Let $gl(n) = \{A \mid A$ is an $n \times n$ matrix over $\mathbb{R}\}$.

Note that $gl(n)$ is a vector space over $\mathbb{R}$ and is an inner product space with inner product defined by

$$\langle A, B \rangle = \sum_{i,j=1}^{n} A_{ij} B_{ij}$$

The norm $\| \ \|$, defined by $\|A\|^2 = \langle A, A \rangle$ has ~~the~~ property ~~that~~ $gl(n)$ is complete w.r.t the norm and moreover $\|AB\| \leq \|A\| \|B\|$ for $A, B \in gl(n)$. Thus $gl(n)$ is a <u>normed algebra</u> and is in fact a <u>Banach algebra</u>.

The function $\det : gl(n) \to \mathbb{R}$ defined by $A \to \det A$ is continuous as it is a polynomial in the entries of A. Since $\mathbb{R} - \{0\}$ is open in $\mathbb{R}$,

$$\det^{-1}(\mathbb{R} - \{0\}) = \{ A \in gl(n) \mid \det A \neq 0 \}$$

is open in $gl(n)$. This subset is denoted by $Gl(n)$. Now $Gl(n)$ is a group under matrix multiplication and the operations

$$(A, B) \to AB \qquad A \to A^{-1}$$

are smooth as AB is a polynomial in the entries of A and B are A^{-1} is a <u>rational</u> function in the entries of A.

So $Gl(n)$ is a Lie group. Moreover the tangent space of $Gl(n)$ at the identity $I \in Gl(n)$

is the same as the tangent space to $gl(n)$ at I. Since $gl(n)$ is a vector space of dimension n^2 $T_I gl(n)$ may be identified with $gl(n)$ again. Thus $T_I \mathcal{GL}(n) = gl(n)$. Recall that a vector v belongs to the tangent space T_pM, for M a manifold and $p \in M$, iff there is a mapping $\gamma : (-a, a) \to M$ such that $\gamma(0) = p$, $\gamma'(0) = v$. Thus for $A \in T_I \mathcal{GL}(n) = gl(n)$ there should be a curve γ_A in $\mathcal{GL}(n)$ through I such that $\gamma_A'(0) = A$. This curve is defined by $\gamma_A(t) = \exp(tA)$ where, for $X \in gl(n)$

$$\exp(X) \equiv I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots$$

To see that this series converges note that if $S_n = \sum_{k=0}^{n} \frac{1}{k!} X^k$ then for $n > m$

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} \frac{1}{k!} X^k \right\| \leq \sum_{k=m+1}^{n} \frac{1}{k!} \|X\|^k$$

using $\|AB\| \leq \|A\|\,\|B\|$ and the triangle inequality. Now the series of numbers $\sum_{k=0}^{\infty} \frac{1}{k!} \|X\|^k$ converges to $e^{\|X\|}$ and so

$s_n = \sum_{k=0}^{n} \frac{1}{k!} \|X\|^k$ is a Cauchy sequence.

But $\|S_n - S_m\| \leq \sum_{k=m+1}^{n} \frac{1}{k!} \|X\|^k = |s_n - s_m| \to 0$

So $\{S_n\}$ is Cauchy in $gl(n)$ + by completeness the series $\sum_{k=0}^{\infty} \frac{1}{k!} X^k$ converges to $\exp(X)$, by definition of $\exp(X)$. The reader is invited to show that if $A, B \in gl(n)$ and $AB = BA$ then

$$\exp(A+B) = \exp A \exp B$$

Consequently if $A \in gl(n)$ and $t, s \in \mathbb{R}$

$$\exp((t+s)A) = \exp(tA)\exp(tB)$$

It requires some analysis such as that above to show that

$$\frac{d}{dt}\left(\exp(tA)\right) = A\exp(tA) = \exp(tA)A.$$

Summary

(1) $Gl(n)$ is an open subgroup of the Banach algebra $gl(n)$. Thus $Gl(n)$ is a Lie group.

(2) $T_I\, Gl(n) = gl(n)$

(3) For each $A \in T_I\, Gl(n) = gl(n)$ the mapping $\gamma_A : \mathbb{R} \to \exp(tA)$ is a group homomorphism from $(\mathbb{R},+)$ to $(Gl(n),\cdot)$. Moreover

$$\gamma_A(0) = I \qquad \gamma_A'(0) = A$$

so γ_A is the curve through I with tangent A.

<u>Proposition</u> If $A \in T_eG$ for some Lie group G, then the unique mapping γ_A from an open interval about 0 in $\mathbb{R}$ into G such that $\gamma_A(0) = e$ and $\gamma_A'(t) = l_A(\gamma(t))$ can be extended to all of $\mathbb{R}$. Moreover the flow of l_A, $\psi_t : G \to G$ is given by $\psi_t(g) = g\,\gamma_A(t)$ and consequently is defined for all $t \in \mathbb{R}$.

<u>Proof</u> l_A is a <u>smooth</u> vector field on G (is this clear?). By the existence and uniqueness theorem of differential equations there exists a function γ from an open interval $I \subseteq \mathbb{R}$ <u>such that</u> $0 \in I$, $\gamma(0) = e$, $\gamma'(t) = l_A(\gamma(t))$. This is only clear from standard theorems of differential equation in a chart (U, x) about e as γ satisfies the conditions iff

$$(x^i \circ \gamma)(0) = x(e)$$

$$\frac{d}{dt}(x^i \circ \gamma)(t) = l_A^i(x(\gamma(t)))$$

where $l_A = l_A^i \frac{\partial}{\partial x^i}$. Some work is required to extend γ beyond the chart domain U but uniqueness of solutions guarantees that one can pass from one chart domain to another. There is a maximal interval I about 0 on which there exists $\gamma : I \to G \ni \gamma(0) = e$ and $\gamma'(t) = l_A(\gamma(t))$!
<u>Suppose</u> I is <u>bounded</u> above. We show this leads to a contradiction. Let $b = \text{lub}\, I$.

Proposition If $A \in T_eG$ for some Lie group G, then the unique mapping γ_A from an open interval about 0 in $\mathbb{R}$ into G such that $\gamma_A(0) = e$ and $\gamma_A'(t) = l_A(\gamma_A(t))$ can be extended to all of $\mathbb{R}$. Moreover the flow of l_A, $\psi_t : G \to G$ is given by $\psi_t(g) = g\,\gamma_A(t)$ and consequently is defined for all $t \in \mathbb{R}$.

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Suppose I is bounded above. We show this leads to a contradiction. Let $b = \text{lub}\, I$.

Then $b>0$. Choose $a<s<0$. Consider

$\gamma_1(t) = \gamma(s+t)$, $\gamma_2(t) = \gamma(s)\gamma(t)$,

$\gamma_1(0) = \gamma(s)$, $\gamma_2(0) = \gamma(s)\gamma(0) = \gamma(s)$

We show that γ_1, γ_2 are both solutions of the differential equation l_A.

$$\gamma_1'(t) = \gamma'(s+t) = l_A(\gamma(s+t)) = l_A(\gamma_1(t))$$

$$\begin{aligned}\gamma_2'(t) &= \frac{d}{dt}\left[L_{\gamma(s)}(\gamma(t))\right] = dL_{\gamma(s)}{}_{\gamma(t)}(\gamma'(t)) \\ &= dL_{\gamma(s)}{}_{\gamma(t)}(l_A(\gamma(t))) \\ &= l_A(\gamma(s)\gamma(t)) \qquad (\text{left invariance}) \\ &= l_A(\gamma_2(t))\end{aligned}$$

So γ_1, γ_2 are both solutions with the same initial condition. Thus $\gamma_1(t) = \gamma_2(t)$ in their common domain. γ_2 has domain I. Let $I = (a,b)$ then γ_1 has domain $\{t \mid s+t \in (a,b)\}$ which is $(a-s, b-s)$. The intersection is $(a-s, b)$ since $b-s>0$. Define

$$\tilde{\gamma}(t) = \begin{cases} \gamma_1(t) & t \in (a-s, b-s) \\ \gamma_2(t) & t \in (a,b) \end{cases}$$

Now $\tilde{\gamma}$ is a solution of the differential equation l_A which extends both γ_1 and γ_2 and has the same initial condition at 0. Now $\tilde{\gamma}$ extends both γ_1 and γ_2 and is defined on $(a, b-s)$. Define $\gamma(t) = \gamma(s)^{-1}\tilde{\gamma}(t)$ on $(a, b-s)$

To see that this is well-defined we show that the mapping $t \to \gamma(s)^{-1}\tilde{\gamma}(t)$ is equal to γ on (a,b). To do this we show it is a solution of the differential equation l_A and since $\gamma(0) = e = \gamma(s)^{-1}\gamma(s) = \gamma(s)^{-1}\tilde{\gamma}(0)$ it will follow from uniqueness of solutions that $\gamma(s)^{-1}\tilde{\gamma}(t) = \gamma(t)$ on (a,b).

But

$$\frac{d}{dt}\left(\gamma(s)^{-1}\tilde{\gamma}(t)\right) = dL_{\gamma(s)^{-1}}{}_{\tilde{\gamma}(t)}\left(\tilde{\gamma}'(t)\right)$$

$$= dL_{\gamma(s)^{-1}}{}_{\tilde{\gamma}(t)}\left(l_A(\tilde{\gamma}(t))\right) = l_A\left(\gamma(s)^{-1}\tilde{\gamma}(t)\right)$$

So we can extend γ to $(a, b-s)$ by defining $\gamma(t) = \gamma(s)^{-1}\tilde{\gamma}(t)$. This contradicts the fact ~~that~~ I was the maximal interval on which a solution of l_A through 0 is defined since $(a,b) \subsetneq (a, b-s)$.

Thus I is not bounded above. This shows that for every $A \in T_eG$ the "maximal" solution through e is defined for all $t \geq 0$. For fixed $A \in T_eG$, we show ~~that~~ the "maximal" solution γ such that $\gamma(0) = e$ and $\gamma'(t) = l_A(\gamma(t))$ exist for all $t \in \mathbb{R}$. To see this consider the "maximal" solution of $\hat{\gamma}(0) = e$, $\hat{\gamma}'(t) = l_{(-A)}(\hat{\gamma}(t))$

Notice that

$$\frac{d}{dt}\left(\hat{\gamma}(-\cdot)\right) = -\hat{\gamma}'(-t) = -l_{(-A)}\left(\hat{\gamma}(-t)\right) = l_A\left(\hat{\gamma}(-t)\right)$$

and $\hat{\gamma}(-t)\big|_{t=0} = e = \gamma(0)$. Thus γ and $t \mapsto \hat{\gamma}(-t)$ agree on their common domain. Since γ is maximal $\gamma(t)$ is defined wherever $\hat{\gamma}(-t)$ is defined. But $\hat{\gamma}(t)$ is defined $\forall\, t \geq 0$. Thus $\hat{\gamma}(-t)$ is defined $\forall\, -t \geq 0$. Thus $\gamma(t)$ is defined for all $-t \geq 0$ or $t \leq 0$. So γ is defined on all of $\mathbb{R}$. The first assertion of the proposition follows.

We now show that if $\gamma_A : \mathbb{R} \to G$ is the solution of $\gamma_A(0) = e$, $\gamma_A'(t) = l_A(\gamma_A(t))$ then $\tilde{\Psi}(t,g) = g\cdot\gamma_A(t)$ is the flow of l_A. Observe that $\tilde{\Psi}(0,g) = g\cdot\gamma_A(0) = ge = g \quad \forall\, g \in G$.

Also
$$\frac{d}{dt}\left(\tilde{\Psi}(t,g)\right) = \frac{d}{dt}\left(g\cdot\gamma_A(t)\right) = dL_g{}_{\gamma_A(t)}\left(\gamma_A'(t)\right)$$
$$= dL_g{}_{\gamma_A(t)}\left(l_A(\gamma_A(t))\right) = l_A\left(g\gamma_A(t)\right)$$
$$= l_A\left(\tilde{\Psi}(t,g)\right)$$

So $\tilde{\Psi}$ is the flow of l_A and it exists for all $(t,g) \in \mathbb{R}\times G$

Assume that M is a manifold, that $\underline{X}$ is a vector field on M and that $\{\varphi_t\}$ is the flow of $\underline{X}$ on M. Let $\mathcal{D} \subseteq \mathbb{R} \times M$ be the maximal open set on which $\tilde{\varphi}(t,q) = \varphi_t(q)$ is defined.

$\boxed{\text{lemma}}$ If f is a smooth real-valued function defined on $\tilde{\varphi}(\mathcal{D}) \subseteq M$ then there is a smooth function $\tilde{g}: \mathcal{D} \to \mathbb{R}$ such that $f(\tilde{\varphi}(t,q)) = f(q) + t\,\tilde{g}(t,q)$ for all $(t,q) \in \mathcal{D}$.
Thus $f(\varphi_t(q)) = f(q) + t\, g_t(q)$ where $g_t(q) = \tilde{g}(t,q)$.

Proof Let $F(t,q) = f(\tilde{\varphi}(t,q)) = f(\varphi_t(q))$, $(t,q) \in \mathcal{D}$.
Then for $0 \le s \le 1$, $(st,q) \in \mathcal{D}$ and

$$\frac{d}{ds}\left(F(st,q)\right) = t\frac{\partial F}{\partial t}(st,q).$$

So
$$F(t,q) - F(0,q) = \int_0^1 \frac{d}{ds}\left(F(st,q)\right)ds = t\int_0^1 \frac{\partial F}{\partial t}(st,q)\,ds$$

If $\tilde{g}(t,q) = \int_0^1 \frac{\partial F}{\partial t}(st,q)\,ds$ we have

$$f(\tilde{\varphi}(t,q)) - f(\tilde{\varphi}(0,q)) = t\,\tilde{g}(t,q)$$
$$f(\tilde{\varphi}(t,q)) = f(q) + t\,\tilde{g}(t,q). \quad \square$$

$\boxed{\text{Theorem}}$ If $\underline{X}$ and $\underline{Y}$ are vector fields on a manifold M and $\{\varphi_t\}$ is the flow of $\underline{X}$ then

$$L_{\underline{X}}\underline{Y} = \underline{X}\circ\underline{Y} - \underline{Y}\circ\underline{X}$$

Proof Let $p \in M$ and let f be a smooth real valued function defined on a neighborhood of p. We show that $(L_{\underline{X}}\underline{Y})_p(f) = \underline{X}_p(\underline{Y}(f)) - \underline{Y}_p(\underline{X}(f))$.

Choose $\tilde{g}$ as in the lemma so that $f(\varphi_t(p)) = f(p) + t\, g_t(p)$.

First note that $\frac{d}{dt}\left(f(\varphi_t(p))\right) = t\,\frac{d}{dt}\left(g_t(p)\right) + g_t(p)$ and if $t=0$, $df_p\left(\frac{d}{dt}(\varphi_t(p))\big|_{t=0}\right) = 0 + g_0(p) = g_0(p)$

But $\frac{d}{dt}(\varphi_t(p)) = X(\varphi_t(p))$ so $g_0(p) = df_p(X_p)$

or $g_0(p) = X_p(f)$.

Next observe that

$$\begin{aligned}
[d\varphi_t(Y)]_p(f) &= d{\varphi_t}_{\varphi_{-t}(p)}\left(Y(\varphi_{-t}(p))\right)(f) \\
&= df_p\left(d\varphi_t\left(Y(\varphi_{-t}(p))\right)\right) \\
&= d(f\circ\varphi_t)\left(Y(\varphi_{-t}(p))\right) \\
&= d(f + t g_t)\left(Y(\varphi_{-t}(p))\right) \\
&= df\left(Y(\varphi_{-t}(p))\right) + t\, dg_t\left(Y(\varphi_{-t}(p))\right)
\end{aligned}$$

$$[Y - d\varphi_t(Y)]_p(f) = Y_p(f) - Y(\varphi_{-t}(p))(f) - t\, Y(\varphi_{-t}(p))(g_t)$$

$$\lim_{t\to 0} \frac{[Y - d\varphi_t(Y)]_p(f)}{t} = \lim_{t\to 0} \frac{Y_p(f) - Y(\varphi_{-t}(p))(f)}{t} - \lim_{t\to 0}\left[Y(\varphi_{-t}(p))(\right.$$

$\boxed{s=-t}$

$$\lim_{s\to 0} \frac{[d\varphi_{-s}(Y) - Y]_p(f)}{s} = \lim_{s\to 0}\left[\frac{Y(\varphi_s(p))(f) - Y_p(f)}{s}\right] - Y(p)(g_0$$

$$\begin{aligned}
&= \frac{d}{ds}\left[Y(\varphi_s(p))(f)\right]\Big|_{s=0} - Y_p\left(X(f)\right) \\
&= \frac{d}{ds}\left[Y(f)(\varphi_s(p))\right]\Big|_{s=0} - Y_p\left(X(f)\right) \\
&= d(Y(f))(X_p) - Y_p(X(f)) = X_p(Y(f)) - Y_p(X(f)). \quad \square
\end{aligned}$$

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<u>Lemma</u> If $A \in \mathfrak{g}$ and $x \in M$ such that $\delta_A(x) = 0$, then $x \cdot \exp(sA) = x$ for all s. If the action of G on M is a <u>free action</u> then $\delta_A(x) = 0$ for <u>some</u> $x \in M$ implies $A = 0$. If the action of G on M is <u>effective</u> then $\delta_A = 0$ implies that $A = 0$.

<u>Proof</u> $\delta_A(x) = \frac{d}{dt}[x \exp(tA)]\big|_{t=0}$, thus $\delta_A(x) = 0$ implies that $\frac{d}{dt}[x \exp(tA)]\big|_{t=0} = 0$. Thus

$$dR_{\exp(sA)}{}_x\left(\frac{d}{dt}[x \exp(tA)]\Big|_{t=0}\right) = 0 \quad \text{for all } s$$

and $\frac{d}{dt}\left(R_{\exp(sA)}(x \exp(tA))\right)\Big|_{t=0} = 0$ for all s.

Thus $\frac{d}{dt}[x \exp((t+s)A)]\Big|_{t=0} = 0$. For $u = t+s$, $\frac{d}{du}[x \exp(uA)]\Big|_{\substack{t=0 \\ u=s}} \frac{du}{dt}(0) = 0$ and

consequently, $\frac{d}{ds}[x \exp(sA)] = 0$. Thus $s \longrightarrow x \cdot \exp(sA)$ is constant and

$$x \cdot \exp(sA) = x \exp(0A) = x.$$

If this is true for <u>some</u> x and the action is free we have $\exp(sA) = 0$ $\forall s$. If the action is effective and $\delta_A = 0$ then $\delta_A(x) = 0$ for all x and $x \cdot \exp(sA) = x$ for all x and all s. Then $\exp(sA) = 0$ for all s. In either case $\exp(sA) = 0$ for all s and for s sufficiently small $sA = 0$ or $A = 0$.

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Definition To say that $P \xrightarrow{\pi} M$ is a principal fiber bundle means that

(1) $P \xrightarrow{\pi} M$ is a fiber bundle with fiber a Lie group G,

(2) the Lie group G acts on the right of P, the action is a free action, and M is the orbit space P/G,

(3) there exists a local trivializing family of mappings $\{\psi_U\}$ such that $\psi_U(u \cdot g) = \psi_U(u) \cdot g$ for $u \in P$, $g \in G$. Here the action of G on $U \times G$ is defined by $(x, g)h = (x, gh)$ for $x \in U$, $g, h \in G$.

Theorem If $P \xrightarrow{\pi} M$ is a principal fiber bundle with fiber a Lie group G, then there is a local trivializing family of mappings $\{\psi_U\}$ whose transition functions $\{\varphi_{UV}\}$ have the property that there exist smooth mappings $g_{UV} : U \cap V \to G$ such that $\varphi_{UV}(x)(g) = g_{UV}(x) g$ for $x \in U \cap V$, $g \in G$. Moreover the mappings $\{g_{UV}\}$ satisfy a cocycle condition:

$$g_{UV} g_{VW} = g_{UW}.$$

Conversely assume that $\mathcal{U}$ is an open cover of M and that $\{g_{UV}\}$ is a family of mappings $g_{UV} : U \cap V \to G$ defined on pairs $U, V \in \mathcal{U}$ such that $U \cap V \neq \phi$. If the family satisfies the cocycle condition $g_{UV} g_{VW} = g_{UW}$ then there is a principal fiber bundle $P \xrightarrow{\pi} M$ with a local trivializing family $\{\Psi_U\}$ whose transition functions $\{\varphi_{UV}\}$ are given by $\varphi_{UV}(x)(g) = g_{UV}(x)\, g \quad \forall x \in U \cap V,\ g \in G$

Proof First assume that $P \xrightarrow{\pi} M$ is a principal fiber bundle with fiber G. Let $\{\Psi_U\}$ be a local trivializing family satisfying (3) of the definition. Recall ~~that~~ the transition functions of $\{\Psi_U\}$ are mappings defined on $U \cap V$ such that $U \cap V \neq \emptyset$ and by property (3),

$$(x, \varphi_{UV}(x)(g)) = (\Psi_U \circ \Psi_V^{-1})(x,g) = (\Psi_U \circ \Psi_V^{-1})(x,e)\, g$$

$$= (x, \varphi_{UV}(x)(e))\, g.$$

Here we use the fact that (3) implies that $\Psi_V^{-1}(x,g) = \Psi_V^{-1}((x,e)g) = \Psi_V^{-1}(x,e)\, g$. Define $g_{UV}: U \cap V \to G$ by $g_{UV}(x) = \varphi_{UV}(x)(e)$.

It follows that $\varphi_{UV}(x)(g) = \varphi_{UV}(x)(e)g = g_{UV}(x)\, g$.

Moreover $\varphi_{UV} \circ \varphi_{VW} = \varphi_{UW}$ implies that

$$g_{UW}(x)\, g = \varphi_{UW}(x)(g) = [\varphi_{UV}(x) \circ \varphi_{VW}(x)](g)$$

$$= \varphi_{UV}(x)(g_{VW}(x)\, g) = g_{UV}(x)\, g_{VW}(x)\, g$$

for all $g \in G$. Let $g = e$ to get

$$g_{UW}(x) = g_{UV}(x)\, g_{VW}(x) = (g_{UV}\, g_{VW})(x)$$

for all $x \in U \cap V \cap W$. Thus

$$g_{UW} = g_{UV}\, g_{VW}.$$

Conversely, assume one has an open

cover $\mathcal{U}$ of M and maps $g_{UV} : U \cap V \to G$ satisfying the cocycle condition. Define mappings $\varphi_{UV} : U \cap V \to \operatorname{Diff} G$ by $\varphi_{UV}^{(x)}(g) = g_{UV}^{(x)} g$ for all $x \in U \cap V$, $g \in G$. Now it is easy to show that the $\{\varphi_{UV}\}$ satisfy a cocycle condition as maps from $U \cap V$ into the group $\operatorname{Diff} M$. By a previous Theorem there exists a fiber bundle $P \xrightarrow{\pi} M$ with fiber G with local trivializing mappings $\{\Psi_U\}$ whose transition functions are the maps $\{\varphi_{UV}\}$. Thus $\Psi_U : \pi^{-1}(U) \to U \times G$ is a diffeomorphism and

$$(\Psi_U \circ \Psi_V^{-1})(x, g) = (x, \varphi_{UV}^{(x)}(g))$$

Define a right action of G on $\pi^{-1}(U)$ by $u \cdot g \equiv \Psi_U^{-1}(\Psi_U(u) \cdot g)$ for $u \in \pi^{-1}(U)$, $g \in G$.
It follows that $\pi^{-1}(U) \xrightarrow{\pi} U$ is a principal fiber bundle which is bundle isomorphic to the trivial bundle $U \times G \to U$. We simply transport the structure of $U \times G \to U$ to $\pi^{-1}(U) \to U$ via Ψ_U. Notice that if $u \in \pi^{-1}(U \cap V)$ then G acts on u in two possibly different ways, one via Ψ_U and the other via Ψ_V. Let $u \cdot g = \Psi_U^{-1}(\Psi_U(u) g)$, $u * g \equiv \Psi_V^{-1}(\Psi_V(u) g)$.
and observe that

$$\Psi_U(u * g) = \Psi_U(\Psi_V^{-1}(\Psi_V(u) g)) = (\Psi_U \circ \Psi_V^{-1})(\pi(u), \Phi_V(u) g)$$

$= \left(\pi(u), \varphi_{UV}^{(\pi(u))}\left(\theta_V(u) g\right)\right) = \left(\pi(u), g_{UV}^{(\pi(u))}\, \theta_V(u)\, g\right)$

where, for each $W \in \mathcal{U}$, $\theta_W : \pi^{-1}(W) \to G$ is defined by $\theta_W(u) = \pi_G(\psi_W(u))$. For $u \in U \cap V$,

$$\psi_U(u) = (\pi(u), \theta_U(u)), \quad \psi_V(u) = (\pi(u), \theta_V(u))$$

and
$$\psi_U^{-1}(\pi(u), \theta_U(u)) = u = \psi_V^{-1}(\pi(u), \theta_V(u)).$$

Thus
$$\begin{aligned}(\pi(u), \theta_U(u)) &= \psi_U\left(\psi_U^{-1}(\pi(u), \theta_U(u))\right) \\ &= \psi_U\left(\psi_V^{-1}(\pi(u), \theta_V(u))\right) \\ &= \left(\pi(u), \varphi_{UV}^{(\pi(u))}\, \theta_V(u)\right)\end{aligned}$$

and $\theta_U(u) = \varphi_{UV}^{(\pi(u))}(\theta_V(u)) = g_{UV}^{(\pi(u))}\, \theta_V(u)$.

It follows from the calculation at the bottom of the previous page and its continuation above that
$$\psi_U(u * g) = (\pi(u), \theta_U(\pi(u)) g) = (\pi(u), \theta_U(\pi(u)))\, g$$
$$= \psi_U(u)\, g = \psi_U(u \cdot g)$$

Since ψ_U is injective, $u * g = u \cdot g$. Thus the action on $\pi^{-1}(U \cap V)$ induced by ψ_U and ψ_V agree. Thus we have a well-defined right action of G on P. The action is free since it is on $\pi^{-1}(U)$ for each U. Moreover the action was defined so that $\psi_U^{-1}(\psi_U(u) \cdot g) = u \cdot g$ and thus so that $\psi_U(ug) = \psi_U(u) \cdot g$. The Theorem fol

Often states of a physical theory are modeled as mappings from a space-time M into a finite dimensional vector space V. This vector space may be a real linear space or it may be complex linear. For example the relevant states in Quantum mechanics are defined by maps from $\mathbb{R} \times \mathbb{R}^3$ into $\mathbb{C}$ where, for $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, t is a time variable and x a space variable. If ψ is such a mapping it is required that

$$\int_{\mathbb{R}^3} |\psi(t,x)|^2 \, d^3x < \infty$$

for each $t \in \mathbb{R}$. The time-evolution of the states $\psi_t : \mathbb{R}^3 \to \mathbb{R}$ is determined by the Schrodinger equation. (States are maps from $\mathbb{R}^3$ to $\mathbb{R}$)

Dirac spinors are states of relativistic quantum mechanics. They are maps from Minkowski space into $\mathbb{C}^4$. Isospin states are maps from Minkowski space M_0 into $\mathbb{C}^2$, although a more detailed analysis would require that such maps ψ satisfy $\psi(x) = (\psi_1(x), \psi_2(x))$ where ψ_1, ψ_2 are Dirac spinors so that ψ is a function from M_0 into $\mathbb{C}^4 \oplus \mathbb{C}^4$.

Mathematicians prefer to formulate these theories in the language of fiber bundles as this provides a unifying language. For example, the fields of general relativity may be taken to be metrics. These metrics are then sections of the tensor bundle $T^0_2 M \to M$ for an appropriate manifold M. On the other hand <u>isospin</u> <u>states</u> $\psi : M_0 \to \mathbb{C}^2$ may also be regarded as section of a vector bundle. In this case we choose $E = M_0 \times \mathbb{C}^2 \longrightarrow M_0$ to be a trivial bundle and ψ is identified with the section $\mathring{\psi} : M_0 \to M_0 \times \mathbb{C}^2$ where $\mathring{\psi}(x) = (x, \psi(x))$, $x \in M_0$. <u>Dirac</u> <u>spinors</u> may be identified as sections of a bundle $S_g(M) \longrightarrow M$ where $S_g(M)$ is a bundle which is essentially a double cover of the orthonormal frame bundle $O_g M \to M$ for some metric g on M.

In the isospin case, $E = M_0 \times \mathbb{C}^2 \to M_0$ one has a metric defined on fibers of E: $g_x((x,z),(x,w)) = z_1 \bar{w}_1 + z_2 \bar{w}_2$. Thus for $x \in M_0$ g_x is a hermitean metric on the fibers E_x.

<u>Theorem</u> Assume that $\varphi : G \to H$ is a continuous homomorphism from a topological group G into a connected topological group H. If there is an open subset O of H containing the identity e of H such that $O \subseteq \varphi(G) \subseteq H$, ~~then~~ φ is <u>subjective</u>.

<u>Proof</u> Since the mapping from H to H defined by $x \to x^{-1}$ is a homeomorphism,

$$O^{-1} = \{ x^{-1} \mid x \in O \}$$

is open in H. Thus $W = O \cup O^{-1}$ is also open. Since the mapping from H to H defined by $x \to ax$ is a homeomorphism for each $a \in H$ we see that $W^2 = \bigcup_{a \in W} (aW)$ is open. Inductively we see that for $n \geq 2$, $W^n = \bigcup_{a \in W} aW^{n-1}$ is open.

Now the subgroup of H generated by O consists of all products

$$x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$$

such that $x_j \in O$ and $i_j \in \{1, -1\}$. Let K denote this subgroup. Notice that K is open since $K = \bigcup_n W^n$. Since $K \subseteq H$ is open so is aK for $a \in H$. Observe that $G - K = \bigcup_{a \notin K} (aK)$ is open. Thus K is both open and closed in a connected space H. Thus $K = H$. But every element of K is of the form $x_1^{i_1} \cdots x_k^{i_k}$ with $x_j \in O \subseteq \varphi(G)$ as above. If $y_j \in G$ such that $\varphi(y_j) = x_j$

We see that $x^{i_1} \cdots x^{i_k} = \varphi(y_1^{i_1} \cdots y^{i_k}) \in \varphi(G)$.
Thus $H = K = \varphi(G)$

Corollary If $\varphi : G \to H$ is a smooth homomorphism from a Lie group G into a connected Lie group H and $d_e\varphi : T_e G \to T_e H$ is invertible, then φ is surjective.

Proof If (U, x) is a chart of G at e and (V, y) is a chart of H at e then $y \circ \varphi \circ x^{-1}$ has the property that it maps an open subset of $\mathbb{R}^n$ into an open subset of $\mathbb{R}^m$. Since $d\varphi : T_e G \to T_e H$ is an isomorphism, $n = m$ and the derivative of $y \circ \varphi \circ x^{-1}$ is invertible at $x(e)$. Thus $y \circ \varphi \circ x^{-1}$ is a local diffeomorphism by the inverse function theorem. So $\exists$ U_0 open in the domain of $y \circ \varphi \circ x^{-1}$ about $x(e)$ such that $(y \circ \varphi \circ x^{-1})(U_0)$ is open and $y \circ \varphi \circ x^{-1} : U_0 \to (y \circ \varphi \circ x^{-1})(U_0)$ is a diffeomorphism. It follows that $\varphi(x^{-1}(U_0))$ is open in H and contains e. The fact that φ is surjective follows from the theorem.

Corollary The mapping $\varphi : SU(2) \to SO(3)$ defined by $\varphi(A)(X) = \tilde{\varphi}(A)(\hat{X})$, where $\tilde{\varphi} : SU(2) \to SO(su_2)$ is given by $\tilde{\varphi}(A)(\hat{X}) = A \hat{X} A^{-1}$, is surjective.

Proof The mapping φ is smooth as it is the composition of smooth maps. Since $SO(3)$ is diffeomorphic to the space $\frac{SO(3)}{SO(2)} = S^2$

it is connected. We show that $d_e\varphi : T_e SU(2) \to T_e SO(3)$ is invertible. To do this it suffices to show that $d_e\tilde{\varphi} : T_e SU(2) \to T_e(SO(su_2))$ is invertible. We first show that $d_e\tilde{\varphi} = \mathrm{ad}$. Let $\lambda \to A_\lambda$ be a curve in $SU(2)$ through the identity $e = I$ and let $B = \frac{d}{d\lambda}(A_\lambda)\big|_{\lambda=0} \in T_e SU(2)$. Then

$$d_e\tilde{\varphi}(B) = \frac{d}{d\lambda}\left[\tilde{\varphi}(A_\lambda)\right]\Big|_{\lambda=0} \text{ and for } \hat{x} \in su_2,$$

$$d_e\tilde{\varphi}(B)(\hat{x}) = \frac{d}{d\lambda}[\tilde{\varphi}(A_\lambda)](\hat{x})\Big|_{\lambda=0} = \frac{d}{d\lambda}\left[A_\lambda \hat{x} A_\lambda^{-1}\right]\Big|_{\lambda=0}$$

$$= \frac{d}{d\lambda}\left(A_\lambda \hat{x} \bar{A}_\lambda^t\right)\Big|_{\lambda=0}$$

$$= B\hat{x} + \hat{x}\bar{B}^t = B\hat{x} - \hat{x}B = [B, \hat{x}] = \mathrm{ad}_B(\hat{x})$$

So $d_e\tilde{\varphi}(B) = \mathrm{ad}_B$. It follows that $B \in \operatorname{Ker} d_e\tilde{\varphi}$ iff $[B, \hat{x}] = 0$ for all $\hat{x} \in su(2)$. Thus

$$B = \begin{pmatrix} i x_3 & -x_2 + i x_1 \\ x_2 + i x_1 & -i x_3 \end{pmatrix} \text{ where } x_1, x_2, x_3 \in \mathbb{R}, \text{ and}$$

$[B, \hat{e}_i] = 0$ for $i = 1, 2, 3$. Here $\{e_i\}$ is the standard basis of $\mathbb{R}^3$. Recall that $\{\hat{e}_i\}$ is a basis of su_2 where

$$\hat{e}_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\hat{e}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{e}_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

By direct computation we have that

$[B, \hat{e}_i] = 0$ for $i = 1, 2, 3$, $B^j[\hat{e}_j, \hat{e}_i] = 0$.

$[B, \hat{e}_1] = 0 \Rightarrow -2B_2\hat{e}_3 + 2B_3\hat{e}_2 = 0 \Rightarrow B_2 = B_3 = 0$

$[B, \hat{e}_2] = 0 \Rightarrow 2B_1\hat{e}_3 - 2B_3\hat{e}_1 = 0 \Rightarrow B_1 = B_3 = 0$,

Thus $B = 0$. Now $\dim T_e SU(2) = 3 = \dim(T_e SO(su_2))$ and $d\tilde{\varphi}$ is injective. It follows that it is invertible. Thus $d_e\varphi$ is also invertible and so φ is a local diffeomorphism. It now follows from the previous corollary that φ is surjective.

Theorem The kernel of $\varphi : SU(2) \to SO(3)$ is $\{\pm I\}$.

Proof Since the mapping from $\mathbb{R}^3$ to $su(2)$ defined by $x \to \hat{x}$ is an isomorphism, $\operatorname{Ker}\varphi = \operatorname{Ker}\tilde{\varphi}$. Let $B \in \operatorname{Ker}\tilde{\varphi}$. Then $\tilde{\varphi}(B) = I \in SO(su(2))$ and $\tilde{\varphi}(B)(\hat{x}) = I(\hat{x}) = \hat{x}$ for all $\hat{x} \in su(2)$. Thus $B \in SU(2)$ such that $B\hat{x}\bar{B}^t = \hat{x}$ for all $\hat{x} \in su(2)$. But $\bar{B}^t = B^{-1}$ and so $B\hat{x} = \hat{x}B$ for all $\hat{x} \in su(2)$.

Write $B = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ for $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$. Setting $\hat{x} = \hat{e}_1, \hat{e}_2, \hat{e}_3$ respectively we get

$$(1)\ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \Rightarrow$$

$$i\begin{pmatrix} -\bar{\beta} & \bar{\alpha} \\ \alpha & \beta \end{pmatrix} = i\begin{pmatrix} \beta & \alpha \\ \bar{\alpha} & -\bar{\beta} \end{pmatrix} \Rightarrow \boxed{\alpha = \bar{\alpha}, \ \beta = -\bar{\beta}}$$

(2) $$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \bar{\beta} & -\bar{\alpha} \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} \beta & -\alpha \\ \bar{\alpha} & -\bar{\beta} \end{pmatrix}$$

$$\Rightarrow \boxed{\alpha = \bar{\alpha} \qquad \beta = \bar{\beta}}$$

From (1) and (2) we see that α is real and $\beta = 0$

Thus $B = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$, $|\alpha|^2 = 1$, $\alpha = \bar{\alpha}$ and

$$B = \pm I.$$