4/1/2

<u>Group Actions:</u>

<u>Ex (1):</u> $Gl(n)$ acts on $T^p_q\mathbb{R}^n$

(i) We first define an action of $Gl(n)$ on $(\mathbb{R}^n)^*$ (right regular rep.)

$A \in Gl(n)$, $\alpha \in (\mathbb{R}^n)^*$ by:

$$(A \circ \alpha)(x) = \alpha(A^{-1}x)$$

note: $(A\circ(B\circ\alpha))(x) = \alpha(B^{-1}A^{-1}x) = \alpha((AB)^{-1}x) = (AB\circ\alpha)(x)$

$(I\circ\alpha)(x) = \alpha(Ix) = \alpha(x)$ $\therefore$ is an action.

(ii) $Gl(n)$ acts on $\mathbb{R}^n$ by matrix mult.

(iii) Combine: $A \in Gl(n)$ $\tau \in T^p_q\mathbb{R}^n$ define

$A\circ\tau \in T^p_q\mathbb{R}^n$ by

$$A\circ\tau(V_1,\dots,V_q,\Theta^1,\dots,\Theta^p) = \tau(A^{-1}\circ V_1,\dots,A^{-1}\circ V_q, A^{-1}\circ\Theta^1,\dots,A^{-1}\circ\Theta^p)$$

$$A^{-1}\circ\tau(e_{i_1},\dots,e_{i_q},e^{j_1},\dots,e^{j_p}) = \tau(A^{-1}{}^{l_1}_{i_1}e_{l_1},\dots,A^{-1}{}^{l_q}_{i_q}e_{l_q}, A^{j_1}_{k_1}e^{k_1},\dots)$$

b/c $$A^{-1}\circ e^{j_1}(e_k) = e^{j_1}(A^m_k e_m) = A^m_k\delta^{j_1}_m = A^{j_1}_k$$

$$(A\circ\tau)^{j_1\dots j_p}_{i_1\dots i_q} = A^{-1}{}^{l_1}_{i_1}\cdots A^{-1}{}^{l_q}_{i_q}A^{j_1}_{k_1}\cdots A^{j_p}_{k_p}\tau^{k_1\dots k_p}_{l_1\dots l_q}$$

Extra Credit Checkout...

$$Gl(n)\times V \longrightarrow V$$

Frame bundle

$$\begin{tikzcd} \mathcal{F}M \arrow[r, "\varphi"] \arrow[d, "\pi"'] & V \\ M \arrow[r, shift left] & \mathcal{F}M\times V/G \arrow[l, shift left, "s_\varphi"] \end{tikzcd}$$

T^p_qM

<u>equivariant mapping</u>: $\varphi(u\cdot g) = g^{-1}\varphi(u)$

<u>Ex (2):</u> $Gl(n)$ acts on $\mathcal{F}M$ on the <u>right</u>.

Let $(p,\{e_i\}) \in \mathcal{F}M$ and $A \in Gl(n)$ define:

$$(p,\{e_i\})\circ A = (p,\{e_iA^i_j\})$$

Should check if action is smooth:

Use chart (U,x), $\mathcal{F}U = \{(p,\{e_i\}) \mid p\in U,\ e_i\in T_pM \text{ a basis}\}$

$$\mathcal{F}x(p,\{e_i\}) = (x(p), (d_px^j(e_i))_{i,j})$$

$$\mathcal{F}x\big((p,\{e_i\})\cdot A\big) = \mathcal{F}x(p,\{e_j A^j_i\}) = \big(x(p), (dx^k e_i A^i_j)_{k,j}\big)$$

$$= \big(x(p), (A^i_j\, dx^k e_i)_{k,j}\big) = \big(x(p), (dx^k e_i)(A^i_j)\big)$$

$$= \underbrace{\big[\mathcal{F}x(p,\{e_i\})\big]}_{\in\, \mathcal{F}x(U) \underset{\text{open}}{\subseteq} \mathcal{F}\mathbb{R}^n} \cdot A \quad \therefore \text{ smooth (after some details)}$$

$$\Rightarrow \underset{\text{right}}{\varphi(u\cdot g)} = \underset{\text{right}}{\varphi(u)\cdot g} \leftarrow \text{equivariant.}$$

$$\underset{\text{right action}}{\varphi(u\cdot g)} = \underset{\text{left action}}{g^{-1}\varphi(u)} \leftarrow$$

$\boxed{\text{Def}}$ If G acts on the left of a manifold M (G a Lie group and the action smooth) $x \in M$ then the <u>orbit</u> of x is

$$G\cdot x = \{g\cdot x \mid g \in G\}$$

note!

$G\cdot x$ and $G\cdot y$ are <u>disjoint</u> or <u>equal</u> (ie partition)

$x \in G\cdot x$, $x \in G\cdot y \Rightarrow x = g\cdot y \Rightarrow g^{-1}\cdot x = y \in G\cdot x$

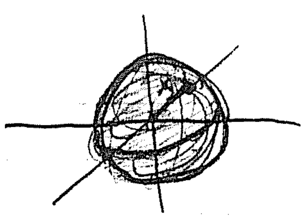
$\Rightarrow x \sim y \Leftrightarrow y \sim x$

+ transitive...

The <u>isotropy subgroup</u> of G determined by $x \in M$ is

$$G_x = \{g \in G \mid g\cdot x = x\} \quad \text{(stabilizer)}$$

<u>Ex:</u> $G = SO(3)$ acts on $\mathbb{R}^3$ by mult. If $x \neq 0$ then $G\cdot x$ is the set of all rotations of x = Sphere of radius $\|x\|$



$G_x \cong SO(2)$ (fixes the line through the origin which also goes through x)

$$S^2 = G\cdot x = G/G_x = \frac{SO(3)}{SO(2)}$$

$SU(2) \xrightarrow{\rho} SO(3)$

$\Rightarrow$ we induce an action of $SU(2)$ on $\mathbb{R}^3$ ... Hopf bundle

4/3/2

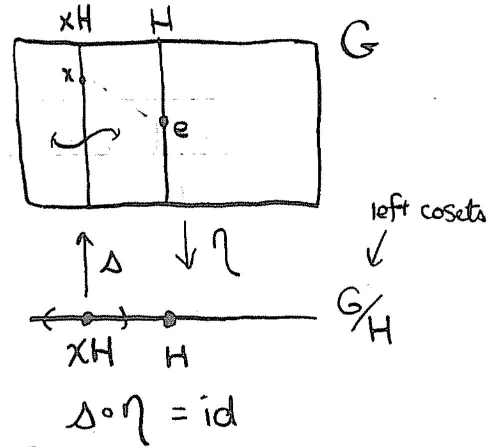
Thm If H is a closed (topologically) subgroup of a Lie group G then,

(1) H is a Lie subgroup of G.

(2) There exists a unique differentiable structure on G/H $\ni$

(i) $\eta : G \to G/H$ is smooth (natural def)

(ii) At each point of G/H $\exists$ a local section of η which is smooth.



proof

(1) See Warner OR Olver

(2) See Warner (maybe Olver also)

Note: We can also use right cosets for G/H etc.

There are 2 natural actions of H on G

$(h,g) \mapsto hg \quad H \times G \to G$

& $(g,h) \mapsto gh \quad G \times H \to G$

For a principle fiber bundle we want to act on the right

Def G acts transitively on M iff $\exists\, x \in M \ni G \cdot x = M$.

Thm If a Lie group G acts transitively on a manifold M then the mapping $\beta : G/G_x \to G\cdot x = M$

defined by $\beta(gG_x) = g \cdot x$ is a diffeomorphism. $\forall\, x \in M$.

(likewise $\beta : G/G_x \to x \cdot G = M$ def by $\beta(g G_x) = x \cdot g$

Ex: $SO(3) \times S^2 \to S^2 \quad S^2 = \{ v \in \mathbb{R}^3 \mid \|v\| = 1 \}$

$\{ A \cdot v \mid A \in SO(3) \} = S^2 \quad S^2 = SO(3) \cdot v$

$\beta : G/G_x \longrightarrow G\cdot x$ makes sense since $gG_x = hG_x$ then $g = hg_x \therefore \beta(gG_x) = g\cdot x = hg_x\cdot x = h\cdot(\underbrace{g_x\cdot x}_{\text{fixed}}) = h\cdot x$

$\therefore \beta(gG_x) = \beta(hG_x)$ hence well def.

= (proof of Thm see Warner...) =

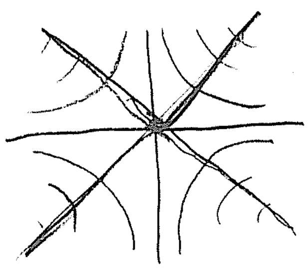
Remark: If $\tilde{f} : G/H \longrightarrow Q$ is smooth (Q a manifold) then $\exists!$ smooth mapping $f : G \longrightarrow Q \ni f = \tilde{f}\circ\eta$ $\tilde{f}$ const. on cosets.

And If $f : G \to Q$ is smooth $\ni f$ is const. on cosets then $\exists!\ \tilde{f} : G/H \longrightarrow Q$ smooth $\ni f = \tilde{f}\circ\eta$

Ex: $\frac{dx}{dt} = x \quad \frac{dx}{x} = dt \Rightarrow x = Ae^t$

$\frac{dy}{dt} = -y \quad \frac{dy}{y} = dt \Rightarrow y = Be^{-t} = B\frac{1}{e^t}$

$\Rightarrow \frac{x}{A} = e^t \ \&\ \frac{B}{y} = e^t \Rightarrow x = \frac{AB}{y}$

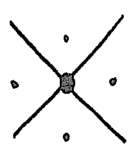


$(\mathbb{R},+)$ acts on $\mathbb{R}^2$

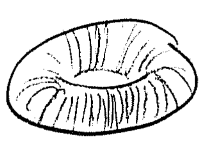
$\varphi : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\varphi(t,(x,y))$ = flow of the sol'n through (x,y)

$\varphi(t,(x,y)) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ ← something like this

quotient looks like  not Hausdorff !!

must be careful with non-transitive actions

Ex:  dense wind on torus ? is another counterexample ?

$\boxed{\text{Def}}$ Let G a Lie group act on a manifold M. The action is called an <u>effective action</u> iff $g \cdot x = x \ \forall x \in M$ then $g = e$

The action is <u>free</u> iff $g \cdot x = x$ for some $x \in M$ then $g = e$.
(no fixed points $\leftarrow$)

- Effective says, $\bigcap_{x \in M} G_x = \{e\}$
- Free says, $\bigcup_{x \in M} G_x = \{e\}$ (ie Free $\Rightarrow$ Effective)

$$G \times M \xrightarrow{\sigma} M$$
$$\tilde{\sigma} : G \to \text{Diff}\, M \quad \tilde{\sigma}(g)(x) = \sigma(g, x)$$

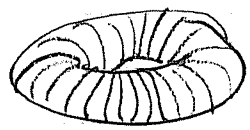
Effective: $g \in \text{Ker}(\tilde{\sigma}) \iff \tilde{\sigma}(g) = id_M$
$$\iff \tilde{\sigma}(g)(x) = x \ \forall x \in M \iff g \cdot x = x \ \forall x \in M$$
$$\iff g = e$$
$\therefore \tilde{\sigma}$ is a monomorphism iff action is effective

free: means $\exists x \ni G_x = \{e\} \Rightarrow G/G_x \cong G \cdot x \Rightarrow G \cong G \cdot x$
(ie the orbit is diffeomorphic to the group.)

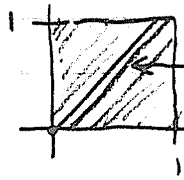
4/5/2

$G \cdot x$ = orbit of x
G_x = stabilizer of x =

Ex: $T^2 = \{(e^{\theta 2\pi i}, e^{\varphi 2\pi i}) \mid \theta, \varphi \in \mathbb{R}\} \subseteq \mathbb{C} \times \mathbb{C}$



$t \in (\mathbb{R}, +)$
$t \cdot (e^{2\pi i \theta}, e^{2\pi i \varphi}) = (e^{2\pi i \theta t}, e^{2\pi i \varphi \alpha t})$ where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$



irrational slope
$(t, \alpha t)$ $\Rightarrow$ can't ever have both rational coords.
but is dense

$\therefore$ we can't get a smooth section (a smooth curve that intersects each orbit in exactly one point).

Assume the Lie group G acts on a manifold M. for $A \in \mathfrak{g}$ (the Lie algebra of G) define a vector field δ_A by

$$\delta_A(x) = \frac{d}{dt}[x \cdot \exp(tA)]\Big|_{t=0} \quad \forall x \in M$$

The mapping $\delta : \mathfrak{g} \to \mathfrak{X}(M)$ defined by $\delta(A) = \delta_A$ is called the <u>infinitesimal generator</u> of the action

$t \mapsto x \cdot \exp(tA)$ is smooth, since tA ✓, exp ✓, action ✓ are smooth. $0 \mapsto x \cdot \exp(0) = x \cdot e = x$
$\therefore$ this is a curve through x, $\Rightarrow \frac{d}{dt}[x \cdot \exp(tA)]\Big|_{t=0} \in T_x M$
$\delta_A(x) \in T_x(M)$ also $x \mapsto \delta_A(x)$ is smooth ✓ (skip)

Note: $t \mapsto x \cdot \exp(tA)$ is in $x \cdot G$ if $x \cdot G$ is a submanifold then $\delta_A(x) \in T_x(x \cdot G)$
$\therefore$ (when it makes sense) δ_A is everywhere tangent to the orbits of the action.

$\boxed{\text{Thm}}$ G a Lie group which acts on the right of the manifold M. Then the infinitesimal generator, $\delta: \mathfrak{g} \to \mathfrak{X}(M)$, is a Lie algebra homomorphism, which is:

- injective when the action is effective.
- δ_A is never zero $\forall A \neq 0$ when the action is free.

proof:

(maybe homework)

If G acts on the right then $M \times G \xrightarrow{\tilde{\sigma}} M$ then we have a map $\sigma: G \to \mathrm{Diff}(M)$ where $\sigma(g)(x) = \tilde{\sigma}(x, g^{-1})$ then σ is a homomorph.

if $\mathrm{Diff}(M)$ where finite dim'l we would have

$$d_e\sigma : T_eG \to T_I(\mathrm{Diff}M) = \mathfrak{X}(M)$$

$\varphi: G \to H \Leftarrow$ Lie group hom. $\Downarrow$ Thm works (but we assumed finite dim.)

$d_e\varphi: T_eG \to T_eH \Leftarrow$ Lie alg. hom.

$T_I(\mathrm{Diff}M) \overset{?}{=} \mathfrak{X}(M)$

A tangent vector to $\mathrm{Diff}(M)$ at I_M should be the derivative of a curve through I_M at 0.

Let $c: (-a,a) \to \mathrm{Diff}(M)$ $\quad c(t): M \to M$

$t \mapsto c(t)(x)$ is a curve in $M \ni c(0)(x) = x$

$\frac{d}{dt} c(t)\Big|_{t=0} : M \to TM \quad \frac{d}{dt}(c(t))(y)\Big|_{t=0} = \frac{d}{dt}(c(t)(y))\Big|_{t=0} \in T_yM$

Sketch:

$\delta_A(x) = \frac{d}{dt}(x \cdot \exp(tA))\Big|_{t=0}$ $\quad x \in M$, $\sigma_x: G \to M$ def by $\sigma(x) = x \cdot g$

$$\delta_A(x) = \frac{d}{dt}\left[\sigma_x \circ \exp(tA)\right]\Big|_{t=0} = d_e\sigma_x\left(\frac{d}{dt}(\exp(tA))\Big|_{t=0}\right)$$

$= d_e\sigma_x(A)$ (obviously linear) $\Rightarrow \delta_{A+B} = \delta_A + \delta_B$, $\delta_{cA} = c\delta_A$

φ $A \in \mathrm{Ker}(\delta)$

$\delta_A = 0 \Rightarrow \delta_A(x) = 0 \;\forall x \quad R_g(x) = x \cdot g \therefore d_xR_g(\delta_A(x)) = 0$

$= d_xR_g\left(\frac{d}{dt}(x \cdot \exp(tA))\Big|_{t=0}\right) = \frac{d}{dt}\left[\;\cdots\;\right]$ finish later.

4/8/2 finish Thm:

$M \times G \longrightarrow M$

$\mathfrak{g} = \mathcal{L}ie\, G$

$A \in \mathfrak{g}$ and $x \in M \ni \delta_A(x) = 0$

Then, $\frac{d}{dt}\left[x \cdot \exp(tA)\right]\Big|_{t=0} = 0$, then use $R_g(x) \equiv x \cdot g \Rightarrow$

$$d_x R_{\exp(sA)}\left(\frac{d}{dt}\left[x \cdot \exp(tA)\right]\Big|_{t=0}\right) = 0 \quad \forall s$$

linear map sends $0 \to 0$

$$\frac{d}{dt}\left[x \cdot \exp(tA)\exp(sA)\right]\Big|_{t=0} = 0 \quad \forall s$$

$$\frac{d}{dt}\left[x \cdot \exp((t+s)A)\right]_{t=0} = 0 \quad \forall s$$

$u = s+t$

$$\frac{d}{du}\left[x \cdot \exp(uA)\right]\Big|_{\substack{t=0\\u=s}} \cdot \cancel{\frac{du}{dt}(0)}^{1} = 0$$

$$\frac{d}{ds}\left[x \cdot \exp(sA)\right] = 0 \qquad s \longmapsto x \cdot \exp(sA) \text{ const.}$$

- If $\delta_A(x) = 0 \;\forall x \in M$ and the action is effective:

  we have $x \cdot \exp(sA) = x \quad \forall s$

  $\Rightarrow x \cdot g = x \quad \Rightarrow g = e$

  $\Rightarrow sA = \exp^{-1}(e)$ for small s (exp is a local diffeom.)

  $\Rightarrow sA = 0 \quad \Rightarrow A = 0$

  hence $\delta_A(x) = 0 \;\forall x \in M$ action effective $\Rightarrow A = 0$

- If $\exists\, x \ni \delta_A(x) = 0$ then

  $x \cdot \exp(sA) = x \;\forall s \quad x \cdot g = x$ free action $\Rightarrow$

  $g = e \Rightarrow A = 0$ hence

  $\delta_A(x) = 0$ for some $x \in M$ action free $\Rightarrow A = 0$

$\therefore$ if the action is free, $A \neq 0$ then δ_A is never zero.

(P.F.B.)

$\boxed{\text{Def}}$ $P \xrightarrow{\pi} M$ is a <u>principal fiber bundle</u> iff

(1) $\pi : P \to M$ is a fiber bundle with fiber G which is a Lie group.

(2) G acts freely on the right of P

(3) $\exists$ local trivializing maps $\{\psi_u\} \ni \psi_u(u \cdot g) = \psi_u(u) \cdot g$

$\psi_u : \pi^{-1}(U) \to U \times G$

(where $(u,g) \in U \times G$ then $h \in G$ define $(u,g) \cdot h = (u, gh)$)

Note: $U \times G$ is a principal fiber in its own right.
$$\begin{array}{c} U \times G \\ \downarrow \\ U \end{array}$$

$\boxed{\text{Thm}}$ Assume $P \xrightarrow{\pi} M$ is a P.F.B. with fiber G, and trivializing maps as in (3) (ie equivariant) then the corresponding transition functions

$\varphi_{uv} : U \cap V \to \text{Diff}\, G$ have the property that

$\varphi_{uv}(x)(g) = g_{uv}(x) g$ for some set of functions $\{g_{uv}\}$

$g_{uv} : U \cap V \to G$ $\quad \varphi_{uv}(x) = L_{g_{uv}(x)}$

proof

$$(x, \varphi_{uv}(x)(g)) = (\Psi_u \circ \Psi_v^{-1})(x,g) = \psi_u \circ \psi_v^{-1}[(x,e)g]$$

note:
$$\psi_u(u \cdot g) = \psi_u(u) \cdot g \Rightarrow \psi_u(\psi_u^{-1}(u) \cdot g) = \psi_u(\psi_u^{-1}(u)) \cdot g$$
$$= u \cdot g \Rightarrow \psi_u^{-1}(u) \cdot g = \psi_u^{-1}(u \cdot g) \quad \psi_u^{-1} \text{ has prop. (3)}$$

$$= \psi_u \left[\psi_v^{-1}(x,e) \cdot g\right] = (\psi_u \circ \psi_v^{-1})(x,e) \cdot g$$

$\pi_G \circ \psi_u \circ \psi_v^{-1}(x,e) \in G$ call this $g_{uv}(x)$

$$\Rightarrow \varphi_{uv}(x)(g) = g_{uv}(x) g$$

Structure group

4/10/2

$P \xrightarrow{\pi} M$ Principal Fiber Bundle (PFB) with group G

- $\psi_U : \pi^{-1}(U) \to U \times G$ are equivariant in the 2nd factor

$\left(\text{ie } \psi_U(u \cdot g) = \psi_U(u) \cdot g \right)$ then we have that $\exists\ g_{UV} : M \to G$

$\varphi_{UV}(x)(g) = g_{UV}(x) g$ where φ_{UV} is a transition function

φ_{UV} satisfy co-cycle condition $\Rightarrow g_{UV}(x)$ do too.

$(\text{ie } g_{UV} g_{VW} = g_{UW})$

thus $g_{UU}(x) = e \quad \forall x \in M$

$\boxed{\text{Thm}}$ Let M a manifold, G a Lie group, $\{U_\alpha\}_{\alpha \in \Lambda}$ an open covering of $M \ni \forall \alpha, \beta \in \Lambda$ for which $U_\alpha \cap U_\beta \neq \emptyset$ one has a mapping $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$. If $\{g_{\alpha\beta}\}$ satisfy the cocycle condition then $\exists$ a P.F.B. $\pi : P \to M$ w/t group G. with transition functions $\{g_{\alpha\beta}\}$.

Sketch proof

$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G \quad \varphi_{\alpha\beta}(x) = g_{\alpha\beta}(x) g$

then $\{\varphi_{\alpha\beta}\}$ are transition functions which satisfy the cocycle cond.

thus we get $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ define action of P

$P \xrightarrow{\pi} M$

$u \cdot g = \psi_\alpha^{-1}(\psi_\alpha(u) \cdot g)$

need to check on overlap (action well def.)

proof 2: detailed

See handout / notes from library.

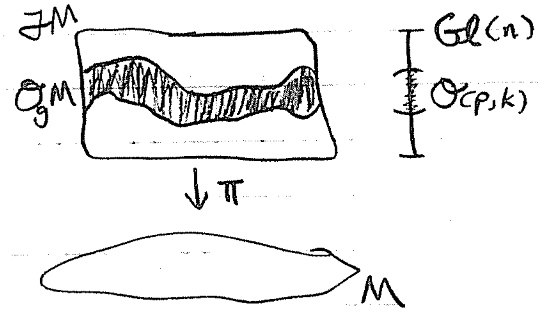
Ex: $\mathcal{F}M = \{(p, \{e_i\}) \mid \{e_i\} \text{ a basis of } T_pM\}$ is the frame bundle over M this is a PFB w/t group $Gl(n)$

choose some metric g

$$\mathcal{O}_gM = \{(p, \{e_i\}) \mid \{e_i\} \text{ an orthoginal basis of } T_pM,\ g_p(e_i, e_j) = \delta_{ij}\}$$

is the orthoginal frame bundle over M is a PFB w/t group $\mathcal{O}(p,k)$.

This is a reduced bundle



$\varphi : P \longrightarrow V$ P a P.F.B., V a vector space.

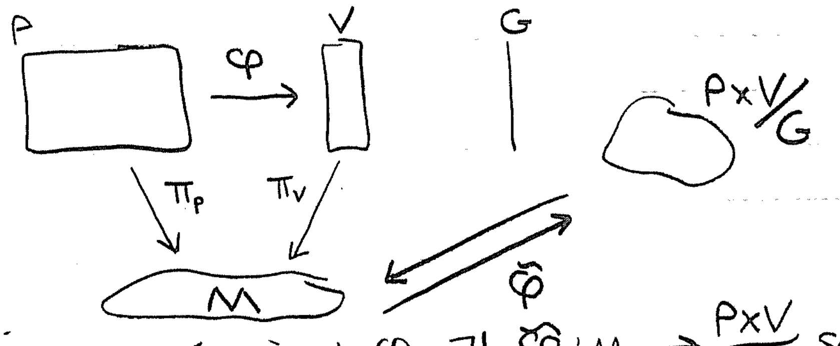
where $G \times V \longrightarrow V$ is an action on V

φ is <u>equivariant</u> iff $\varphi(u \circ a) = a^{-1} \circ \varphi(u)$

$$\frac{P \times V}{G} \qquad (u, x) \circ g = (u \circ g, g^{-1} \circ x)$$

$= \{\text{orbit space}\}$

$P \times V / G$ is a vector bundle with fiber V.



given equivariant φ $\exists!\ \tilde{\varphi} : M \longrightarrow \frac{P \times V}{G}$ section & conversely.

Handout: pgs 227, 232, 233, 238-246

4/12/2

$$TM \xrightarrow{L} \mathbb{R} \quad \text{(Lagrangian)} \qquad T^*M \xrightarrow{H} \mathbb{R} \quad \text{(Hamiltonian)}$$

- An internal theory is modeled by a vector bundle

$$E \xrightarrow{\pi} M$$

M is spacetime

In many theories you have a state-space parameterized by state vectors are not (definable in terms of tangent vectors) ie not tensor bundles,
So E is a vector bundle and $\pi^{-1}(x) = E_x$ for $x \in M$ is the set of all states of the given theory.

If M is Minkowski then $E = M \times V$, V a finite dim'l vector space.

(*) Example: (like isospin) $\begin{pmatrix} p \\ n \end{pmatrix}$ ← proton, ← neutron; $^+$ ⇝ nucleon

$$\begin{array}{c} E = M \times \mathbb{C}^2 \\ \downarrow \pi \\ M \end{array} \qquad \text{then } E_x \cong \mathbb{C}^2 \qquad p \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad n \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\psi : M \to \mathbb{C}^2 \qquad \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$$

← protonness, ← neutronness

$$g_x(z, w) = z_1 \bar{w}_1 + z_2 \bar{w}_2$$

$\boxed{\text{Def}}$ A fiber metric is a map $g \ni \forall x \in M$ g_x is a metric on E_x.

$$\tilde{\psi}(x) = (x, \psi(x))$$

→ Math types associate fields with sections of E.

(fiber V)

$\boxed{\text{Def}}$ $\mathcal{F}E$ is the <u>frame bundle</u> of the <u>vector bundle</u> $E \xrightarrow{\pi} M$

$$\mathcal{F}E = \left\{ (p, \{e_i\}) \,\middle|\, p \in M,\ \{e_i\} \text{ is a basis for } E_p \,(= \pi^{-1}\{p\}) \right\}$$

$\mathcal{F}E \xrightarrow{\pi} M$ is a principal fiber bundle with group $\mathcal{G}l(k)$
$(k = \dim V)$

$$(p, \{e_i\}) \cdot A = (p, \{e_j A^j_i\})$$

could be η_{ij}

If $\exists$ a fiber metric g we can define $\mathcal{O}_g E = \{ (p, \{e_i\}) \mid g_p(e_i, e_j) = \delta_{ij} \}$

$\mathcal{O}_g E \xrightarrow{\pi} M$ orthoginal frame bundle.

- back to example $(*)$ $g(z,w) = z_1 \bar{w}_1 + z_2 \bar{w}_2$
  then the group of $\mathcal{O}_g E$ is $U(2)$
  where $U(2) = \{ A \in \mathcal{G}l(2, \mathbb{C}) \mid \bar{A}^t A = I \}$

When M is Minkowski space, then E is trivial $(E = M \times \mathbb{C}^2)$
then $\mathcal{O}_g E \equiv \{ (p, z, w) \mid z, w \in \mathbb{C}^2 \ni g_p(z,w) = \delta_{ij} \}$
$= M \times U(2)$

These are trivial:

- $$\begin{tikzcd} \mathcal{O}_g E \arrow[dr] & & M \times U(2) \arrow[ll, "\eta"'] \arrow[dl] \\ & M & \end{tikzcd} \qquad \eta(p, A) = (p, (1,0)A, (0,1)A)$$

- $$\begin{tikzcd} \mathcal{F}E \arrow[dr] & & M \times \mathcal{G}l(m) \arrow[ll] \arrow[dl] \\ & M & \end{tikzcd} \qquad \zeta(p, A) = (p, (A^j_i \lambda^i) X_j(p))$$

$\boxed{\text{Def}}$ if $\exists$ m linear independent (at each pt.) vector fields on M $(\dim(M) = m)$ then we say M is <u>parallelizable</u>.

$\boxed{\text{Thm}}$ M parallelizable then TM & $\mathcal{F}M$ are trivial bundles.

$\boxed{\text{Thm}}$ If $\exists$ a global section of a principal fiber bundle then it must be trivial

$\boxed{\text{Thm}}$ If $\exists$ $\underline{n}$ lin. indep. sections of a vector bundle it is trivial (where $\dim V = n$)

Isospin Case:

$\exists$ sections s_1, s_2 of $E = M \times \mathbb{C}^2$

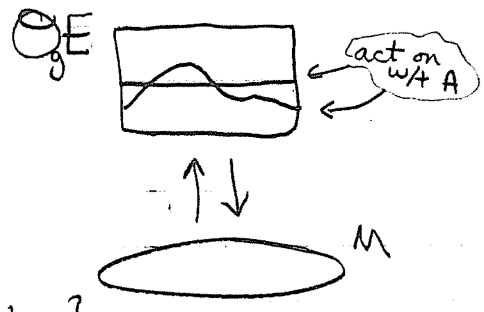
$s_1(x) = (x, (1,0))$
$s_2(x) = (x, (0,1))$

$\exists$ a secton $s: M \to O_g M$

$s(x) = (x, \{(1,0), (0,1)\})$

Let $A: M \to U(2)$

$\bar{s}(x) = (x, \{(1,0), (0,1)\}) \cdot A = (x, \{(1,0)A, (0,1)A\})$



In general M a manifold, $E \xrightarrow{\pi} M$ a vector bundle, g a fiber metric we can consider $O_g E \to M$

it has group $G = \{\text{orthoginal w/t resp. to } g\}$

4/15/2

$\boxed{\text{Lemma}^{pg}\ 239}$ If $\pi : P \to M$ is a PFB w/t group G, and $U \subseteq M$ open then $\exists\ s : U \to P$ a section (ie $\pi \circ s = \mathrm{Id}_U$) iff $\exists$ local trivializing map $\psi_U : \pi^{-1}(U) \to U \times G$

<u>Sketch proof</u>

$\Rightarrow$ $s : U \to P$ is a section (<u>note</u>: $s(U) \subseteq \pi^{-1}(U)$)

Define $\Phi : U \times G \to \pi^{-1}(U)$ by $\Phi(x, g) = s(x) \circ g$

$\Rightarrow$ $s(x) \circ g_1 = s(x) \circ g_2 \Rightarrow g_1 = g_2$ by freeness of action $\Rightarrow$ 1-1

onto by transitivity. Hence Φ is a bijection, smooth since s is smooth & the action is smooth.

Using the inverse function theorem we get locally smooth inverse ... and it has a global inverse $\Phi^{-1} \equiv \psi_U$ hence must agree w/t local inverse hence smooth. Also $\psi_U(z) \circ g = \psi_U(z \circ g)$ $\checkmark$ //

$\Leftarrow$ $\psi_U : \pi^{-1}(U) \to U \times G$ given then

define $s : U \to \pi^{-1}(U)$ by $s(x) = \psi_U^{-1}(x, e)$

(<u>note</u>: ψ_U^{-1} smooth & $\mathrm{id}_U \times \underline{e}$ is smooth ($\underline{e}(x) \equiv e$)

$\Rightarrow s$ smooth.) $\pi \circ s(x) = \pi \circ \psi_U^{-1}(x, e) = \pi_U(x, e) = x$ //

$\boxed{\text{Cor}}$. P has a global section iff it is trivial

<u>proof</u>

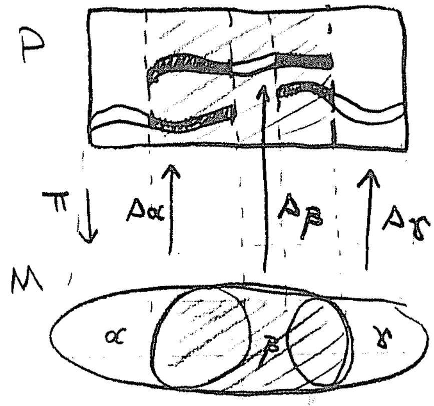
$s : M \to P \iff \psi_U : P \to M \times G$ diffeomorphism.

$\boxed{\text{Thm}}$ Let $\pi : P \to M$ be principal G-bundle, and let $\{U_\alpha\}$ be an open cover of M. $\exists$ <u>local trivializing maps</u> $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ iff $\exists$ sections $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$

If one has $\{\psi_\alpha\}$ & $\{s_\alpha\}$ and transition maps $\{g_{\alpha\beta}\}$ (of ψ_α's)

then
$$s_\alpha(x) = s_\beta(x)\, g_\beta(x)^{-1}\, g_{\beta\alpha}(x)\, g_\alpha(x)$$

where
$$g_\gamma(y) = \pi_G(\psi_\gamma(s_\gamma(y)))$$



proof
first part is immed. from Lemma.
- again $g_\gamma(y) = \pi_G \circ \psi_\gamma \circ \Delta_\gamma(y)$
$\Leftrightarrow \psi_\gamma \circ \Delta_\gamma(y) = (y, \underline{g_\gamma(y)})$

$$\psi_\beta \circ \Delta_\alpha(x) = \psi_\beta \circ \psi_\alpha^{-1}(x, g_\alpha(x)) = \psi_\beta \circ \psi_\alpha^{-1}(x, e) \circ g_\alpha(x)$$
$$= (x, g_{\beta\alpha}(x)) \circ g_\alpha(x) = (x, e) \circ g_{\beta\alpha}(x) g_\alpha(x)$$
$$\Rightarrow \Delta_\alpha(x) = \psi_\beta^{-1}(x, e) \circ g_{\beta\alpha}(x) g_\alpha(x) = \psi_\beta^{-1}(x, g_\beta(x)) \circ g_\beta(x)^{-1} g_{\beta\alpha}(x) g_\alpha(x)$$
$$= \Delta_\beta(x) g_\beta(x)^{-1} g_{\beta\alpha}(x) g_\alpha(x) \,//$$

Ex 2.6 (c)

$$\begin{array}{ll} SU(2) & G \\ & \downarrow \\ S^2 & G/H \end{array}$$

$\boxed{\text{Thm}}$ $\exists$ unique manifold structure on G/H ...

$$\psi_\alpha^{-1}(x, gh) = \Delta_\alpha(x) gh = \psi_\alpha^{-1}(x, g) h$$

$G_x \subseteq G$ $\leftarrow$ closed subgroup & $G/G_x \cong x \cdot G$

Handout:
"*Lemma* Of $A\in\mathfrak{g}$ and ...
& "*Definition* To say that $P \xrightarrow{\pi} M$ is a ...

4/17/2

Let $E \xrightarrow{\pi} M$ be a vector bundle with fiber V (a finite dim'l vector space). Let $\{n_i\}_{i=1}^{N}$ be a basis for V

Define $\mathcal{GL}(N)\times V \to V$ by $A\circ n_j \equiv A^i_j n_i$
(ie $v = v^i n_i$ then $A\cdot v = v^i A^j_i n_j = n^i(v) A^j_i n_j$)
$\Rightarrow A\cdot v = (A^j_i n^i)(v) n_j$ where $\{n^i\}$ is the dual basis.

$$\mathcal{F}E = \{ (x,\{e_i\}) \mid \{e_i\} \text{ is a basis of } \pi^{-1}\{x\} \ (=E_x)\}$$

$(\text{ie } \mathcal{F}M = \mathcal{F}(TM) \text{ from before})$ ← Bad Notation

$\boxed{\text{Thm}}$ $\exists$ 1-1 correspondence between sections φ of $E \xrightarrow{\pi} M$ and equivariant maps $\hat{\varphi} : \mathcal{F}E \to V$
(note: $(x\{e_i\})\cdot A = (x,\{e_i A^i_j\})$ the fibers are $\mathcal{GL}(N)$)
$\hat{\varphi}(x,\{e_i\}\cdot A) = A^{-1}\cdot\hat{\varphi}(x,\{e_i\})$ ← equivariance

- Physicists start with $\varphi : M \to V$ from this we induce $\tilde{\varphi} : M \to M\times V$ def by $\tilde{\varphi}(x) = (x,\varphi(x))$ from which we can induce $\hat{\varphi} : \mathcal{F}(M\times V) \to V$

Note: This Thm can be generalized to
$P \xrightarrow{\pi} M$ principal, $G\times F \to F$, then sections of $P\times F \to M$ are in 1-1 correspondance w/t equiv. maps from P to F.

proof
Let $\varphi : M \to E$ be a section of π. Define $\hat{\varphi} : \mathcal{F}E \to V$ by $\hat{\varphi}(x,\{e_i\}) = e^i(\varphi(x))\, n_i$ (this is ok because $\varphi(x)\in E_x$)

$$\hat{\varphi}[(x,\{e_i\})\cdot A] = \hat{\varphi}(x,\{\underbrace{e_i A^i_j}_{f_j}\}) = f^i(\varphi(x))\, n_i$$

but $f^i = (A^{-1})^i_j e^j \quad \therefore \quad f^i(\varphi(x))r_i = ((A^{-1})^i_j e^j)(\varphi(x))r_i$

$= A^{-1} \circ e^i(\varphi(x))r_i$ (from before)

$\therefore \hat{\varphi}$ is equivariant

Given $\hat{\varphi} : \mathcal{F}E \to V$ define $\varphi : M \to E$ a section of π as follows:

$x \in M$ then choose $\{e_i\}$ a basis of E_x then

$\varphi(x) = r^j(\hat{\varphi}(x, \{e_i\}))e_i$

If $\{f_i\}$ is another basis then $\exists\, A \ni e_i = A^j_i f_j$ and $e^i = A^{-1\,i}_{\;j} f^j$

$r^j(\hat{\varphi}(x, \{e_i\}))e_j = r^i(\hat{\varphi}((x, \{f_k\}) \cdot A)\, A^j_i f_j$

$= r^i(A^{-1} \circ \hat{\varphi}(x, \{f_k\}))\, A^j_i f_j = r^i(A^{-1\,k}_{\;\ell} r^\ell(\hat{\varphi}(x, \{f_c\}))r_k) A^j_i f_j$

$= A^{-1\,k}_{\;\ell} r^\ell(\hat{\varphi}(x, \{f_c\}))\, \delta^i_k A^j_i f_j = \cancel{A^{-1\,k}_{\;\ell} A^j_k}^{\;\delta^j_\ell}\, r^\ell(\hat{\varphi}(x, \{f_c\}))\, f_j$

$= r^j(\hat{\varphi}(x, \{f_i\}))\, f_j \quad \therefore$ well defined

finally $\pi \circ \varphi(x) = \pi(\underbrace{r^i(\hat{\varphi}(x, \{e_i\}))e_i}_{\in E_x}) = x$ ✓

Skip: φ, $\hat{\varphi}$ smooth and 1-1-ness of this correspondance

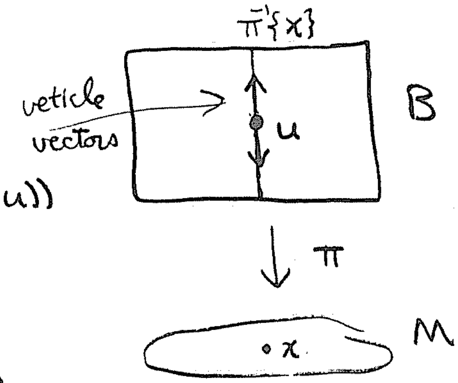
---

Skip Exercise 2.7 for Now

? See Ex 2.8 do for $T^1_1 M$?

Connections

Def Let $B \xrightarrow{\pi} M$ be a fiber bundle with fiber F.
It is possible to select a subspace from T_uB for $u \in B$ which we call <u>vertical vectors</u>. These vertical vectors are tangent to the fiber of π at u.



If $X \in T_uB$ is tangent to $\pi^{-1}(\pi(u))$ then $\exists$ a curve $\gamma : (-a,a) \to \pi^{-1}(\pi(u))$ $\ni \gamma(0) = u$ & $\gamma'(0) = X$

Also, note that $d_u\pi : T_uB \to T_{\pi(u)}M$

$$d_u\pi(X) = d_u\pi(\gamma'(0)) = \frac{d}{dt}(\overbrace{\pi \circ \gamma(t)}^{\text{const.}})\Big|_{t=0} = \frac{d}{dt}(u)\Big|_{t=0} = 0$$

denote the set of vertical vectors by

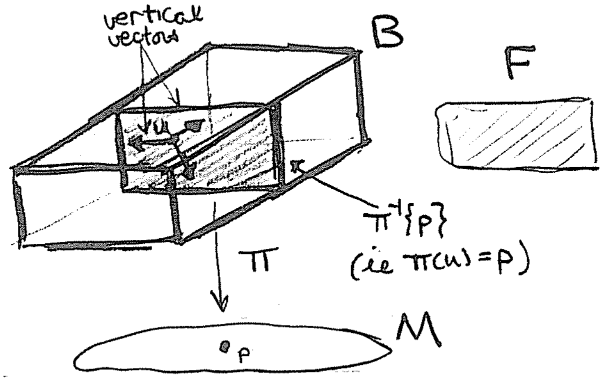
$$VT_uB = \{ X \in T_uB \mid d_u\pi(X) = 0 \}$$

Our choice of complement: $VT_uB \oplus \underline{\quad\quad} = T_uB$
will give rise to our connection

$$\dim M + \dim F = \dim B$$

$$\dim F \neq 0 \Rightarrow VT_uB \subsetneq T_uB$$

4/19/2



$B \xrightarrow{\pi} M$ with fiber F

$V T_u B = \{ X \in T_u B \mid d_u\pi(X) = 0 \}$ = <u>vertical vectors</u>.

$\boxed{\text{Def}}$ An <u>Ehresmann connection</u> on B is a mapping
$H : B \longrightarrow \{$ subspaces of T_uB where $u \in B\}$
(1) $\forall\, u \in B$ $H(u) = HT_uB$ is a subspace of T_uB
$\ni T_uB = H(u) \oplus VT_uB = HT_uB \oplus VT_uB$
we write $X_u = hX_u + vX_u = Y_u + Z_u$
(2) The mapping H is "smooth" $\equiv \forall$ vector field X on B
the vector fields hX & vX are smooth.

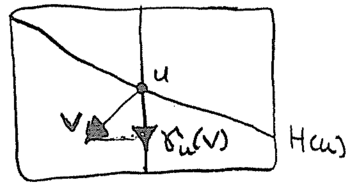
If $B \xrightarrow{\pi} M$ is a P.F.B. with group G, then H is called a <u>(principal) connection</u> iff H is an Ehresmann connection $\ni$
(3) $d_u r_g(H(u)) = H(u \cdot g)\ \forall\, u \in B,\ g \in G$
where $r_g : B \to B$ defined by $r_g(x) = x \cdot g$

<u>note</u>: $d_u r_g : T_uB \to T_{u\cdot g}B$ $\therefore$ $d_u r_g(H(u)) \subseteq T_{u\cdot g}B$

- $\dim(B) = \dim(M \times F) = \dim(M) + \dim(F)$
and $\dim(VT_uB) = \dim(F)$ & $\dim(H(u)) = \dim(M)$.

$\mathcal{F}(TM)$ group is $GL(n)$ where $n = \dim(M)$
$\downarrow$ (<u>note</u>: prev. we wrote $\mathcal{F}M$ instead of $\mathcal{F}(TM)$)
M

connections in Reimannian geometry live here



Let H be an Ehresmann connection we can define a map
$\gamma_u : T_uB \to T_uB \ni \gamma_u(X_u) = vX_u + \cancel{hX_u}$ just keep the vert. part.
(i) γ_u is a projection and $\gamma_u(T_uB) = VT_uB$
(ii) $\mathrm{Ker}(\gamma_u) = H(u)$
γ is smooth since it sends smooth vector fields to smooth v.f.'s
and conversely,
given a projection γ_u onto $VT_uB \ni \gamma$ is smooth.
we get $H(u) \equiv \mathrm{Ker}(\gamma_u)$ a connection.

If $\pi : B \to M$ is principal and H is a prin. connection
we can induce a projection γ.
which induces a $\mathfrak{g}$-valued 1-form.

$$\gamma_u : T_uB \to VT_uB \xleftrightarrow{\delta_u^{-1}} \mathfrak{g}$$

this induced form is the "connection form"

note: $\begin{array}{c} VTB \\ \downarrow \\ B \end{array}$ is a vector bundle

Handout: pgs 254–266

4/22/2

$\boxed{\text{Thm}}$ $\pi : P \to M$ is a P.F.B. with structure group G.
If $H : P \to \{\text{subspaces of } T_uP \text{ for } u \in P\}$ is a principal connection then $\exists$ $\mathfrak{g}$-valued one form ω on P $\ni$

① $\mathrm{Ker}(\omega_u) = H(u)$ $\forall u \in P$

② $\omega_u(\delta_A(u)) = A$ $\forall u \in P, A \in \mathfrak{g}$

③ $r_g^* \omega_{ug} = Ad(g^{-1})\omega_u$. $\forall g \in G$

proof

$\sigma(u,g) = u \cdot g$

Let H be a connection. For $A \in \mathfrak{g}$, $u \in P$

$$\delta_A(u) = \frac{d}{dt}\left[u \cdot exp(tA)\right]\Big|_{t=0} = \frac{d}{dt}\left[\sigma_u(exp(tA))\right]\Big|_{t=0}$$

$$= d_e\sigma_u\left(\frac{d}{dt}(exp(tA))\Big|_{t=0}\right) = d_e\sigma_u(A) \quad (\delta_A(u) \text{ linear in } A)$$

Since the action is free: $\delta_A(u) = 0$ for some u then $A = 0$

If we define $\delta_u : \mathfrak{g} \to VT_uP$ by $\delta_u(A) = \delta_A(u)$

Note: δ_u is linear & injective ($\delta_u(A) = 0 \Rightarrow A = 0$)

$\dim VT_uP = \dim T_u\pi^{-1}(\pi(u)) = \dim(\pi^{-1}(\pi(u))) = \dim(G) = \dim(\mathfrak{g})$

hence δ_u is an isomorphism. (vector space) $\forall u \in P$.

Let $\omega_u = \delta_u^{-1} \circ \gamma_u$ where γ_u is the E. Connection induced by H.

$X \in \mathrm{Ker}(\omega_u) \Leftrightarrow \omega_u(X) = \delta_u^{-1} \circ \gamma_u(X) = 0 \Leftrightarrow \gamma_u(X) = 0 \Leftrightarrow X \in H(u)$

① $\therefore \mathrm{Ker}(\omega_u) = H(u)$ ✓

Since γ_u is a projection of T_uP onto VT_uP, we have $\gamma_u | Im(\gamma_u)$

$= \gamma_u|_{VT_uP} = id_{VT_uP}$ and so, $\omega_u(\delta_A(u)) = \delta_u^{-1}\gamma_u(\delta_A(u)) = \delta_u^{-1}(\delta_A(u))$

② (since $\delta_A(u) \in VT_uP$) $= \delta_u^{-1}\delta_u(A) = A$ ✓

Since H is principal $d_u r_g(H(u)) = H(u \cdot g)$ $\forall u \in P, g \in G$.

$r_g^*(\omega)(h) = \omega_{ug}(d_u r_g(h)) = 0$ since $d_u r_g(h) \in H(u \cdot g) = \mathrm{Ker}(\omega_{ug})$

$\forall h \in H(u)$ hence $Ad(g^{-1})\omega_u(h) = Ad(g^{-1})(0) = 0$ ✓

$r_g^*(\omega)_u(\delta_A(u)) = \omega_{ug}(d_u r_g(\delta_A(u))) \overset{\text{lemma}}{\underset{\text{pg256}}{=}} \omega_{ug}\, \delta_{Ad(g^{-1})A}(ug)$

$= Ad(g^{-1})(A) = Ad(g^{-1})\,\omega_u(\delta_A(u)) \Rightarrow r_g^*\omega_{ug} = Ad(g^{-1})\omega_u$ on VT_uP

$\Rightarrow X \in T_uP \quad r_g^* \omega(vX + hX) = r_g^*\omega(vX) + \cancel{r_g^*\omega}(hX)^{0}$

$= Ad(g^{-1})\omega(vX) + Ad(g^{-1})\overset{(=0)}{\omega}(hX) = Ad(g^{-1})\omega(vX + hX)$ ✓

Converse:

$\boxed{\text{Thm}}$ Let ω be a Lie algebra valued 1-form on TP ∋

① $\omega_u(\delta_A(u)) = A \quad \forall\, u \in P, A \in \mathfrak{g}$

② $r_g^*\omega = Ad(g^{-1})\omega$

then

③ $H(u) \equiv Ker(\omega_u)$ is a principal connection.

Proof

First show $T_uP = H(u) + VT_uP$ then show sum is direct:

Let $w \in T_uP$ and let $w_v \equiv \delta_{\omega_u(w)}(u) \in VT_uP$

need $w - w_v \in H(u) = Ker(\omega_u)$: $\omega_u(w - w_v) =$

$\omega_u(w) - \omega(\delta_{\omega_u(w)}(u)) = \omega_u(w) - \omega_u(w) = 0$ ✓

& $w \in H(u) \cap VT_uP \Rightarrow \omega_u(w) = 0$ & $\delta_A(u) = w$ some $A \in \mathfrak{g}$

$\Rightarrow \omega_u(\delta_A(u)) = A = 0 \Rightarrow w = 0 \quad \therefore T_uP = H(u) \oplus VT_uP$

Note: $vX_u = \delta_u(\omega_u(X_u))$ is smooth b/c δ & ω are.

and $hX_u = X_u - vX_u$ is smooth ✓ $\quad Ad(g^{-1}) = Ad(g)^{-1}$ $\quad$ invertible map

$w \in H(u) \Leftrightarrow w \in Ker(\omega_u) \Leftrightarrow \omega_u(w) = 0 \Leftrightarrow Ad(g^{-1})\omega_u(w) = 0$

$\Leftrightarrow \omega_{ug}(d_u r_g w)) = 0 \Leftrightarrow d_u r_g(w) \in Ker(\omega(ug)) = H(ug)$

$\Rightarrow$ The connection is principal

Handout:
"**Theorem** Assume that $\varphi: G \to H$ is a ..."

4/24/2

Physics

A gauge potential is a Lie algebra valued 1-form on a space-time
(Some must be modified if the space-time is not contractible.)
M be a space-time manifold (say Minkowski)
Two such potentials $A, \bar{A}$ are gauge equivalent $\begin{pmatrix} G \subseteq Gl(n) \\ \mathfrak{g} \subseteq gl(n) \end{pmatrix}$

$$\iff \exists\, g: M \to G \ni \bar{A} = g^{-1}Ag + g^{-1}dg, \quad (= g^{-1}[Ag + dg\,g^{-1}]g)$$

ie
$$\hookrightarrow \bar{A}_x(V) = g^{-1}(x)\,A_x(V)\,g(x) + g^{-1}(x)\,d_x g(V)$$

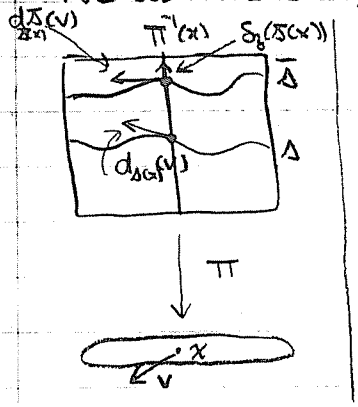
Math

A connection ω on a P.F.B. $P \xrightarrow{\pi} M$ with structure group G.
If s & $\bar{s}$ are sections of π define $A = s^*\omega$ & $\bar{A} = \bar{s}^*\omega$
(think global for now)

$$s(x), \bar{s}(x) \in \pi^{-1}\{x\} = P_x = s(x)\cdot G \implies s(x)\cdot g_x = \bar{s}(x)$$

$g: M \to G$ is smooth (skip)

We can show $d_x\bar{s} = d_{s(x)}r_{g(x)} \circ d_x s + \delta_z(\bar{s}(x))$ for some $z \in \mathfrak{g}$ vertical vector



ie $d_{\bar{s}(x)}\pi\left(d_x\bar{s}(V) - d_{s(x)}r_{g(x)} \circ d_x s(V)\right) = 0$

$$z = dl_{g(x)^{-1}} \circ d_x g$$

$$\omega_{\bar{s}(x)}(d_x\bar{s}(V)) = \omega_{\bar{s}(x)}(d_{s(x)}r_{g(x)} \circ d_x s(V)) + \omega_{\bar{s}(x)}(\delta_z(\bar{s}(x)))$$

$$\omega_{\bar{s}(x)}(d_x\bar{s}(V)) = (\bar{s}^*\omega)_x(V)$$

$$= Ad(g(x)^{-1})(s^*\omega)_x(V) + dl_{g(x)^{-1}} \circ d_x g(V)$$

$$\implies \bar{A}_x(V) = Ad(g(x)^{-1})A_x(V) + g(x)^{-1}d_x g(V)$$

$\hookrightarrow$ so our P.F.B. & Connection ... encode the gauge equiv.

Next Semester:

$\mathcal{G} = \{ g : M \to G \}$ $\quad \mathcal{A} = \{ \text{gauge potentials} \}$

$\mathcal{A} \times \mathcal{G} \to \mathcal{A}$ by $g \cdot A = g^{-1} A g + g^{-1} dg$

$\hookrightarrow$ work to get an ∞-dim'l P.F.B. $\quad \mathcal{A} \searrow \mathcal{A}/\mathcal{G}$ w/ fiber $\mathcal{G}$

Gets messy b/c ∞-dim'l.

$\boxed{\text{Thm}}$ $\pi : P \to M$ P.F.B. with group G,
ω a connection.
$s : U \to P$ & $\bar{s} : \bar{U} \to P$ $\quad U \cap \bar{U} \neq \emptyset$ are local sections
then $\exists\, g : U \cap \bar{U} \to G \ni \bar{s}(x) = s(x) \cdot g(x)$ with g smooth &
$(\bar{s}^* \omega)_x = Ad(g(x)^{-1})(s^* \omega)_x + d_{g(x)} l_{g(x)}^{-1} \circ d_x g$

proof (Rough Sketch)
$\gamma(0) = p \quad \bar{s}(\gamma(t)) = s(\gamma(t)) g(\gamma(t))$
$\gamma'(0) = v \quad d_{\gamma(t)} \bar{s}(\gamma'(t)) = \frac{d}{dt}\left[s(\gamma(t)) g(\gamma(t)) \right]$
define $f(t_1, t_2) = s(\gamma(t_1)) \cdot g(\gamma(t_2))$ look for $\left. \frac{d}{dt} f(t,t) \right|_{t=0}$

$$= \frac{\partial}{\partial t_1} f(0,0) + \frac{\partial}{\partial t_2} f(0,0) = d\, r_{g(\gamma(0))} (d_{\gamma(0)} s(\gamma'(0))) + \dots$$

details in notes pg 264 etc. //

We can define equivalence classes of gauge potentials
$A_\alpha = s_\alpha^* \omega \quad A_\beta = s_\beta^* \omega \quad U_\alpha \cap U_\beta \neq \emptyset$ then
$A_\alpha = g^{-1} A_\beta g + g^{-1} dg$ some g then $A_\alpha \sim A_\beta$

Given an open cover $\{U_\alpha\}$ with local sections $\{s_\alpha\}$
and we have $\{A_\alpha\} \ni A_\alpha = g_{\alpha\beta}^{-1} A_\beta g_{\alpha\beta} + g_{\alpha\beta}^{-1} d g_{\alpha\beta}$ some $g_{\alpha\beta}$
we get a unique connection ω
we also have enough data to both the P.F.B. & ω.

4/26/2 How to Build a Connection:

Given $A: TU \to \mathfrak{g}$ where U is the domain of a section $s: U \to P$ of some PFB $\pi: P \to M$.
How do you define a connection on $\pi^{-1}(U)$?

Define ω at points of $s(U)$ first.
Let $x \in U$ and $W \in T_{s(x)} s(U)$ $\quad W = d_x s(V)$ for some $V \in T_x M$

Vectors tangent to $s(U)$ ∘ Define $\omega_{s(x)}(W) = \omega_{s(x)}(d_x s(V)) = A_x(V)$ $\quad (s^*\omega = A)$
Let $\hat{W} \in T_{s(x)} \pi^{-1}(U)$ ($\in T_{s(x)}P$ since $\pi^{-1}(U)$ is an open submanifold of P)

$\hat{V} = d_{s(x)}\pi(\hat{W}) \in T_x M \quad d_{s(x)}\pi\left[\hat{W} - d_x s(\hat{V})\right] = d_{s(x)}\pi(\hat{W}) -$

$d_{s(x)}\pi(d_x s(\hat{V})) = \hat{V} - \underbrace{d_x \pi \circ s}_{id}(\hat{V}) = \hat{V} - \hat{V} = 0$

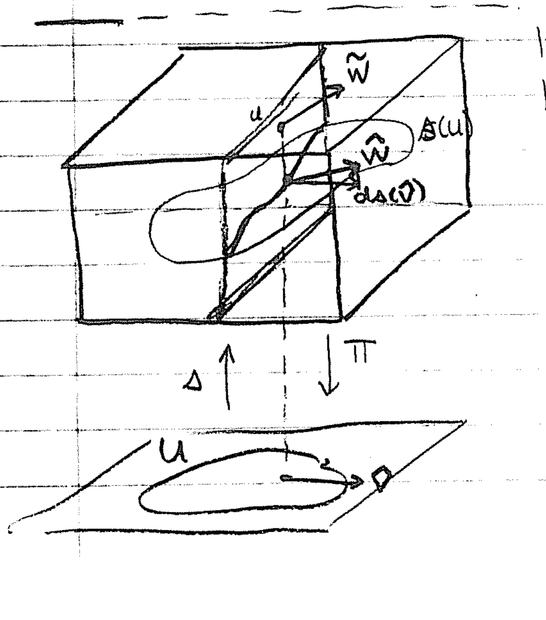
$\Rightarrow \hat{W} - d_x s(\hat{V}) \in VT_{s(x)}P$

$\therefore \hat{W} - d_x s(\hat{V}) = \delta_{s(x)}(\xi)$ some $\xi \in \mathfrak{g}$

Vectors at points in $s(U)$ ∘ $\omega_{s(x)}(\hat{W}) = \omega(d_x s(\hat{V}) + \delta_{s(x)}(\xi)) \equiv A_x(\hat{V}) + \xi$

Now let $u \in \pi^{-1}(x)$, $x \in U$ then $u = s(x)\cdot g$ some $g \in G$

In general ∘ For $\tilde{W} \in T_u P$ where $\pi(u) \in U$ then
$\omega_u(\tilde{W}) \equiv Ad(g^{-1})\,\omega_{s(x)}(d_u R_{g^{-1}}(\tilde{W}))$
$(= s(x)\cdot g)$



? Greg ?
Magnetic Monopole ... (see Neighbors)
...
QFT & Anomalies by Bertlemann
...

$A^a_\mu \, dx^\mu \otimes T_a$ ← basis for Lie algebra.

↖ Lie algebra-valued 1-form on (U, x)

What is the <u>horizontal lift</u> of a vector?

$x \in M$

$\boxed{\text{Def}}$ $u \in \pi^{-1}(x)$ & $v \in T_xM$ the <u>horizontal lift</u> of v to u is $\hat{v} \in T_uP$, $\hat{v} = [d_u\pi|_{H(u)}]^{-1}(v) \in H(u) \subseteq T_uP$

$\exists\!\!\!/\ d_u\pi(\hat{v}) = 0 \Rightarrow \hat{v} \in VT_uP \cap H(u) = \{0\} \Rightarrow d_u\pi|_{H(u)}$ is 1-1.

$\dim(H(u)) = \dim(M) = \dim(T_xM)$ hence $d_u\pi|_{H(u)}$ is a linear isomorphism. ✓

Alt. Def. properties

① $d_u\pi(\hat{v}) = d_u\pi([d_u\pi|_{H(u)}]^{-1}(v)) = v$

② $\omega_n(\hat{v}) = 0$

$\exists\!\!\!/\ \hat{v} \in T_uP \ni d_u\pi(\hat{v}) = v$ & $\omega_n(\hat{v}) = 0$

$\Rightarrow \hat{v} \in H(u) \therefore [d_u\pi|_{H(u)}]^{-1}(v) = \hat{v}$

look at Prop on pg 2 of Handout

$\boxed{\text{Prop}}$ X is v.f. on M then horizontal lift of X is a v.f. $\ni$ at $u \in P$ it is $[d_u\pi|_{H(u)}]^{-1}(X_{\pi(u)})$

this is smooth ✓

⋮

Will use horizontal lift to talk about covariant derivative.

4/29/2

$P \xrightarrow{\pi} M$ P.F.B. with connection

$G \times V \to V$, V a finite dimensional vector space.

$\hat{\varphi}: P \to V$ equivariant

Def $D\hat{\varphi}: TP \to V$ horizontal comp.

$(D\hat{\varphi})_u(Y_u) = d_u \hat{\varphi}(h(Y_u))$

(dim $M = n$)

Ex: $P = \mathcal{F}(TM) = \{(p, \{e_i\}) \mid p \in M, \{e_i\}$ a basis of $T_pM\}$

$$Gl(n) \times (\mathbb{R}^n)^* \longrightarrow (\mathbb{R}^n)^* \quad (g \cdot f)(x) \equiv f(g^{-1}x)$$

$$\hat{\varphi}: \mathcal{F}(TM) \longrightarrow (\mathbb{R}^n)^* \quad \alpha \in (\mathbb{R}^n)^*, \alpha = \alpha_i r^i$$ $\{r^i\}$ dual standard basis of $\mathbb{R}^n$

$\varphi: M \longrightarrow T^*M$

$\varphi(p) = \varphi_i(p)\, d_p x^i$

equivariant map

$\hat{\varphi}((p, \{e_i\})) = \hat{\varphi}_j(p, \{e_i\}) r^j$

$\varphi(p) = \hat{\varphi}_j(p, \{e_i\}) e^j \in (T_p^* M)$

section of the bundle TM

equivariant maps.

section of the bundle.

Prop $D_u\hat{\varphi}(Y) = d_u\hat{\varphi}(Y_u) + \omega_u(Y_u) \cdot \hat{\varphi}(u)$

(better formula for covariant derivative)

Def $\mathfrak{g} \times V \to V$ by $(z, x) \longmapsto \frac{d}{dt}(\exp(tz) \cdot x)\Big|_{t=0}$

(Lie algebra)

If $\{e_\alpha\}$ is a basis of $\mathfrak{g}$ and $\{V_b\}$ is a basis of V

then $e_\alpha \cdot V_b = h^c_{\alpha b} V_c$ where $h^c_{\alpha b}$ are constants.

where $b, c = 1, \ldots, \dim V$ $\alpha = 1, \ldots, \dim \mathfrak{g}$

representation of Lie algebra on V

Weinberg "I" vol. 2

$$\omega(\Upsilon) = \omega^{\alpha}(\Upsilon)e_{\alpha} \qquad \hat{\varphi}(u) = \hat{\varphi}^{a}(u)V_a$$

then

$$D_u\hat{\varphi}(\Upsilon_u) = d_u\hat{\varphi}(\Upsilon_u) + \omega^{\alpha}_u(\Upsilon)e_{\alpha}\cdot(\hat{\varphi}^{a}(u)V_a)$$

$$D_u\hat{\varphi}(\Upsilon_u) = (d_u\hat{\varphi})^{b}(\Upsilon_u)V_b + \omega^{\alpha}_u(\Upsilon)\hat{\varphi}^{b}(u)h^{c}_{\alpha b}V_c$$

$$\therefore D_u\hat{\varphi} = \left(d_u\hat{\varphi}^{b} + \omega^{\alpha}_u\hat{\varphi}^{c}(u)h^{b}_{\alpha c}\right)V_b$$

covariant derivative components

$$\boxed{D_u\hat{\varphi}^{c} = d_u\hat{\varphi}^{c} + h^{c}_{\alpha b}\hat{\varphi}^{b}(u)\omega^{\alpha}_u}$$

$\boxed{\text{Prop}}$ $D_u\hat{\varphi}(\Upsilon) = d_u\hat{\varphi}(\Upsilon_u) + \omega_u(\Upsilon_u)\cdot\hat{\varphi}(u)$

proof

$\Upsilon_u \in T_uP \Rightarrow v\Upsilon_u \in VT_uP$ and so $(v\Upsilon)_u = \delta_u(A(u))$

for some $A(u) \in \mathfrak{g}$

$$\omega_u(\Upsilon(u)) = \underbrace{\cancel{\omega(h\Upsilon_u)}}_{\to 0} + \omega(v\Upsilon_u) = \omega(\delta_u(A(u))) = A(u)$$

$$v\Upsilon_u = \delta_u(\omega(\Upsilon_u)) = \frac{d}{dt}\left[u\cdot\exp(t\,\omega(\Upsilon_u))\right]\Big|_{t=0}$$

$$d_u\hat{\varphi}(v\Upsilon_u) = d_u\hat{\varphi}\left(\frac{d}{dt}\left[u\cdot\exp(t\,\omega(\Upsilon_u))\right]\Big|_{t=0}\right)$$

$$= \frac{d}{dt}\left[\hat{\varphi}(u\cdot\exp(t\,\omega(\Upsilon_u)))\right]\Big|_{t=0} =$$ use equivar. prop of $\hat{\varphi}$ $\Rightarrow$

$$\frac{d}{dt}\left[\exp(-t\,\omega(\Upsilon_u))\cdot\hat{\varphi}(u)\right]\Big|_{t=0} \underset{\text{by def}}{=} -\omega(\Upsilon_u)\cdot\hat{\varphi}(u)$$

$$\Rightarrow D_u\hat{\varphi}(\Upsilon) = d_u\hat{\varphi}(h\Upsilon_u) = d_u\hat{\varphi}(\Upsilon_u) - d_u\hat{\varphi}(v\Upsilon_u)$$

$$= d_u\hat{\varphi}(\Upsilon_u) + \omega(\Upsilon_u)\cdot\hat{\varphi}(u) \quad //$$

$$\begin{tikzcd} P \arrow[r, "\hat{\varphi}"] \arrow[d, "\pi"'] & V \\ M & \end{tikzcd} \qquad \begin{tikzcd} P\times_G V \arrow[d] \\ M \end{tikzcd}$$

G-orbit

$$P\times_G V = \{ (u,v)\cdot G \mid (u,v)\in P\times V \}$$

$$(u,v)\cdot g = (ug, g^{-1}v)$$

vector field on M

$$\begin{tikzcd} \varphi \in \Gamma_M(P\times_G V) \arrow[d] & \longleftarrow \text{sections} \longrightarrow & X \in \Gamma_M(TM) \arrow[d] \\ \hat{\varphi} \in \text{Equivariant Maps}_G(P,V) \arrow[d] & & \tilde{X} \text{ horizontal lift} \arrow[d] \\ D\hat{\varphi} : TP \to V \arrow[rr] & & D\hat{\varphi}(\tilde{X}) \in \text{Equivar. Maps}_G(P,V) \arrow[d] \\ & & \nabla_X \varphi \text{ covar. der.} \end{tikzcd}$$

Wed.: Curvature
Fri.: Holonomy

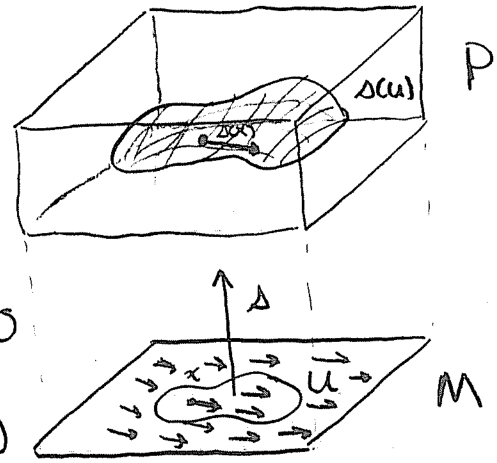
5/1/2

1. Find a formula for the horizontal lift of a vector field.

$P \xrightarrow{\pi} M$ P.F.B., X a vector field on M, $\omega: TP \to \mathfrak{g}$ a connection

Let $s: U \to P$ be a local section. Then,

$$\tilde{X}(s(x)) = d_x s(X_x) - \delta_{s(x)}(A_x(X_x)) \quad \text{where } A = s^*\omega$$



proof

$$d_{s(x)}\pi\left(\tilde{X}(s(x)) - d_x s(X_x)\right)$$

$$= X_{\pi s(x)} - d_{s(x)}\pi(d_x s(X_x))$$

$$= X_x - d_x \underbrace{\pi \circ s}_{id}(X_x) = X_x - X_x = 0$$

$$\therefore \tilde{X}(s(x)) = d_x s(X_x) + \underbrace{\delta_{s(x)}(z)}_{\text{vertical part}}$$

$$\underset{\overset{\|}{0}}{\omega(\tilde{X}(s(x)))} = \omega(d_x s(X_x)) + \underset{\overset{\|}{z}}{\omega(\delta_{s(x)}(z))}$$

$$0 = s^*\omega(X_x) + z \Rightarrow z = s^*\omega(X_x) = A_x(X_x)$$

Assume (x^μ) are coordinates on U. We can lift ∂_μ:

$$\tilde{\partial}_\mu(x) = d_x s(\partial_\mu(x)) - \delta_{s(x)}(A_x(\partial_\mu(x)))$$

$$\therefore \tilde{\partial}_\mu(x) = (\partial_\mu s)(x) - \delta_{s(x)}(A_\mu(x)) \quad \text{where } A_x = A_\mu(x)\, d_x x^\mu$$

- So $s: U \to P$ is horizontal if $\tilde{\partial}_\mu(x) = (\partial_\mu s)(x)$ that is $\delta_{s(x)}(A_\mu(x))$ ie $A_\mu = 0$ ie $A = 0$

2. If $P \xrightarrow{\pi} M$ is a P.F.B w/ group G, $G \times V \to V$ is an action on a finite dim'l vector space V.
then $\tilde{X}(\hat{\varphi}) \in Maps_G(P,V)$ (= Equivariant Maps from P to V) $\forall \hat{\varphi} \in Maps_G(P,V)$ and $\forall X$ a vector field on M.

section $\quad$ (equiv. map) $\quad$ section

$$\varphi \xleftrightarrow[\text{corresp.}]{1-1} \hat{\varphi} \longrightarrow \tilde{X}(\hat{\varphi}) \xleftrightarrow[\text{corresp.}]{1-1} \nabla_X \varphi$$

---

covariant derivative

different notations:

$$L_{\tilde{X}}(\hat{\varphi}) = \tilde{X}(\hat{\varphi}) = d\hat{\varphi}(\tilde{X}) = D\hat{\varphi}(\tilde{X}) = d\hat{\varphi}(h\tilde{X})$$

Note: $d_u r_g(\tilde{X}(u)) = \tilde{X}(ug)$ & $d\pi \circ dr_g = d\pi$

$\boxed{\text{Lemma}}$ $\tilde{X}(\hat{\varphi}) \in Maps_G(P,V)$ (ie $\tilde{X}(\hat{\varphi})$ is equivariant)

proof see notes.

3. If $\varphi \in \Gamma_M(P \times_G V)$ and X is a vector field on M. we define $\nabla_X \varphi$ to be the element of $\Gamma_M(P \times_G M)$ $\ni$

$$(\nabla_X \varphi)(x) = (u_x, \underline{\tilde{X}(\hat{\varphi})}(u_x)) \cdot G \quad u_x \in \pi^{-1}(x)$$

(well defined b/c equivar.)

$$= (u_x, D\hat{\varphi}(\tilde{X}(u_x))) \cdot G$$

Let $\{V_a\}$ be a basis for V, $\{e_\alpha\}$ a basis for $\mathfrak{g}$ and $s: U \to P$ a local section of π.

Define:

$$W_a(x) = (s(x), V_a) \cdot G$$

$$\varphi(x) = (s(x), \hat{\varphi}(s(x))) \cdot G = (s(x), \hat{\varphi}^a(s(x)) V_a) \cdot G$$
$$= \hat{\varphi}^a(s(x)) (s(x), V_a) \cdot G = \hat{\varphi}^a(s(x)) W_a(x)$$

$\therefore \varphi^a(x) = \hat{\varphi}^a(s(x))$ where $\varphi^a(x) W_a(x) = \varphi(x)$, $\hat{\varphi}^a(s(x)) V_a = \hat{\varphi}(s(x))$

$$\Rightarrow \boxed{\varphi^a = \hat{\varphi}^a \circ s}$$

Claim: If $(\nabla_X \varphi)(x) = (\nabla_X \varphi)^a(x) W_a(x)$, then

$$(\nabla_X \varphi)^a(x) = d\varphi^a(X_x) + h^a_{\alpha c} A^\alpha(X_x) \varphi^c(x), \quad A = s^* \omega$$

$$e_\alpha \cdot V_a = h^b_{\alpha a} V_b$$

$$(\nabla_{\partial_\mu} \varphi)^a(x) = (\partial_\mu \varphi^a)(x) + h^a_{\alpha c} A^\alpha_\mu(x) \varphi^c(x) \quad \text{where } A^\alpha = A^\alpha_\mu dx^\mu$$
$$A = A^\alpha e_\alpha$$

$$(\nabla_{\mathfrak{X}}\varphi)(x) = (s(x), d\hat{\varphi}(\tilde{\mathfrak{X}}(s(x))))\cdot G = (s(x), d\hat{\varphi}^a(\tilde{\mathfrak{X}}(s(x)))V_a)G$$

$$= d\hat{\varphi}^a(\tilde{\mathfrak{X}}(s(x)))\,(s(x), V_a)\cdot G = d\hat{\varphi}^a(\tilde{\mathfrak{X}}(s(x)))\,W_a(x)$$

$$= d\hat{\varphi}^a(d_x s(\mathfrak{X}_x) - \delta_{s(x)}((s^*\omega)(\mathfrak{X}_x)))\,W_a(x)$$

$$= d_x\hat{\varphi}^a\circ s(\mathfrak{X}_x) - d\hat{\varphi}^a(\delta_{s(x)}((s^*\omega)(\mathfrak{X}_x)))\,W_a(x)$$

(use (*))

$$= \left(d\varphi^a(\mathfrak{X}_x) + [(s^*\omega)(\mathfrak{X}_x)\cdot\hat{\varphi}(s(x))]^a\right)W_a(x)$$

$$\boxed{(\nabla_{\mathfrak{X}}\varphi)(x) = \left(d\varphi^a(\mathfrak{X}_x) + [(s^*\omega)(\mathfrak{X}_x)\cdot\varphi(x)]^a\right)W_a(x)}$$

...

Problem #2: Uses $T^*M \cong \mathcal{F}(TM)\times(\mathbb{R}^n)^*$

↪ identify: $\alpha_i e^i = ((p,\{e_i\}), \alpha_i \eta^i)Gl(n)$

$$\nabla_{\partial_\mu}\alpha = ((d\alpha_\nu)(\partial_\mu) + (\Gamma_\mu\cdot\alpha)_\nu)dx^\nu$$

$$= \partial_\mu\alpha_\nu + \cdots$$

$$= \left(d\varphi^a(\mathfrak{X}_x) + [A(\mathfrak{X}_x)\cdot\varphi(x)]^a\right)W_a(x)$$

$$= \left[d\varphi^a(\mathfrak{X}_x) + \left(A^\alpha(\mathfrak{X}_x)e_\alpha\cdot\varphi^b(x)V_b\right)^a\right]W_a(x)$$

$$= \left\{d\varphi^a(\mathfrak{X}_x) + [A^\alpha(\mathfrak{X}_x)\varphi^b(x)h^c_{\alpha b}V_c]^a\right\}W_a(x)$$

$$= \left(d\varphi^a(\mathfrak{X}_x) + \underbrace{A^\alpha(\mathfrak{X}_x)}_{\text{Connection stuff}}\varphi^b(x)\underbrace{h^a_{\alpha b}}_{\text{Lie algebra action}}\right)W_a(x) \checkmark$$

5/3/2

Def $\varphi \in \Lambda^p(M,\mathfrak{g})$, $\psi \in \Lambda^q(M,\mathfrak{g})$ (ie $\varphi(X_1,\dots,X_p) \in \mathfrak{g}$ for $X_i \in T_xM$)
Define $\underline{[\varphi,\psi]} \in \Lambda^{p+q}(M,\mathfrak{g})$ by

$$[\varphi,\psi]_x(X_1,\dots,X_{p+q}) = \frac{1}{p!}\frac{1}{q!}\sum_{\sigma\in S_{p+q}}(-1)^{\sigma}\Big[\varphi(X_{\sigma(1)},\dots X_{\sigma(p)}),\ \psi(X_{\sigma(p+1)},\dots X_{\sigma(p+q)})\Big]$$

$\forall x \in M$ & $X_i \in T_xM$

Aside:

$\left.\begin{array}{l} r_g^*\varphi = \varphi \\ \varphi(\dots,V,\dots) = 0 \end{array}\right\}$ Tensoral form
vertical

$\omega(\delta_A) = A \leftarrow$ Pseudo-Tensoral

note: ω_1, ω_2 connections $\Rightarrow$ $\omega_1 - \omega_2$ is tensoral

$\mathcal{C} = \{\text{connections}\}$ is an affine space
$\mathcal{C} = \omega_0 + \{\text{Tensoral Stuff}\}$

$$[\omega,\omega](X,Y) = \frac{1}{1!}\frac{1}{1!}\Big([\omega(X),\omega(Y)] - [\omega(Y),\omega(X)]\Big)$$

$$\Rightarrow \boxed{[\omega,\omega](X,Y) = 2[\omega(X),\omega(Y)]}$$

Def ω is a connection on a P.F.B. $P \xrightarrow{\pi} M$ w/ group G, then the <u>curvature</u> of ω is $\Omega = D\omega$
$(D\omega)(X,Y) = d\omega(hX, hY)$
↖ Exterior Covariant Derivative

Thm $\Omega = D\omega = d\omega + \frac{1}{2}[\omega,\omega]$

proof

lemma $[\delta_A, \tilde{X}] = 0$

$[\delta_A,\tilde{X}]_u = (L_{\delta_A}\tilde{X})_u = \frac{d}{dt}\Big[d_u\varphi_{-t}(\tilde{X}_{\varphi_t(u)})\Big]\Big|_{t=0}$ where φ is the flow of δ_A

now use $dr_g(\tilde{X}_u) = \tilde{X}_{ug}$

$\varphi_t(u) = r_{\exp(tA)}(u) \Rightarrow d_u\varphi_{-t} = d_u r_{\exp(tA)}$

$$\Rightarrow \frac{d}{dt}\left[d_u\varphi_{-t}\left(\tilde{X}_{\varphi_t(u)}\right)\right]\Big|_{t=0} = \frac{d}{dt}\left[d_u r_{\exp(-tA)}\tilde{X}_{\varphi_t(u)}\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[\tilde{X}_{\varphi_t(u)\circ\exp(-tA)}\right]\Big|_{t=0} = \frac{d}{dt}\left[\tilde{X}_{\varphi_{-t}\varphi_t(u)}\right]\Big|_{t=0} = \cancelto{0}{\frac{d}{dt}\left[\tilde{X}_u\right]}\Big|_{t=0}$$

$(\varphi_{t-t}(u) = \varphi_0(u) = u)$

Y, Z horizontal. $D\omega(Y,Z) = d\omega(hY, hZ) = d\omega(Y,Z)$

$$\left(d\omega + \tfrac{1}{2}[\omega,\omega]\right)(Y,Z) = d\omega(Y,Z) + \tfrac{1}{2}[\omega,\omega](Y,Z)$$

$$= d\omega(Y,Z) + [\cancelto{0}{\omega(Y)}, \omega(Z)]$$

$Y = \delta_A$, $Z = \delta_B$ where Y, Z vertical

$$D\omega(\delta_A,\delta_B) = d\omega(\cancelto{0}{h\delta_A}, \cancelto{0}{h\delta_B}) = 0$$

$$d\omega(\delta_A,\delta_B) + \tfrac{1}{2}[\omega,\omega](\delta_A,\delta_B) = d\omega(\delta_A,\delta_B) + [\omega\delta_A, \omega\delta_B]$$

$$= \delta_A\omega(\delta_B) - \delta_B\omega(\delta_A) - \omega([\delta_A,\delta_B]) + [A,B]$$

(Lie hom. δ)

$$= \cancelto{0}{\delta_A B} - \cancelto{0}{\delta_B A} - \omega\,\delta_{[A,B]} + [A,B] = -[A,B] + [A,B] = 0$$

const

[vector field acting on costants] $\Rightarrow$ Curvature kills vertical vectors

Mixed Case: $Y = \delta_A$, $Z = \tilde{X}$ 1-vert. 1-horz.

$$D\omega(Y,Z) = d\omega(\cancelto{0}{h\delta_A}, h\tilde{X}) = 0$$

$$d\omega(\delta_A,\tilde{X}) + \tfrac{1}{2}[\omega,\omega](\delta_A,\tilde{X}) = d\omega(\delta_A,\tilde{X}) + [\omega\delta_A, \cancelto{0}{\omega(\tilde{X})}]$$

$$= \delta_A(\cancelto{0}{\omega(\tilde{X})}) - \tilde{X}(\cancelto{0}{\omega\delta_A}) - \omega(\cancel{[\delta_A,\tilde{X}]}) = 0 \checkmark$$

Const

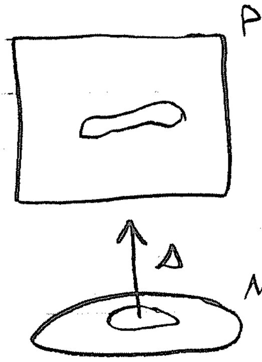
$$\Omega = d\omega + \tfrac{1}{2}[\omega,\omega]$$

$$(s^*\Omega)(X,Y) = \Omega(ds(X), ds(Y)) =$$

$$d\omega(ds(X), ds(Y)) + [\omega(ds(X)), \omega(ds(Y))]$$

$$= s^*(d\omega)(X,Y) + [(s^*\omega)(X), (s^*\omega)(Y)]$$

$$= d(s^*\omega)(X,Y) + [(s^*\omega)(X), (s^*\omega)(Y)]$$



We call $s^*\omega = A$ thus

$$= dA(X,Y) + [A(X), A(Y)]$$

Also call $F = s^*\Omega$ then $F = dA + \tfrac{1}{2}[A,A]$

now use $\partial_\mu = X$, $\partial_\nu = Y$:

$$F(\partial_\mu,\partial_\nu) = dA(\partial_\mu,\partial_\nu) + [A(\partial_\mu), A(\partial_\nu)]$$

write:

$$F_{\mu\nu} = \partial_\mu(A\partial_\nu) - \partial_\nu(A\partial_\mu) - A\cancelto{0}{[\partial_\mu,\partial_\nu]} + [A\partial_\mu, A\partial_\nu]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$\boxed{\text{Lie algebra valued function } F_{\mu\nu} : M \to \mathfrak{g}}$

& $A^a_\mu e_a$ where e_a is a basis of $\mathfrak{g}$

then

$$F_{\mu\nu} = (\partial_\mu A^a_\nu)e_a - (\partial_\nu A^a_\mu)e_a + A^a_\mu A^b_\nu [e_a, e_b]$$

$\Rightarrow$ If $[e_a, e_b] = f^c_{ab} e_c$

$$\therefore F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + A^a_\mu A^b_\nu f^c_{ab}$$

$P = \mathcal{F}(TM) \quad G = \mathcal{G}l(n) \quad \mathfrak{g} = gl(n)$

then $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

$$R_{\mu\nu\lambda}{}^{\kappa} = (F_{\mu\nu})_\lambda^\kappa = \partial_\mu A_{\nu\lambda}{}^{\kappa} - \partial_\nu A_{\mu\lambda}{}^{\kappa} + [A_\mu, A_\nu]_\lambda^\kappa$$

$$A_\nu = \Gamma_\nu \Rightarrow \boxed{R_{\mu\nu\lambda}{}^{\kappa} = \partial_\mu \Gamma_{\nu\lambda}^\kappa - \partial_\nu \Gamma_{\mu\lambda}^\kappa + \Gamma_{\mu\lambda}^a \Gamma_{\nu a}^\kappa - \Gamma_{\nu\lambda}^a \Gamma_{\mu a}^\kappa}$$

Like Reimannian Curvature ✓

(is R-Curvature if $\Gamma_{\nu\lambda}^\kappa$ are Cristoffel Symbols)