a chart on U_α, z a chart of $F\}$

then show $\mathcal{Q}_E$ is an atlas & ψ_α are smooth & π smooth

finally the diagram commutes (like $O_g M$ argument) //

$\boxed{\text{Def}}$ Let M and F be manifolds, $\mathcal{U}$ an open cover of M and that $\varphi_{UV} : U \cap V \to \text{Diff}(M)$ is a family of smooth mappings $(\forall U, V \in \mathcal{U}) \ni \varphi_{WU}\varphi_{UV} = \varphi_{WV}$ then we say $\{\varphi_{UV}\}$ are <u>transition functions</u> of <u>some</u> fiber bundle over M with fiber F and φ_{UV} are <u>Čech cocycles</u>.

$\boxed{\text{Def}}$ A fiber bundle $\pi : E \to M$ is a <u>vector bundle</u> iff $\exists$ local trivializing maps $\psi_U : \pi^{-1}(U) \to U \times F$ and the transition functions φ_{UV} are linear. The fiber F is a vector space ie. $\varphi_{UV} : U \cap V \to \text{Diff}(F)$ then

$$\varphi_{UV}(m)(\alpha x + y) = \alpha\, \varphi_{UV}(m)(x) + \varphi_{UV}(m)(y)$$

- We assume F is finite dim'l (as a vector space).

If the fiber F is a Lie group then it may be that

$$\varphi_{UV} : U \cap V \to \text{Diff}(F)$$

can be written as

$$\varphi_{UV}(m)(f) = g_{UV}(m) f \quad \text{for } g_{UV} : U \cap V \to F$$

this is true for Principle Fiber bundles.

$\boxed{\text{Def}}$ Let G be a group and S a set then <u>G acts on S</u> from the left iff $\exists$ a map $\sigma : G \times S \to S \ni \sigma(g_1 g_2)(x) = \sigma(g_1)(\sigma(g_2)(x))$ and $\sigma(e)(x) = x$ ie $\sigma(e) = id_S$

we will write $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$
$$e \cdot x = x$$

- likewise right actions $x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2$ & $x \cdot e = x$