__Ex 1.1:__ $\mu \in \Lambda^n V^* \Rightarrow \mu_{|i_1,\dots,i_n|} e^{i_1} \wedge \dots \wedge e^{i_n}$ (but if we are choosing n strictly inc. indices from n integers they must be $i_1 = 1, \dots, i_n = n$. $\Rightarrow \mu = c\, e^1 \wedge \dots \wedge e^n$

$\mu(e_1, \dots, e_n) = 1$ & $e^1 \wedge \dots \wedge e^n (e_1, \dots, e_n) =$

$\boxed{\frac{1}{n!}} \sum_{\sigma \in S_n} (-1)^\sigma e^{\sigma(1)} \otimes \dots \otimes e^{\sigma(n)} (e_1, \dots, e_n) =$

My error! $\frac{1}{n!}$ should not be part of definition of $e^1 \wedge \dots \wedge e^n$

$\frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma e^{\sigma(1)}(e_1) \dots e^{\sigma(n)}(e_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \delta_1^{\sigma(1)} \dots \delta_n^{\sigma(n)}$

the only non-zero term would be where $\sigma(1) = 1, \dots, \sigma(n) = n$ (ie $\sigma = 1$) $\therefore$ $e^1 \wedge \dots \wedge e^n (e_1, \dots, e_n) = \frac{1}{n!} \Rightarrow c = n!$

$\therefore \mu = n!\, e^1 \wedge \dots \wedge e^n$

by definition $e^1 \wedge \dots \wedge e^n = \boxed{\frac{1}{n!}} \sum_{\sigma \in S_n} (-1)^\sigma e^{\sigma(1)} \otimes \dots \otimes e^{\sigma(n)}$

OK

$= \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon_{\sigma(1) \dots \sigma(n)} e^{\sigma(1)} \otimes \dots \otimes e^{\sigma(n)} = \frac{1}{n!} \sum_{i_1, \dots, i_n} \epsilon_{i_1 \dots i_n} e^{i_1} \otimes \dots \otimes e^{i_n}$

(because if $i_1, \dots, i_n$ is not a permutation $\Rightarrow i_s = i_t$ some $s \neq t$ but this means $\epsilon_{i_1 \dots i_n} = 0$ ✓)

$\therefore \mu = n!\, e^1 \wedge \dots \wedge e^n = \cancel{n!} \cancel{\frac{1}{n!}} \epsilon_{i_1 \dots i_n} e^{i_1} \otimes \dots \otimes e^{i_n}$

$= \epsilon_{i_1 \dots i_p} e^{i_1} \wedge \dots \wedge e^{i_n}$ (as in notes)

Ex 1.2: $\tilde{g} : \Lambda^p V^* \times \Lambda^p V^* \to \mathbb{R}$ defined by $\tilde{g}(\alpha,\beta) = \frac{1}{p!}\alpha^{i_1 \cdots i_p}\beta_{i_1 \cdots i_p}$
where $\alpha = \alpha_{i_1 \cdots i_p} e^{i_1} \wedge \cdots \wedge e^{i_p}$ and $\beta = \beta_{i_1 \cdots i_p} e^{i_1} \wedge \cdots \wedge e^{i_p} \in \Lambda^p V^*$

- Sp $c \in \mathbb{R}$ and $\gamma = \gamma_{i_1 \cdots i_p} e^{i_1} \wedge \cdots \wedge e^{i_p} \in \Lambda^p V^*$ then

$$\tilde{g}(\alpha, c\beta + \gamma) = \frac{1}{p!}\alpha^{i_1 \cdots i_p}(c\beta + \gamma)_{i_1 \cdots i_p} =$$
$$\frac{1}{p!}\alpha^{i_1 \cdots i_p}(c\beta_{i_1 \cdots i_p} + \gamma_{i_1 \cdots i_p}) = c\frac{1}{p!}\alpha^{i_1 \cdots i_p}\beta_{i_1 \cdots i_p} +$$
$$\frac{1}{p!}\alpha^{i_1 \cdots i_p}\gamma_{i_1 \cdots i_p} = c\tilde{g}(\alpha,\beta) + \tilde{g}(\alpha,\gamma)$$

- $$\tilde{g}(\alpha,\beta) = \frac{1}{p!}\alpha^{i_1 \cdots i_p}\beta_{i_1 \cdots i_p} = \frac{1}{p!}\alpha^{i_1 \cdots i_p}\beta^{j_1 \cdots j_p} g_{i_1 j_1} \cdots g_{i_p j_p}$$
$$= \frac{1}{p!}\alpha_{k_1 \cdots k_p} g^{i_1 k_1} \cdots g^{i_p k_p} \beta^{j_1 \cdots j_p} g_{i_1 j_1} \cdots g_{i_p j_p}$$
$$= \frac{1}{p!}\beta^{j_1 \cdots j_p}\alpha_{k_1 \cdots k_p}(g^{i_1 k_1} g_{i_1 j_1}) \cdots (g^{i_p k_p} g_{i_p j_p})$$
$$= \frac{1}{p!}\beta^{j_1 \cdots j_p}\alpha_{k_1 \cdots k_p}\delta^{k_1}_{j_1} \cdots \delta^{k_p}_{j_p} = \frac{1}{p!}\beta^{j_1 \cdots j_p}\alpha_{j_1 \cdots j_p} = \tilde{g}(\beta,\alpha)$$

$\Rightarrow \tilde{g}$ is symmetric & bilinear

Sp $\tilde{g}(\alpha,\beta) = 0 \ \forall \beta \in \Lambda^p V^*$ Choose $j_1, \ldots j_p \in \{1, \ldots, n\}$
let $\beta = \delta^{j_1}_{i_1} \cdots \delta^{j_p}_{i_p} e^{i_1} \wedge \cdots \wedge e^{i_p}$ $(\in \Lambda^p V^*)$ then
← these are not skew in $i_1, i_2, \cdots i_p$

(/2)

$$\tilde{g}(\alpha,\beta) = \frac{1}{p!}\alpha^{i_1 \cdots i_p}\delta^{j_1}_{i_1} \cdots \delta^{j_p}_{i_p} = \frac{1}{p!}\alpha^{j_1 \cdots j_p} = 0 \Rightarrow \alpha^{j_1 \cdots j_p} = 0 \Rightarrow \alpha = 0$$

hence $\tilde{g}$ is non-degenerate.

Need a little extra work here!

$\therefore \tilde{g}$ is a metric ✓

Ex 1.3: $\$ \; \alpha \in \Lambda^{p} V^{*}$ and $\alpha \wedge \gamma = 0 \;\forall\, \gamma \in \Lambda^{n-p} V^{*}$ where $\dim(V) = n$. let $\{e_i\}$ be a basis for V

Write $\alpha_{|i_1,\dots i_p|} e^{i_1} \wedge \dots \wedge e^{i_p}$. Let $k_1 < \dots < k_p$ be indices $(1 \leq k_i \leq n)$ let $j_1 < \dots < j_{n-p}$ be the remaining $n-p$ integers between 1 and n. (ie $\{k_1, \dots, k_p\} \cup \{j_1, \dots, j_{n-p}\} = \{1, \dots, n\}$)

Now $e^{j_1} \wedge \dots \wedge e^{j_{n-p}} \in \Lambda^{n-p} V^{*} \;\therefore\; \alpha \wedge (e^{j_1} \wedge \dots \wedge e^{j_{n-p}}) = 0$
but $(e^{i_1} \wedge \dots \wedge e^{i_p}) \wedge (e^{j_1} \wedge \dots \wedge e^{j_{n-p}}) = 0$ if $i_s = j_t$ some s, t
$\therefore (e^{i_1} \wedge \dots \wedge e^{i_p}) \wedge (e^{j_1} \wedge \dots \wedge e^{j_{n-p}}) \neq 0$ if $i_s \neq j_t \;\forall\, s, t$
if we also requires $i_1 < \dots < i_p$ this forces $i_1 = k_1, \dots, i_p = k_p$

$$\Rightarrow \alpha_{|i_1,\dots i_p|} e^{i_1} \wedge \dots \wedge e^{i_p} \wedge e^{j_1} \wedge \dots \wedge e^{j_{n-p}} = \alpha_{k_1 \dots k_p} e^{k_1} \wedge \dots \wedge e^{k_p} \wedge e^{j_1} \wedge \dots \wedge e^{j_{n-p}}$$

Now, $e^{k_1} \wedge \dots \wedge e^{k_p} \wedge e^{j_1} \wedge \dots \wedge e^{j_{n-p}} \neq 0$ since k_s, j_t's are all distinct hence $\alpha \wedge (e^{j_1} \wedge \dots \wedge e^{j_{n-p}}) = 0 \Rightarrow \alpha_{k_1 \dots k_p} = 0$
but the $k_1 < \dots < k_p$ were arbitrary (set of inc. indices)
$\Rightarrow \alpha = 0$ //

Exercise 1.1

Let $\mu = \mu_g$ be induced by a metric g and let $\{e_i\}$ be a g-orthonormal basis such that $\mu(e_1, e_2, \ldots, e_n) = 1$. Show that

$$\mu = e^1 \wedge e^2 \wedge \cdots \wedge e^n = \boxed{\frac{1}{n!}}^{\,?} \varepsilon_{i_1 \cdots i_n} \left(e^{i_1} \wedge \cdots \wedge e^{i_n} \right)$$

where

$$\varepsilon_{i_1 \cdots i_n} = \begin{cases} 1 & \text{if } i \text{ is even permutation} \\ 0 & \text{if } i_k = i_l \text{ for } k \neq l \\ -1 & \text{if } i \text{ is odd permutation} \end{cases}$$

Exercise 1.2

Show that the natural bilinear mapping $\tilde{g} : \Lambda^p V^* \times \Lambda^p V^* \to \mathbb{R}$ induced by a metric g on a finite dimensional vector space V is itself a metric.

$$\tilde{g}(\alpha, \beta) \equiv \frac{1}{p!} \alpha^{i_1 i_2 \cdots i_p} \beta_{i_1 i_2 \cdots i_p}$$

Exercise 1.3

If $\alpha \in \Lambda^p V^*$ and $\alpha \wedge \gamma = 0$ for every $\gamma \in \Lambda^{n-p} V^*$ where $n = \dim V$, then show that $\alpha = 0$.

Exercise 1.4 Let x, y, z be coordinates on all $\mathbb{R}^3$ and $x = x^1$, $y = x^2$, and $z = x^3$. for $\partial/\partial x^i$ write ∂_i
Let X be a vector field with $X = X^i \partial_i$

(1) Let $\omega_X = X^1 dx + X^2 dy + X^3 dz$ then

$$d\omega_X = d(X^1 dx) + d(X^2 dy) + d(X^3 dz) \quad (d \text{ is linear})$$

$$= \partial_1 X^1 \cancelto{0}{dx \wedge dx} + \partial_2 X^1 dy \wedge dx + \partial_3 X^1 dz \wedge dx + \partial_1 X^2 dx \wedge dy + \partial_2 X^2 \cancelto{0}{dy \wedge dy} + \partial_3 X^2 dz \wedge dy + \partial_1 X^3 dx \wedge dz + \partial_2 X^3 dy \wedge dz + \partial_3 X^3 \cancelto{0}{dz \wedge dz}$$

now we use: $dx^i \wedge dx^j = -dx^j \wedge dx^i$

$$= (\partial_2 X^3 - \partial_3 X^2) dy \wedge dz + (\partial_3 X^1 - \partial_1 X^3) dz \wedge dx + (\partial_1 X^2 - \partial_2 X^1) dx \wedge dy$$

$$= (\text{Curl } X)^1 dy \wedge dz + (\text{Curl } X)^2 dz \wedge dx + (\text{Curl } X)^3 dx \wedge dy$$

(2) Let $\tau_X = X^1 dy \wedge dz + X^2 dz \wedge dx + X^3 dx \wedge dy$ then

$$d\tau_X = d(X^1 dy \wedge dz) + d(X^2 dz \wedge dx) + d(X^3 dx \wedge dy)$$

$$= \partial_1 X^1 dx \wedge dy \wedge dz + \partial_2 X^1 \cancelto{0}{dy \wedge dy \wedge dz} + \partial_3 X^1 \cancelto{0}{dz \wedge dy \wedge dz} + \partial_1 X^2 \cancelto{0}{dx \wedge dz \wedge dx} + \partial_2 X^2 dy \wedge dz \wedge dx + \partial_3 X^2 \cancelto{0}{dz \wedge dz \wedge dx} + \partial_1 X^3 \cancelto{0}{dx \wedge dx \wedge dy} + \partial_2 X^3 \cancelto{0}{dy \wedge dx \wedge dy} + \partial_3 X^3 dz \wedge dx \wedge dy$$

$$= \partial_1 X^1 dx \wedge dy \wedge dz + \partial_2 X \underbrace{dy \wedge dz \wedge dx}_{(-1)^2 = 1 \checkmark} + \partial_3 X^3 \underbrace{dz \wedge dx \wedge dy}_{(-1)^2 = 1 \checkmark}$$

$$= (\partial_1 X^1 + \partial_2 X^2 + \partial_3 X^3) dx \wedge dy \wedge dz = (\text{Div } X) dx \wedge dy \wedge dz$$

(3) Define a metric g by $g(\partial_i, \partial_j) = \delta_{ij}$

(a) $\hat{g}(dx^i, dx^j) = \hat{g}(\delta^i_a dx^a, \delta^j_b dx^b) = 1/1!(g^{ak} \delta^i_k) \cdot \delta^j_a$

but $g_{ij} = \delta_{ij}$ hence $g^{ij} = \delta^{ij}$ $\therefore$

$\hat{g}(dx^i, dx^j) = \delta^{ak} \delta^i_k \delta^j_a = \delta^i_a \delta^j_a = \delta^{ij}$ ✓

(b) $g^b(X)(\partial_j) = g(X^i \partial_i, \partial_j) = X^i g(\partial_i, \partial_j) = X^i \delta_{ij} = X^j$

$\Rightarrow g^b(X) = X^1 dx^1 + X^2 dx^2 + X^3 dx^3 \quad (= \omega_X)$

(3) (b) continued...

$$\mathrm{Curl}\,X = (\partial_2 X^3 - \partial_3 X^2)\partial_1 + (\partial_3 X^1 - \partial_1 X^3)\partial_2 + (\partial_1 X^2 - \partial_2 X^1)\partial_3$$

hence $g^\flat(\mathrm{Curl}\,X) = (\mathrm{Curl}\,X)^1 dx^1 + (\mathrm{Curl}\,X)^2 dx^2 + (\mathrm{Curl}\,X)^3 dx^3$

$$\Rightarrow * g^\flat(\mathrm{Curl}\,X) = \sqrt{|\det I|}\sum_{j<k}\sum_{i,l}(\mathrm{Curl}\,X)^i g^{il}\varepsilon_{ljk}\,dx^j \wedge dx^k$$

<u>note</u>: $I = (g_{ij}) = (\delta_{ij})$ hence $\sqrt{|\det I|} = \sqrt{1} = 1$ and $(\mathrm{Curl}\,X)^i g^{il} = (\mathrm{Curl}\,X)^i \delta^{il}$ ∴

$$* g^\flat(\mathrm{Curl}\,X) = \sum_{j<k}\sum_{i}(\mathrm{Curl}\,X)^i \varepsilon_{ijk}\,dx^j \wedge dx^k$$

We need i,j,k distinct for $\epsilon_{ijk} \neq 0$ but $j<k$ ∴ we have

$$= (\mathrm{Curl}\,X)^1 \cancelto{+1}{\epsilon_{123}}\,dx^2\wedge dx^3 + (\mathrm{Curl}\,X)^2 \cancelto{-1}{\epsilon_{213}}\,dx^1\wedge dx^3$$
$$+ (\mathrm{Curl}\,X)^3 \cancelto{+1}{\epsilon_{312}}\,dx^1\wedge dx^2 \quad \text{use } -dx^1\wedge dx^3 = dx^3\wedge dx^1$$
$$= (\mathrm{Curl}\,X)^1 dx^2\wedge dx^3 + (\mathrm{Curl}\,X)^2 dx^3\wedge dx^1 + (\mathrm{Curl}\,X)^3 dx^1\wedge dx^2$$
$$= d(\omega_X) = d(g^\flat X) \checkmark$$

(c) $$* \omega_X = \sqrt{|\det I|}\sum_{j<k}\sum_{i,l} X^i \delta^{il}\epsilon_{ljk}\,dx^j\wedge dx^k \quad \text{(as before)}$$
$$= X^1 dx^2\wedge dx^3 + X^2 dx^3\wedge dx^1 + X^3 dx^1\wedge dx^2 = \tau_X$$

and $d\tau_X = (\mathrm{Div}\,X)\,dx^1\wedge dx^2\wedge dx^3$

$$\Rightarrow d(* g^\flat(X)) = (\mathrm{Div}\,X)\,dx^1\wedge dx^2\wedge dx^3$$

$$*(\mathrm{Div}\,X) = \cancelto{+1}{\sqrt{|\det I|}}\sum_{i<j<k}(\mathrm{Div}\,X)\,\epsilon_{ijk}\,dx^i\wedge dx^j\wedge dx^k$$
$$= (\mathrm{Div}\,X)\,\epsilon_{123}\,dx^1\wedge dx^2\wedge dx^3$$
$$= (\mathrm{Div}\,X)\,dx^1\wedge dx^2\wedge dx^3 = d(* g^\flat(X)) \checkmark$$

Exercise 1.4

Let X be a vector field on $\mathbb{R}^3$,

$$X = X^1 \frac{\partial}{\partial x} + X^2 \frac{\partial}{\partial y} + X^3 \frac{\partial}{\partial z}.$$

(1) Show that if

$$\omega_X = X^1 dx + X^2 dy + X^3 dz$$

then

$$d\omega_X = (\mathrm{Curl}\, X)^1 (dy \wedge dz) + (\mathrm{Curl}\, X)^2 (dz \wedge dx) + (\mathrm{Curl}\, X)^3 (dx \wedge dy$$

(2) Show that if

$$\tau_X = X^1 (dy \wedge dz) + X^2 (dz \wedge dx) + X^3 (dx \wedge dy)$$

then

$$d\tau_X = (\mathrm{Div}\, X)(dx \wedge dy \wedge dz)$$

(3) Let $x^1 = x$, $x^2 = y$, $x^3 = z$ and define a metric g by

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij}$$

(a) Show that $\hat{g}(dx^i, dx^j) = \delta^{ij}$

(b) Show that

$$g^b(X) = X^1 dx^1 + X^2 dx^2 + X^3 dx^3$$

and that

$$d(g^b(X)) = *(g^b(\mathrm{Curl}\, X))$$

(c) $d(*(g^b(X))) = *(\mathrm{Div}\, X)$

Exercise 1.5: Let M be a manifold with $\dim(M)=n$, a metric g, and a metric compatible volume μ_g.

✓ (1) $\delta\Delta-\Delta\delta = \delta(d\delta+\delta d)-(d\delta+\delta d)\delta =$
$\delta d\delta+\cancelto{0}{\delta^2} d - d\cancelto{0}{\delta^2}-\delta d\delta = \delta d\delta-\delta d\delta = 0$
$\therefore\ \delta\Delta=\Delta\delta$

✓ (2) $d\Delta-\Delta d = d(d\delta+\delta d)-(d\delta+\delta d)d =$
$\cancelto{0}{d^2}\delta+d\delta d-d\delta d-\delta\cancelto{0}{d^2} = d\delta d-d\delta d=0$
$\therefore\ d\Delta=\Delta d$

✓ (3) First, let $\alpha\in\Omega^k_0 M$ and $\beta\in\Omega^{k+1}_0 M$ then
$\tilde{g}(d\alpha,\beta)\mu_g = \tilde{g}(\alpha,\delta\beta)\mu_g + d(\alpha\wedge(*\beta))$
$\therefore\ \underline{\hat{g}(d\alpha,\beta)} = \int_M \tilde{g}(d\alpha,\beta)\mu_g = \int_M\tilde{g}(\alpha,\delta\beta)\mu_g+\int_M d(\alpha\wedge(*\beta))$
$= \int_M\tilde{g}(\alpha,\delta\beta)\mu_g + 0$ (Baby Stoke's Thm) $= \underline{\hat{g}(\alpha,\delta\beta)}$
$\hookrightarrow$ d & δ are adjoint.
$\therefore$ Let $\alpha,\beta\in\Omega^k_0 M$ then $\hat{g}(\Delta\alpha,\beta)=\hat{g}((d\delta+\delta d)(\alpha),\beta)$

✓ $= \hat{g}(d(\delta\alpha),\beta)+\hat{g}(\delta(d\alpha),\beta) = \underline{\hat{g}(\delta\alpha,\delta\beta)+\hat{g}(d\alpha,d\beta)}$
$= \hat{g}(\alpha,d\delta\beta)+\hat{g}(\alpha,\delta d\beta) = \hat{g}(\alpha,(d\delta+\delta d)(\beta))$
$= \hat{g}(\alpha,\Delta\beta)\ \therefore\ \boxed{\hat{g}(\Delta\alpha,\beta)=\hat{g}(\alpha,\Delta\beta)}$
$\hookrightarrow$ Δ is self-adjoint.

✓ also we have $\hat{g}(\Delta\alpha,\beta)=\hat{g}(\delta\alpha,\delta\beta)+\hat{g}(d\alpha,d\beta)$
$\because$ g is positive-definite $\therefore\ \forall\, p\in M\ \exists$ a chart $(\mathcal{U},z=(x^i))$ about $p\ni g_{ij}=\delta_{ij} \Rightarrow \hat{g}(\alpha,\alpha)=\frac{1}{k!}\alpha^{i_1\cdots i_k}\alpha_{i_1\cdots i_k}$

definitely need $\tilde{g}$ pos. def.

$= \frac{1}{k!}g^{i_1j_1}\cdots g^{i_kj_k}\alpha_{j_1\cdots j_k}\alpha_{i_1\cdots i_k} = \frac{1}{k!}\delta^{i_1j_1}\cdots\delta^{i_kj_k}\alpha_{j_1\cdots j_k}\alpha_{i_1\cdots i_k}$
$= \sum_{i_1,\cdots i_k}\frac{1}{k!}\alpha_{i_1\cdots i_k}\alpha_{i_1\cdots i_k} = \frac{1}{k!}\sum_{i_1,\cdots,i_k}(\alpha_{i_1\cdots i_k})^2 \geq 0$
(equal only if $\alpha=0$) $\Rightarrow \tilde{g}$ is pos. def. $\Rightarrow \hat{g}$ is pos. def.
$\therefore\ \hat{g}(\Delta\alpha,\alpha)=\hat{g}(\delta\alpha,\delta\alpha)+\hat{g}(d\alpha,d\alpha)\geq 0$ ✓

Exercise 1.6: Let (x^i) be the standard coordinates on $\mathbb{R}^3$ also let $\mathbb{R}^3$ have the standard Euclidean metric $(g_{ij} = \delta_{ij})$ Let $\omega = \omega_i dx^i$ be a 1-form on $\mathbb{R}^3$

$$\delta\omega = \frac{(-1)^{0+1}}{\cancel{0!}\,|\cancelto{1}{\det I}|^{1/2}}\partial_j(|\cancelto{1}{\det I}|^{1/2}\delta^{ij}\omega_i) = -\sum_{j=1}^{3}\partial_j(\omega_j)$$

$$\therefore d\delta\omega = -\sum_{j=1}^{3} d(\partial_j(\omega_j)) = -\sum_{i,j=1}^{3}\partial_i\partial_j(\omega_j)dx^i$$

$$d\omega = (\mathrm{Curl}(\vec{\omega}))^1 dx^2\wedge dx^3 + (\mathrm{Curl}(\vec{\omega}))^2 dx^3\wedge dx^1 + (\mathrm{Curl}(\vec{\omega}))^3 dx^1\wedge dx^2$$

where $\vec{\omega} = \sum_{i=1}^{3}\omega_i\partial_i$ (using prev. homework problem)

$$= \tfrac{1}{2!}\epsilon_{ijk}(\mathrm{Curl}(\vec{\omega}))^i dx^j\wedge dx^k$$

$$\Rightarrow (d\omega)_{jk} = \epsilon_{ijk}(\mathrm{Curl}(\vec{\omega}))^i$$

$$(\delta d\omega)^i = \frac{\cancelto{1}{(-1)^{1+1}}}{\cancel{1!}\,|\cancelto{1}{\det I}|^{1/2}}\partial_j(|\cancelto{1}{\det I}|^{1/2}\delta^{ik}\delta^{jl}(d\omega)_{kl}) = \sum_{j=1}^{3}\partial_j((d\omega)_{ij})$$

$$\Rightarrow \delta d\omega = \sum_{i,j,k=1}^{3}\epsilon_{kij}\,\partial_j((\mathrm{Curl}(\vec{\omega}))^k)dx^i$$

$$(\Delta\omega)_1 = (d\delta\omega)_1 + (\delta d\omega)_1 = [-\partial_1\partial_1\omega_1 - \partial_1\partial_2\omega_2 - \partial_1\partial_3\omega_3] + [\epsilon_{213}\partial_3((\mathrm{Curl}(\vec{\omega}))^2) + \epsilon_{312}\partial_2((\mathrm{Curl}(\vec{\omega}))^3)]$$

$$= -\partial_1\partial_1\omega_1 \cancel{-\partial_1\partial_2\omega_2} \cancel{-\partial_1\partial_3\omega_3} - \partial_3\partial_3\omega_1 \cancel{+\partial_3\partial_1\omega_3} \cancel{+\partial_2\partial_1\omega_2} - \partial_2\partial_2\omega_1$$

$$(\Delta\omega)_2 = (d\delta\omega)_2 + (\delta d\omega)_2 = [-\partial_2\partial_1\omega_1 - \partial_2\partial_2\omega_2 - \partial_2\partial_3\omega_3] + [\epsilon_{123}\partial_3((\mathrm{Curl}(\vec{\omega}))^1) + \epsilon_{321}\partial_1((\mathrm{Curl}(\vec{\omega}))^3)]$$

$$= \cancel{-\partial_2\partial_1\omega_1} - \partial_2\partial_2\omega_2 \cancel{-\partial_2\partial_3\omega_3} \cancel{+\partial_3\partial_2\omega_3} - \partial_3\partial_3\omega_2 - \partial_1\partial_1\omega_2 \cancel{+\partial_1\partial_2\omega_1}$$

(next page →)

$$(\Delta\omega)_3 = (d\delta\omega)_3 + (\delta d\omega)_3 = \left[-\partial_3\partial_1\omega_1 - \partial_3\partial_2\omega_2 - \partial_3\partial_3\omega_3\right] + \left[\epsilon_{132}\partial_2((\mathrm{Curl}(\vec{\omega}))^1) + \epsilon_{231}\partial_1((\mathrm{Curl}(\vec{\omega}))^2)\right]$$

$$= -\cancel{\partial_3\partial_1\omega_1} - \cancel{\partial_3\partial_2\omega_2} - \partial_3\partial_3\omega_3 - \partial_2\partial_2\omega_3 + \cancel{\partial_2\partial_3\omega_2} + \cancel{\partial_1\partial_3\omega_1} - \partial_1\partial_1\omega_3$$

$$\Rightarrow (\Delta\omega)_i = -\sum_{j=1}^{3}\partial_j\partial_j(\omega_i) \quad \therefore \Delta\omega = -\sum_{i,j=1}^{3}\frac{\partial^2\omega_j}{(\partial x^i)^2}dx^j \; //$$

Extra Credit: $*\Delta = \Delta *$

note: $** = (-1)^{p(n-p)+s+1} \equiv (-1)^k$ (or $**(-1)^k = 1$)

sign?

Same k?

$$*\Delta = *(\delta d + d\delta) = *\delta d + *d\delta = \cancel{*}*d*d + \cancel{(*d*)}d*$$

$(-1)^k$

$(-1)^k$

$$= \underline{(-1)^k} d*d\left[(-1)^k **\right] + \delta d* = \cancel{(-1)}^{2k+1} d(\cancel{*d*})* + \delta d*$$

$$= d\delta * + \delta d * = (\delta d + d\delta)* = \Delta * \; //$$

Needs a little more detail regarding signs

$$p \xrightarrow{d} p+1 \xrightarrow{*} n-p-1$$

$$(-1)^{(n-p')(p\ )+s+1}$$

$$(-1)^{p(n-p)+s+1}$$

$$\|$$

$$(-1)^{(p(n-p)+s+1)2}$$

$** = \pm 1$ but it depends on size of form

it's not clear your k's are all the same.

Exercise 1.5 Let M be an n-dimensional manifold, g a metric on M and μ_g a volume compatible with g. Show that

(1) $\delta \Delta = \cancel{\delta\Delta}\ \Delta \delta$

(2) $d \Delta = \cancel{d\Delta}\ \Delta d$

(3) If g is positive definite and $\hat{g}$ is the metric on the space $\Omega^p_0 M$ of p-forms with compact support defined by

$$\hat{g}(\alpha, \beta) = \int_M (\alpha \wedge * \beta) = \int_M \tilde{g}(\alpha, \beta) \mu_g$$

then show that $\hat{g}(\Delta\alpha, \alpha) \geq 0$ for all α.

Exercise 1.6 Let ω be a 1-form on $\mathbb{R}^3$ (with the Euclidean metric). Show that

$$\Delta \omega = - \cancel{\sum_l \frac{\partial^2 \omega_l}{(\partial x^l)^2} dx^l} .$$

↪ Extra Credit $* \Delta = \Delta *$

$$\Delta \omega = - \sum_{i,j} \frac{\partial^2 \omega_j}{(\partial x^i)^2} dx^j$$

$$= - \sum_j \Delta \omega_j dx^j$$

add $\hat{g}(\Delta \alpha, \beta) = \hat{g}(\alpha, \Delta \beta)$

Let V be a vector space equipped with metric g.
If $L: V \to V$ is a linear bijection then we call L an isometry iff $g(v,w) = g(L(v), L(w))$ $\forall\, v,w \in V$

Let $Gl(V) = \{L: V \to V \mid L \text{ is a linear bijection}\}$

Exercise 1.7: Define $L \cdot \alpha \in \Lambda^p V^*$ $\forall\, L \in \mathcal{G}l(V)$, $\alpha \in \Lambda^p V^*$ by $L \cdot \alpha(v_1, \dots, v_p) = \alpha(L^{-1}(v_1), \dots, L^{-1}(v_p))$ $\forall\, v_1, \dots, v_p \in V$
(note: obviously $L \cdot \alpha \in \Lambda^p V^*$ since L linear $\Rightarrow$ L^{-1} linear etc.)

✓ 1) $(L_1 \circ L_2) \cdot \alpha(v_1, \dots, v_p) = \alpha((L_1 \circ L_2)^{-1}(v_1), \dots, (L_1 \circ L_2)^{-1}(v_p)) = \alpha(L_2^{-1} \circ L_1^{-1}(v_1), \dots, L_2^{-1} \circ L_1^{-1}(v_p)) = L_2 \cdot \alpha(L_1^{-1}(v_1), \dots, L_1^{-1}(v_p)) = L_1 \cdot (L_2 \cdot \alpha)(v_1, \dots, v_p)$ $\forall\, v_1, \dots, v_p \in V$ hence we have $\forall\, L_1, L_2 \in \mathcal{G}l(V)$ and $\alpha \in \Lambda^p V^*$ $(L_1 \circ L_2) \cdot \alpha = L_1 \cdot (L_2 \cdot \alpha)$

2) Let $I_V: V \to V$ be the identity $(I_V \in \mathcal{G}l(V))$ then
$I_V \cdot \alpha(v_1, \dots, v_p) = \alpha(I_V^{-1}(v_1), \dots, I_V^{-1}(v_p)) = \alpha(v_1, \dots, v_p)$ $\forall\, v_1, \dots, v_p \in V$
$\Rightarrow I_V \cdot \alpha = \alpha$ $\forall\, \alpha \in \Lambda^p V^*$ $\therefore$ This is an action of $\mathcal{G}l(V)$ on $\Lambda^p V^*$

Exercise 1.8: Let $\alpha \in \Lambda^1 V^*$ and let L be an isometry
✓ $\alpha(v) = g(g^\# \alpha, v)$ for $v \in V$ (by definition)
$g(g^\#(L \cdot \alpha), v) = (L \cdot \alpha)(v) = \alpha(L^{-1}(v)) = g(g^\# \alpha, L^{-1}(v))$
$= g(L(g^\# \alpha), L(L^{-1}(v))) = g(L(g^\# \alpha), v)$ $\forall\, v \in V$
$\Rightarrow g^\#(L \cdot \alpha) = L(g^\# \alpha)$ since g is non-deg.

If $L \in \mathcal{G}l(V)$ and $L(g^\# \alpha) = g^\#(L \cdot \alpha)$ $\forall\, \alpha \in \Lambda^1 V^*$ then
✓ $g(g^\#(L \cdot \alpha), L(v)) = L \cdot \alpha(L(v)) = \alpha(L^{-1}(L(v))) = \alpha(v)$
Nice $= g(g^\# \alpha, v)$ but $g^\#(L \cdot \alpha) = L(g^\# \alpha)$ $\Rightarrow$
$g(L(g^\# \alpha), L(v)) = g(g^\#(L \cdot \alpha), L(v)) = g(g^\# \alpha, v)$ $\forall\, v \in V$
but $g^\#$ is an isomorphism $\Rightarrow$ $\forall\, w \in V$ $\exists\, \alpha \in V^*$ $\ni$ $g^\# \alpha = w$
$\Rightarrow g(L(w), L(v)) = g(L(g^\# \alpha), L(v)) = g(g^\# \alpha, v) = g(w, v)$
$\Rightarrow L$ is an isometry (thus isometry is a necessary assumption)

Continue in this manner :

$$= \frac{1}{p!} g^{i_1 j_1} g^{i_2 j_2} \alpha_{\underline{i_1 i_2} k_3 \dots k_p} \beta_{\underline{j_1 j_2} l_3 \dots l_p} g^{i_3 j_3} \dots g^{i_p j_p} (L^{-1})^{k_3}_{j_3} \dots (L^{-1})^{k_p}_{j_p} (L^{-1})^{l_3}_{i_3} \dots (L^{-1})^{l_p}_{i_p}$$

$\circ \circ \circ$

$$= \frac{1}{p!} g^{i_1 j_1} \dots g^{i_p j_p} \alpha_{i_1 \dots i_p} \beta_{j_1 \dots j_p} = 1/p!\, \alpha^{j_1 \dots j_p} \beta_{j_1 \dots j_p}$$

$$= \tilde{g}(\alpha, \beta) \quad \therefore \quad \tilde{g}(\alpha, \beta) = \tilde{g}(L \cdot \alpha, L \cdot \beta)$$

$$\forall \alpha, \beta \in \Lambda^p V^*$$

Exercise 1.9: Let $L: V \to V$ be an isometry and fix a basis (e_i) of V. Then we have $\forall\, \alpha \in \Lambda^p V^*$

$$(L\cdot\alpha)_{i_1\cdots i_p} = (L\cdot\alpha)(e_{i_1},\dots,e_{i_p}) = \alpha(L^{-1}(e_{i_1}),\dots,L^{-1}(e_{i_p}))$$
$$= \alpha((L^{-1})^{j_1}_{i_1}e_{j_1},\dots,(L^{-1})^{j_p}_{i_p}e_{j_p}) = (L^{-1})^{j_1}_{i_1}\cdots(L^{-1})^{j_p}_{i_p}\alpha(e_{j_1},\dots,e_{j_p})$$
$$= (L^{-1})^{j_1}_{i_1}\cdots(L^{-1})^{j_p}_{i_p}\alpha_{j_1\cdots j_p}$$

Consider $\alpha, \beta \in \Lambda^1 V^*$ <u>note</u>: $g^\#\alpha = g^{ij}\alpha_i e_j$ where $\alpha = \alpha_i e^i$ (of course (e^i) is the dual of (e_i)) then we have

$$\tilde{g}(\alpha,\beta) = \tfrac{1}{1!}\alpha^j\beta_j = g^{ij}\alpha_i\beta_j \quad \text{but } g(g^\#\alpha, g^\#\beta) =$$
$$g(g^{ij}\alpha_i e_j, g^{kl}\beta_k e_l) = g^{ij}g^{kl}\alpha_i\beta_k g(e_j,e_l) = g^{ij}g^{kl}g_{jl}\alpha_i\beta_k$$
$$= g^{ij}\delta^k_j\alpha_i\beta_k = g^{ij}\alpha_i\beta_j = \tilde{g}(\alpha,\beta)$$

Thus $\tilde{g}(\alpha,\beta) = g(g^\#\alpha, g^\#\beta)$ $\forall\, \alpha,\beta \in \Lambda^1 V^*$ now use <u>Ex 1.8</u>

$$\tilde{g}(\alpha,\beta) = g(g^\#\alpha, g^\#\beta) = g(L(g^\#\alpha), L(g^\#\beta)) \quad \text{(isometry)}$$
$$= g(g^\#(L\cdot\alpha), g^\#(L\cdot\beta)) = \tilde{g}(L\cdot\alpha, L\cdot\beta)$$

$$\Rightarrow\ g^{ij}\alpha_i\beta_j = \tilde{g}(\alpha,\beta) = \tilde{g}(L\cdot\alpha, L\cdot\beta) = \tfrac{1}{1!}(L\cdot\alpha)^i(L\cdot\beta)_i$$
$$= g^{ij}(L\cdot\alpha)_j(L\cdot\beta)_i = g^{ij}(L^{-1})^k_j\alpha_k(L^{-1})^l_i\beta_l$$

$$\therefore\ g^{ij}\alpha_i\beta_j = g^{ij}(L^{-1})^k_j(L^{-1})^l_i\alpha_k\beta_l$$

Let $\alpha,\beta \in \Lambda^p V^*$ $\tilde{g}(L\cdot\alpha, L\cdot\beta) = \frac{1}{p!}(L\cdot\alpha)^{i_1\cdots i_p}(L\cdot\beta)_{i_1\cdots i_p}$

$$= \frac{1}{p!}g^{i_1j_1}\cdots g^{i_pj_p}(L\cdot\alpha)_{j_1\cdots j_p}(L\cdot\beta)_{i_1\cdots i_p} =$$
$$= \frac{1}{p!}g^{i_1j_1}\cdots g^{i_pj_p}(L^{-1})^{k_1}_{j_1}\cdots(L^{-1})^{k_p}_{j_p}\alpha_{k_1\cdots k_p}(L^{-1})^{l_1}_{i_1}\cdots(L^{-1})^{l_p}_{i_p}\beta_{l_1\cdots l_p}$$
$$\to\ = \frac{1}{p!}\underline{g^{i_1j_1}(L^{-1})^{k_1}_{j_1}(L^{-1})^{l_1}_{i_1}\alpha_{k_1\cdots k_p}\beta_{l_1\cdots l_p}}\cdot g^{i_2j_2}\cdots g^{i_pj_p}(L^{-1})^{k_2}_{j_2}\cdots(L^{-1})^{k_p}_{j_p}(L^{-1})^{l_2}_{i_2}\cdots(L^{-1})^{l_p}_{i_p}$$
$$= \frac{1}{p!}g^{i_1j_1}\alpha_{\underline{i_1}k_2\cdots k_p}\beta_{\underline{j_1}l_2\cdots l_p}\cdot g^{i_2j_2}\cdots g^{i_pj_p}(L^{-1})^{k_2}_{j_2}\cdots(L^{-1})^{k_p}_{j_p}(L^{-1})^{l_2}_{i_2}\cdots(L^{-1})^{l_p}_{i_p}$$

would be a little clearer to first show
$g^{ij}(L^{-1})^k_j(L^{-1})^l_i = g^{kl}$

Exercise 1.10 Let G be a Lie group. $\forall \alpha \in T_e^*G$ let l_α be a 1-form on G defined by: $(l_\alpha)_g \equiv (L_g^{-1})^*(\alpha) = (L_{g^{-1}})^*(\alpha)$ $\forall g \in G$. Note: $(l_\alpha)_g(v) = \alpha(d_g L_{g^{-1}}(v))$ $\forall g \in G,\ v \in T_gG$

① Let $h \in G$, $\alpha \in T_e^*G$. Then for $g \in G$, $v \in T_gG$ we have

$$(L_h^*(l_\alpha))_g(v) = (l_\alpha)_{L_h(g)}((d_gL_h)(v)) = (l_\alpha)_{hg}((d_gL_h)(v))$$

$$= \alpha\left(d_{hg}L_{(hg)^{-1}}[(d_gL_h)(v)]\right) = \alpha\left(d_g(L_{(hg)^{-1}}\circ L_h)(v)\right)$$

✓ $= \alpha((d_gL_{(hg)^{-1}h})(v))$ but $(hg)^{-1}h = g^{-1}h^{-1}h = g^{-1}$ hence

$$= \alpha(d_gL_{g^{-1}}(v)) = (l_\alpha)_g(v) \quad \therefore\ L_h^*(l_\alpha) = l_\alpha$$

② $\beta \in \Omega^1G$ is a left-invariant form iff $L_h^*\beta = \beta$ $\forall h \in G$.
Let $\Omega^1_{inv}G = \{\beta \in \Omega^1G \mid \beta \text{ left-invariant}\}$

Ᵽ $\beta \in \Omega^1_{inv}G$ then $\beta_e \in T_e^*G$ let $g \in G$, $v \in T_gG$ then

Nice ✓ $(l_{\beta_e})_g(v) = \beta_e((d_gL_{g^{-1}})(v)) = \beta_{gg^{-1}}(d_gL_{g^{-1}}(v)) = (L_{g^{-1}}^*\beta)_g(v)$

$= \beta_g(v)$ because β is left-inv. $\therefore\ l_{\beta_e} = \beta$

Define $\varphi : T_e^*G \to \Omega^1_{inv}G$ by $\varphi(\alpha) = l_\alpha$ (① $\Rightarrow l_\alpha \in \Omega^1_{inv}G$)
we have just shown that $\varphi(\beta_e) = l_{\beta_e} = \beta$ where $\beta_e \in T_e^*G$
$\therefore\ \varphi$ is onto.

✓ Ᵽ $k \in \mathbb{R}$, $\alpha, \beta \in T_e^*G$ then $\forall g \in G$, $v \in T_gG$

$$\varphi(k\alpha+\beta)_g(v) = (l_{k\alpha+\beta})_g(v) = (k\alpha+\beta)(d_gL_{g^{-1}}(v)) =$$
$$k\cdot\alpha(d_gL_{g^{-1}}(v)) + \beta(d_gL_{g^{-1}}(v)) = k(l_\alpha)_g(v) + (l_\beta)_g(v)$$

Let V be a vector space and g a metric on V. An isometry of V is an invertible linear mapping $L : V \to V$ such that

$$g(L(v), L(w)) = g(v, w)$$

for all $v, w \in V$.

Let $GL(V)$ denote the group of all invertible linear mappings from V to V.

Exercise 1.7 Define an action of $GL(V)$ on $\Lambda^p V^*$ by

$$(L \cdot \alpha)(v_1, v_2, \cdots v_p) = \alpha(L^{-1}(v_1), L^{-1}(v_2), \cdots, L^{-1}(v_p))$$

for $\alpha \in \Lambda^p V$, $L \in GL(V)$, $v_1, v_2, \cdots v_p \in V$. Show that this is indeed an action:

$$(L_1 \circ L_2) \cdot \alpha = L_1 \cdot (L_2 \cdot \alpha)$$

$$I_V \cdot \alpha = \alpha.$$

identity $I_V : V \to V$

Exercise 1.8 If $\alpha \in \Lambda^1 V^*$ show that

$$L((g^\# \alpha)) = g^\# (L \cdot \alpha)$$

for each $L \in GL(V)$

Exercise 1.9 If $L : V \to V$ is an isometry and $\alpha, \beta \in \Lambda^p V^*$ show that $\tilde{g}(L \cdot \alpha, L \cdot \beta) = \tilde{g}(\alpha, \beta)$. Hint: First show that if $\tilde{g}$ is the metric on $\Lambda^1 V^*$ then $\tilde{g}(L \cdot \alpha, L \cdot \beta) = \tilde{g}(\alpha, \beta)$. Write the result in terms of an orthonormal basis of V and write out a proof of the case when $\alpha, \beta \in \Lambda^p V^*$ using components.

Continue 2 ...

$\therefore \varphi(k\alpha + \beta) = k\, l_\alpha + l_\beta = k\varphi(\alpha) + \varphi(\beta) \Rightarrow \varphi$ is linear.

Let $\alpha \in \mathrm{Ker}(\varphi) \Rightarrow \varphi(\alpha) = l_\alpha = 0 \therefore \forall\, v \in T_eG$ we have

$0 = (l_\alpha)_e(v) = \alpha(d_eL_e(v)) = \alpha(v)$ because L_e is the identity map on G hence d_eL_e is the identity map on T_eG

$\therefore \alpha(v) = 0 \;\forall\, v \in T_eG \Rightarrow \alpha = 0 \;\therefore \mathrm{Ker}(\varphi) = \{0\} \;\therefore \varphi$ is 1-1

$\therefore \varphi$ is a linear isomorphism from T_e^*G to $\Omega^1_{inv}G$.

③ Let $\beta \in \Omega^1_{inv}G$ and $X \in \mathfrak{X}_{inv}G$ then $\forall\, g \in G$

We know from 2 that $\beta = l_{\beta_e} \therefore \beta_g(X_g) = (l_{\beta_e})_g(X_g)$

$= \beta_e(d_gL_{g^{-1}}(X_g)) = \beta_e(X_{gg^{-1}})$ (because X is left-invar.)

$= \beta_e(X_e) \;\therefore \beta_g(X_g) = \beta_e(X_e)$ hence $g \longmapsto \beta_g(X_g)$ is a constant map.

④ Let $\{e_i\}$ be a basis for T_eG and let $\{e^i\}$ be the corresponding dual basis for T_e^*G. Let $\beta^i = l_{e^i}$ and $X_j = l_{e_j}$

First note that $\beta^i \in \Omega^1_{inv}G$ and $X_j \in \mathfrak{X}_{inv}G$

(a) $\beta^i_g(X_j(g)) = (l_{e^i})_g(l_{e_j}(g)) = e^i(d_gL_{g^{-1}}(d_eL_g(e_j)))$

$= e^i(d_e(L_{g^{-1}} \circ L_g)(e_j)) = e^i(d_eL_{\cancel{g^{-1}g}^{\,e}}(e_j)) = e^i(\overset{\text{identity}}{\underline{d_eL_e}}(e_j))$

$= e^i(e_j) = \delta^i_j \;\forall\, g \in G \;\therefore \beta^i_g(X_j(g)) = \delta^i_j \;\forall g \in G$

Continue 4...

(b) Choose $f^i_{jk} \in \mathbb{R} \ni [X_i, X_j] = f^k_{ij} X_k$ ← is it clear that there always exist such numbers

$$d\beta^i(X_j, X_k) = X_j(\beta^i(X_k)) - X_k(\beta^i(X_j)) - \beta^i([X_j, X_k])$$

$$= \cancelto{0}{X_j(\delta^i_k)} - \cancelto{0}{X_k(\delta^i_j)} - \beta^i(f^l_{jk} X_l) \quad \text{(vector field on const, is 0)}$$

$$= -f^l_{jk}\beta^i(X_l) = -f^l_{jk}\delta^i_l = -f^i_{jk}$$

$$-\tfrac{1}{2} f^i_{ab}\beta^a \wedge \beta^b(X_j, X_k) = -\tfrac{1}{2} f^i_{ab}\left(\beta^a(X_j)\beta^b(X_k) - \beta^b(X_k)\beta^a(X_j)\right)$$

$$= -\tfrac{1}{2} f^i_{ab}\delta^a_j\delta^b_k + \tfrac{1}{2} f^i_{ab}\delta^b_k\delta^a_j = -\tfrac{1}{2} f^i_{jk} + \tfrac{1}{2} f^i_{kj} \quad \text{but}$$

$$-[X_j, X_k] = [X_k, X_j] \Rightarrow -f^i_{jk} = f^i_{kj} \quad \therefore \quad = -\tfrac{1}{2} f^i_{jk} - \tfrac{1}{2} f^i_{jk}$$

hence $-\frac{1}{2} f^i_{ab}\beta^a \wedge \beta^b(X_j, X_k) = -f^i_{jk}$

$$\therefore \; d\beta^i = -\tfrac{1}{2} f^i_{jk}\beta^j \wedge \beta^k$$

⑤ Define Θ a 1-form on G with values in the Lie algebra $\mathfrak{X}_{inv}(G)$ by $\Theta_g(v) = \beta^i_g(v) X_i \quad \forall g \in G, v \in T_g G$

(a) Let $\{f_i\}$ be a basis for $T_e G$ and let $\{f^j\}$ be its corresponding dual basis (basis for $T^*_e G$) also define $\overline{\beta}^i = l_{f^i}$ and $\overline{X}_j = l_{f_j}$ (note: again $\overline{\beta}^i \in \Omega^1_{inv} G$ and $\overline{X}_j \in \mathfrak{X}_{inv} G$)

Now we can write the f_i's as a linear comb. of e_i's $f_i = B^l_i e_l$ then we know that $(C)^k_j e^j = f^k$ since they are dual bases we have:

$$\delta^i_j = f^i(f_j) = C^i_l e^l(B^k_j e_k) = C^i_l B^k_j \cancelto{\delta^l_k}{e^l(e_k)} = C^i_l B^l_j$$

Continue 5...

for $g, h \in G$ and $v \in T_gG$ we have:

$$\bar{\beta}^i_g(v)\bar{\mathbb{X}}_i(h) = (\ell_{f^i})_g(v)\,\ell_{f_i}(h) = f^i(d_gL_{g^{-1}}(v))\,d_eL_h(f_i)$$

$$= C^i_\ell e^\ell(d_gL_{g^{-1}}(v))\,d_eL_h(B^k_i e_k) = C^i_\ell(\ell_{e^\ell})_g(v)\,B^k_i d_eL_h(e_k)$$

$$= C^i_\ell B^k_i(\ell_{e^\ell})_g(v)\,\ell_{e_k}(h) = \delta^k_\ell\,\beta^\ell_g(v)\,\mathbb{X}_k(h) = \beta^\ell_g(v)\,\mathbb{X}_\ell(h)$$

$\therefore\ \bar{\beta}^i_g(v)\bar{\mathbb{X}}_i = \beta^i_g(v)\mathbb{X}_i$ $\therefore\ \Theta$ is independent of basis choices.

(b) Let $\mathbb{X} \in \mathfrak{X}_{inv}(G) \Rightarrow \mathbb{X} = \ell_\alpha$ for some $\alpha \in T_eG$ hence $\alpha = k^i e_i$ where $k^i \in \mathbb{R}$, $\therefore\ \mathbb{X} = \ell_{k^ie_i} = k^i\ell_{e_i} = k^i\mathbb{X}_i$

$$\Theta_g(\mathbb{X}_g) = \beta^i_g(k^j\mathbb{X}_j(g))\mathbb{X}_i = k^j\beta^i_g(\mathbb{X}_j(g))\mathbb{X}_i = k^j\delta^i_j\mathbb{X}_i =$$

$$k^i\mathbb{X}_i = \mathbb{X} \quad \therefore\ \Theta(\mathbb{X}) = \mathbb{X}$$ ← Note; This means $\Theta_g(\mathbb{X}_g) = \mathbb{X}$ a Const.

(c) Let $\mathbb{Y} \in \mathfrak{X}(G)$ then $\mathbb{Y} = \lambda^i\mathbb{X}_i$ but λ^i are non-constant functions unless $\mathbb{Y} \in \mathfrak{X}_{inv}G$

So we can't say $\Theta(\lambda^i\mathbb{X}_i) = \lambda^i\Theta(\mathbb{X}_i)$. ← actually I misled you here. I will discuss it in Class

Exercise 1.10

Let G be an arbitrary but fixed Lie group. For each $\alpha \in T_e^*G$ let $l\alpha$ be the 1-form on G defined by

$$(l\alpha)_g \equiv (L_g^{-1})^*(\alpha) = L_{g^{-1}}^*(\alpha), \quad g \in G.$$

Note that $(l\alpha)_g(v) = \alpha(dL_{g^{-1}}(v))$ for $g \in G$, $v \in T_gG$.

✓ ① Show that for $h \in G$ and $\alpha \in T_e^*G$, $L_h^*(l\alpha) = l\alpha$

½ ② We say that β is a <u>left-invariant 1-form</u> on G iff $\beta \in \Omega^1 G$ and $L_h^*\beta = \beta$ for all $h \in G$. Denote the set of all left-invariant 1-forms on G by $\Omega^1_{inv}(G)$. Show that <u>for every left-invariant</u> 1-form β there exists $\alpha \in T_e^*G$ such that $\beta = l\alpha$. Show that the mapping from T_e^*G onto $\Omega^1_{inv}G$ defined by $\alpha \to l\alpha$ is a vector space isomorphism,

✓ ③ Assume that $\beta \in \Omega^1_{inv}(G)$ and $X \in \mathfrak{X}_{inv}(G)$. Show that $\beta_g(X_g) = \beta_e(X_e)$ for all $g \in G$. Thus the mapping from G into $\mathbb{R}$ defined by $g \to \beta_g(X_g)$ is constant.

✓ ④ Let $\{e_i\}$ be a basis of T_eG and $\{e^i\}$ the corresponding basis of T_e^*G dual to $\{e_i\}$. Let $\beta^i = le^i$ and $X_i = le_i$.

(a) Show that $\beta^i_g(X_j(g)) = \delta^i_j \quad \forall g \in G$

(b) Choose $f^i_{jk} \in \mathbb{R}$ such that $[X_j, X_k] = f^i_{jk} X_i$. Show that $d\beta^i = -\frac{1}{2} f^i_{jk}(\beta^j \wedge \beta^k)$.

Hint: Evaluate $d\beta^i$ and $-\frac{1}{2} f^i_{jk}(\beta^j \wedge \beta^k)$ at an arbitrary pair of basis vector fields (X_p, X_q). You will need (a) and the identity $d\beta(X,Y) = X(\beta(Y)) - Y(\beta(X)) - \beta([X,Y])$.

⑤ Using the same notation as in 4., define a 1-form Θ on G with values in the Lie algebra $\mathfrak{X}_{inv}(G)$ by

$$\Theta_g(v) = \beta^i_g(v) X_i.$$

(a) Show that if $\{f_i\}$ is another basis of T_eG with dual basis $\{f^i\}$ and if $\bar{\beta}^i = lf^i$, $\bar{X}_j = lf_j$ then

$$\bar{\beta}^i_g(v)\bar{X}_i = \beta^j_g(v) X_j \qquad \forall g \in G,\ v \in T_eG.$$

(Over)

It follows that Θ is independent of the choice of basis.

(b) Show that $\Theta(\underline{X}) = \underline{X}$ for every $\underline{X} \in \mathfrak{X}_{inv}(G)$.

(c) Explain what is wrong with the following argument. Let $\underline{Y} \in \mathfrak{X}(G)$ and write $\underline{Y} = \lambda^i (e_i) = \lambda^i \underline{X}_i$ which holds since $\underline{Y}_g \in T_gG$ and $\{\underline{X}_i(g)\}$ is a basis of T_gG for each $g \in G$. Now

$$\Theta(\underline{Y}) = \underline{\Theta(\lambda^i \underline{X}_i) \neq \lambda^i \Theta(\underline{X}_i)} = \lambda^i \underline{X}_i = \underline{Y}.$$

So $\Theta(\underline{Y}) = \underline{Y}$ for every vector field $\underline{Y}$. This result is false. What's the problem?

depend on g

→ not $\lambda^i \in \mathbb{R}$
but $\lambda^i \in \mathfrak{F}(M)$ //

$$\Theta_g(v) = \beta^i_g(v)\, \underline{X}_i(g)$$

$$\Theta_g(\lambda^i(g)\, \underline{X}_i(g))$$
$$= \beta^j_g(\lambda^i(g)\, \underline{X}_i(g))\, \underline{X}_j(g)$$

Exercise 2.1: Let U, F be manifolds, and let $\varphi: U \to F$ be a smooth mapping. Let $s: U \to U \times F$ be defined by $s(u) = (u, \varphi(u))$ $\forall\, u \in U$.

(a) $\oint$ $x: U_x \to \mathbb{R}^m$, $y: U_y \to \mathbb{R}^m$ are charts on U, and $z: U_z \to \mathbb{R}^d$ is a chart on F then

y is a chart $\exists\, x^{-1}(t) \in U_y$

If $t \in x(U_x) \subseteq \mathbb{R}^m$ then

$(y \times z) \circ s \circ x^{-1}(t) = y \times z(x^{-1}(t), \varphi \circ x^{-1}(t)) = (y \circ x^{-1}) \times (z \circ \varphi \circ x^{-1})(t)$

but $y \circ x^{-1}$ is smooth (both are charts on U) and $z \circ \varphi \circ x^{-1}$ is smooth (composition of smooth maps)

$\Rightarrow (y \circ z) \circ s \circ x^{-1} = (y \circ x^{-1}) \times (z \circ \varphi \circ x^{-1})$ is smooth

hence s is a smooth map.

$\pi \circ s(u) = \pi(u, \varphi(u)) = u \Rightarrow s$ is a smooth section of the trivial bundle $\pi: U \times F \to U$.

(b) Let $u_0 \in U$, $x: U_0 \to \mathbb{R}^m$ be a chart of $U \ni u_0 \in U_0$ and $y: V_0 \to \mathbb{R}^d$ a chart of $F \ni \varphi(u_0) \in V_0$.

define $z: U_0 \times V_0 \to \mathbb{R}^m \times \mathbb{R}^d$ by $z(u,f) = (x(u), y(f) - y(\varphi(u)))$ $\forall\, (u,f) \in U_0 \times V_0$

then $\mathrm{id} \circ z \circ (x \times y)^{-1}(s,t) = z(x^{-1}(s), y^{-1}(t)) = (x \circ x^{-1}(s), y \circ y^{-1}(t) - y \circ \varphi \circ x^{-1}(s)) = (s, t - y \circ \varphi \circ x^{-1}(s))$

now $y \circ \varphi \circ x^{-1}$ is smooth and subtraction is smooth

$\Rightarrow (s,t) \xmapsto{\Theta} t - y \circ \varphi \circ x^{-1}(s)$ is smooth $\therefore \mathrm{id} \circ z \circ (x \times y)^{-1} = \mathrm{id} \times \Theta$ is smooth $\Rightarrow z$ is smooth.

$z^{-1}(s,t) = (x^{-1}(s), y^{-1}(t + y \circ \varphi \circ x^{-1}(s)))$

$z \circ z^{-1}(s,t) = (x \circ x^{-1}(s), y(y^{-1}(t + y \circ \varphi \circ x^{-1}(s))) - y \circ \varphi \circ x^{-1}(s))$

$= (s, t + y \circ \varphi \circ \cancel{x^{-1}(s) -} y \circ \varphi \circ x^{-1}(s)) = (s,t)$

$$z^{-1}\circ z(u,f) = (x^{-1}\circ x(u),\ y^{-1}(y(f) - y\circ\varphi(u) + y\circ\varphi\circ x^{-1}\circ x(u))$$
$$= (u,\ y^{-1}(y(f) - y\circ\varphi\cancel{(u) + y\circ\varphi}(u)) = (u, y^{-1}\circ y(f)) = (u,f)$$

$\Rightarrow$ z^{-1} is in fact the inverse of z,

✓ and $(x\times y)\circ z^{-1}\circ \mathrm{id}(s,t) = (x\times y)\circ z^{-1}(s,t) =$
$(x\times y)(x^{-1}(s), y^{-1}(t + y\circ\varphi\circ x^{-1}(s))) = (x\circ x^{-1}(s), y\circ y^{-1}(t + y\circ\varphi\circ x^{-1}(s)))$
$= (s, t + y\circ\varphi\circ x^{-1}(s))$ again, $(s,t) \xrightarrow{\Theta'} t + y\circ\varphi\circ x^{-1}(s)$
is smooth since $y\circ\varphi\circ x^{-1}$ is smooth (composition of smooth maps) and addition is smooth. $\therefore$ $(x\times y)\circ z^{-1}\circ \mathrm{id} = \mathrm{id}\times\Theta'$ is smooth hence z^{-1} is smooth.

$u\in U_0$ then $z\circ s(u) = z(u, \varphi(u)) = (x(u), y\circ\varphi(u) - y\circ\varphi(u))$
$= (x(u), 0) \equiv x(u) \Rightarrow z\circ s = x$

Now let $\bar{x} : \bar{U}_0 \to \mathbb{R}^m$, $\bar{y} : \bar{V}_0 \to \mathbb{R}^d$ be charts on U and F resp.
$\ni u_0 \in \bar{U}_0$, $\varphi(u_0)\in\bar{V}_0$ and $\bar{z} : \bar{U}_0\times\bar{V}_0 \to \mathbb{R}^m\times\mathbb{R}^d$ is defined by $\bar{z}(u,f) = (\bar{x}(u), \bar{y}(f) - \bar{y}\circ\varphi(u))$ then

$\bar{z}\circ z^{-1}(s,t) = \bar{z}(x^{-1}(s), y^{-1}(t + y\circ\varphi\circ x^{-1}(s))) =$
$(\bar{x}\circ x^{-1}(s), \bar{y}\circ y^{-1}(t + y\circ\varphi\circ x^{-1}(s)) - \bar{y}\circ\varphi\circ x^{-1}(s))$
✓ $= (\bar{x}\circ x^{-1})\times(\bar{y}\circ y^{-1}\circ\Theta' - \bar{y}\circ\varphi\circ x^{-1})(s,t)$, which is smooth since it is compositions of smooth maps etc.

$\Rightarrow$ z and $\bar{z}$ are compatible charts

note:
It is obvious that $d_p z$ is full rank $\forall\, p\in U_0\times V_0$
since $z(u,f) = \underset{\substack{\uparrow \\ \text{rank } m}}{x(u)} \times (\underset{\substack{\uparrow \\ \text{rank } d}}{y(f)} - \underset{\substack{\uparrow \\ \text{depends only on } u}}{y\circ\varphi(u)})$

(in the right basis)
Jacobian looks like $\begin{bmatrix} I_m & * \\ 0 & I_d \end{bmatrix}$ $\therefore$ by the inverse function thm $\exists$ nbhd V_p of $p \ni z(V_p)$ open thus $z(U_0\times V_0) = \bigcup_{p\in U_0\times V_0} z(V_p)$ is open

Exercise 2.2: Let $\pi : E \to M$ be a fiber bundle with fiber F. Let $s : U \to E$ be a local section of π, $U \subseteq E$ open

If $x \in s(U)$ then $\exists\, u \in U \ni s(u) = x \Rightarrow \pi(x) = \pi \circ s(u) = u \in U$
$\therefore s(U) \subseteq \pi^{-1}(U)$ but π is smooth hence $\pi^{-1}(U)$ open $\therefore$ an open submanifold of E.

Choose coordinates on $\pi^{-1}(U)$, say $y : V_0 \to \mathbb{R}^{m+d}$ (let $\dim(M) = m$ and $\dim(F) = d$) $\ni$ we also have a local trivializing map $\psi_{U_0} : \boxed{V_0} \to U_0 \times F$ and finally we also want $V_0 \cap s(U) \neq \phi$.
U_0? maybe $V_0 \subseteq \pi^{-1}(U_0)$ $\psi_{U_0} : \pi^{-1}(U_0) \to U_0 \times F$

Note $s^{-1}|s(U)$ is a homeomorphism

Thus we have for $u \in s^{-1}(s(U) \cap V_0) = W_0 (\subseteq M)$ is W_0 open? Does it need to be?

$$\pi \circ s = id_U \ , \ \pi|_{V_0} = \pi_{U_0} \circ (\psi_{U_0}|V_0) \Rightarrow \pi|_{V_0} \circ s|_{W_0} = id_{W_0}$$

$$\therefore id_{W_0} = \pi_{U_0} \circ \psi_{U_0} \circ s|_{W_0} \Rightarrow d_p(id_{W_0}) = d_p(\pi_{U_0} \circ \psi_{U_0} \circ s)$$

$$= (d_{\psi_{U_0} \circ s(p)} \pi_{U_p})(d_{s(p)} \psi_{U_0}) \circ (d_p s)$$

$$\text{rank}(d_p id_{W_0}) = \dim(M) = m = \text{rank}((d_{\psi_{U_0} \circ s(p)} \pi_{U_0}) \circ (d_{s(p)} \psi_{U_0})(d_p s))$$

$$\leq \min\{\text{rank}(d_{\psi_{U_0} \circ s(p)} \pi_{U_0}), \text{rank}(d_{s(p)} \psi_{U_0}), \text{rank}(d_p s)\}$$

$= \min\{m, m+d, \text{rank}(d_p s)\}$ since ψ_{U_0} is a diff. and π_{U_0} is a proj. onto U_0. but $s : U \to E$, $U \subseteq M$
$\Rightarrow \text{rank}(d_p s) \leq \dim(M) = m$
hence $\min\{m, m+d, \text{rank}(d_p s)\} = \text{rank}(d_p s) \leq m$
$\Rightarrow m \leq \text{rank}(d_p s) \leq m \Rightarrow \text{rank}(d_p s) = m$

$\therefore \forall p \in U$ we have $\text{rank}(d_p s) = \dim(U) (= \dim(M))$
$\Rightarrow d_p s$ is 1-1

Small gap

<It doesn't make sense to say j is smooth unless $s(\bar{U})$ is a manifold. What manifold structure are you putting on $s(\bar{U})$?✓

needs a little work here

Finally the inclusion map $\boxed{j : s(U) \hookrightarrow E}$ is smooth since $j = s\circ\pi$ ($j(x) = j(s(u)) = s\circ\cancel{\pi\circ} s(u) = s(u) = x$ ✓) which is the composition of smooth maps.

Note you could define the manifold structure on $s(\bar{U})$ so as to force $s : \bar{U} \to s(\bar{U})$ to be a diffeom.

$\Longrightarrow s(U)$ is a submanifold of E. (* need relative top.) ↙ see below

$\$$ $p \in U \Rightarrow s(p) \in U$ and $\pi\circ s(p) = p \Rightarrow s(p) \in \pi^{-1}\{p\}$
$\therefore \{s(p)\} \subseteq s(U) \cap \pi^{-1}\{p\}$

✓ $\$$ $x, y \in s(U) \cap \pi^{-1}\{p\} \Rightarrow \exists\, u, v \in U \ni s(u) = x,\ s(v) = y$
$\Rightarrow u = \pi\circ s(u) = \pi(x) = p = \pi(y) = \pi\circ s(v) = v$
$\Rightarrow x = s(u) = s(v) = y$

$\therefore s(U) \cap \pi^{-1}\{p\} = \{s(p)\}$

(*) ⌈ $\Theta \subseteq U$ open then $\pi_U^{-1}(\Theta) = \Theta \times F$ open $\Rightarrow \psi_U^{-1}(\Theta \times F)$ open

✓ $\$$ $x \in \psi_U^{-1}(\Theta \times F) \Rightarrow \pi_U \circ \psi_U(x) \in \Theta \Rightarrow \pi(x) \in \Theta$
hence $s\circ\pi(x) = x$ ie $x \in s(\Theta)$

⌊ $\Rightarrow s(\Theta) \cap \psi^{-1}(\Theta \times F) = s(\Theta)$ $\therefore$ open in relative top.

Exercise 2.3: Let M be a manifold, $\dim(M) = m$

Let $x^i : \mathbb{R}^m \to \mathbb{R}$ be $x^i(p_1, \dots, p_m) = p_i$ (global coords on $\mathbb{R}^m$)

Let $y : V \to \mathbb{R}^m$ be a chart on M then if $(p, \alpha) \in \Lambda^k_p M$ we can write $\alpha = \alpha_{|i_1 \dots i_k|} d_p y^{i_1} \wedge \dots \wedge d_p y^{i_k}$

thus our chart on $\Lambda^k M$ is $\Lambda^k y : \Lambda^k V \to \mathbb{R}^m \times \mathbb{R}^{\binom{m}{k}}$ defined by $\Lambda^k y(p, \alpha) = (y^1(p), \dots, y^m(p), \alpha_{|i_1 \dots i_k|})$

<u>note</u>: $\Lambda^k y(\Lambda^k V) = V \times \mathbb{R}^{\binom{m}{k}}$ open
and $\Lambda^k y^{-1}(t_1, \dots, t_m, \alpha_{|i_1 \dots i_k|}) = (\overset{\overset{p}{\shortparallel}}{y^{-1}(t_1, \dots, t_m)}, \alpha_{|i_1 \dots i_k|} d_p y^{i_1} \wedge \dots d_p y^{i_k})$
thus $\Lambda^k y$ is invertible.

Sp $z : W \to \mathbb{R}^m$ is another chart on M giving rise to another chart $\Lambda^k z : \Lambda^k W \to \mathbb{R}^m \times \mathbb{R}^{\binom{m}{k}}$ on $\Lambda^k M$
then

$$\Lambda^k y \circ \Lambda^k z^{-1}(t_1, \dots, t_m, \alpha_{|i_1 \dots i_k|}) = \Lambda^k y(\overset{\overset{p}{\shortparallel}}{z^{-1}(t_1, \dots, t_m)}, \alpha_{|i_1 \dots i_k|} d_p z^{i_1} \wedge \dots \wedge d_p z^{i_k})$$

$$= \Lambda^k y(p, \alpha_{|i_1 \dots i_k|} \frac{\partial z^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial z^{i_k}}{\partial y^{j_k}} d_p y^{j_1} \wedge \dots \wedge d_p y^{j_k})$$

$$= \Lambda^k y(p, \beta_{|j_1 \dots j_k|} d_p y^{j_1} \wedge \dots d_p y^{j_k}) = (y(p), \beta_{|j_1 \dots j_k|})$$

$= (y \circ z^{-1}(t_1, \dots, t_m), \beta_{|j_1 \dots j_k|})$ now $y \circ z^{-1}$ is smooth (both are charts of M) and $\beta_{|j_1 \dots j_k|}$ depend smoothly because they are sums and products of smooth objects.

$\Rightarrow$ $\Lambda^k y$ & $\Lambda^k z$ are compatible charts

$\Rightarrow$ $\Lambda^k M$ is a manifold.

- Let $\pi : \Lambda^k M \to M$ be defined by $\pi(p, \alpha) = p$

  let $y : V \to \mathbb{R}^m$ be a chart on M then

  $$y \circ \pi \circ \Lambda^k y^{-1}(t_1, \dots, t_m, \alpha_{|i_1 \dots i_k|}) = y \circ \pi(y^{-1}(t_1, \dots, t_m) = p, \alpha_{|i_1 \dots i_k|} d_p y^{i_1} \wedge \dots \wedge d_p y^{i_k}) = y \circ y^{-1}(t_1, \dots, t_m) = (t_1, \dots, t_m)$$

  $\Rightarrow y \circ \pi \circ \Lambda^k y^{-1} = id$ which is smooth $\Rightarrow$ π is smooth ✓

- Define $\psi_V : \Lambda^k V \to V \times \Lambda^k \mathbb{R}^m$ by $\psi_V(p, \alpha) =$

  $(p, \alpha_{|i_1 \dots i_k|} d_{y(p)} x^{i_1} \wedge \dots \wedge d_{y(p)} x^{i_k})$ ($\alpha = \alpha_{|i_1 \dots i_k|} d_p y^{i_1} \wedge \dots \wedge d_p y^{i_k}$ and recall x^i's are global coords on $\mathbb{R}^m$)

  Obviously $\alpha_{|i_1 \dots i_k|} d_{y(p)} x^{i_1} \wedge \dots \wedge d_{y(p)} x^{i_k} \in \Lambda^k \mathbb{R}^m$ and ψ_V is onto $V \times \Lambda^k \mathbb{R}^m$.

  $$(y \times \Lambda^k x) \circ \psi_V \circ \Lambda^k y^{-1}(t_1, \dots, t_m, \alpha_{|i_1 \dots i_k|}) = (y \times \Lambda^k x) \circ \psi_V(p = y^{-1}(t_1, \dots, t_m), \alpha_{|i_1 \dots i_k|} d_p y^{i_1} \wedge \dots \wedge d_p y^{i_k}) =$$

  $$(y \times \Lambda^k x)(p, \alpha_{|i_1 \dots i_k|} d_{y(p)} x^{i_1} \wedge \dots \wedge d_{y(p)} x^{i_k}) = (y(p), \alpha_{|i_1 \dots i_k|})$$

  $$= (y \circ y^{-1}(t_1, \dots, t_m), \alpha_{|i_1 \dots i_k|}) = (t_1, \dots, t_m, \alpha_{|i_1 \dots i_k|})$$

  $\therefore (y \times \Lambda^k x) \circ \psi_V \circ \Lambda^k y^{-1} = id$ which is smooth $\Rightarrow \psi_V$ is smooth

  Obviously, $\psi_V^{-1}(p, \alpha_{|i_1 \dots i_k|} d_{y(p)} x^{i_1} \wedge \dots \wedge d_{y(p)} x^{i_k}) = (p, \alpha)$ where $\alpha_{|i_1 \dots i_k|} d_p y^{i_1} \wedge \dots \wedge d_p y^{i_k}$

  and $\Lambda^k y \circ \psi_V^{-1} \circ (y \times \Lambda^k x)^{-1} = [(y \times \Lambda^k x) \circ \psi_V \circ \Lambda^k y^{-1}]^{-1} = id^{-1} = id$ hence smooth $\Rightarrow \psi_V^{-1}$ is smooth $\Rightarrow \psi_V$ is a diffeomorphism finally we need that

$$\begin{tikzcd}\Lambda^k M \arrow[rr, "\psi_V"] \arrow[dr, "\pi"'] & & V\times\Lambda^k\mathbb{R}^m \arrow[dl, "\pi_V"] \\ & M & \end{tikzcd}$$

commutes.

$$\pi_V \circ \psi_V(p,\alpha) = \pi_V(p, \alpha_{|i_1\ldots i_k|} d_{y(p)}x^{i_1}\wedge\ldots\wedge d_{y(p)}x^{i_k}) = p$$

where $\alpha = \alpha_{|i_1\ldots i_k|} d_p y^{i_1}\wedge\ldots\wedge d_p y^{i_k}$.

but $\pi(p,\alpha) = p$ hence $\pi = \pi_V \circ \psi_V$ $\therefore$ the diagram commutes.

Exercise 2.1 Let U, F be manifolds and $\varphi : U \to F$ a smooth mapping. Let $s : U \to U \times F$ be defined by $s(u) = (u, \varphi(u))$, $u \in U$.

(a) Show that s is a smooth section of the trivial bundle $U \times F \xrightarrow{\pi} U$.

(b) Let $u_0 \in U$. Choose charts $x : U_0 \to \mathbb{R}^m$, $y : V_0 \to \mathbb{R}^d$ of U and F respectively such that $u_0 \in U_0$, $\varphi(u_0) \in V_0$. Show that the mapping $z : U_0 \times V_0 \to \mathbb{R}^m \times \mathbb{R}^d$ defined by

$$z(u,f) = (x(u), y(f) - y\varphi(u)) \qquad (u,f) \in U_0 \times V_0$$

is smooth and has a smooth inverse. Also show ~~that~~ that $z(s(u)) = x(u)$ for each $u \in U_0$. Finally show that if $\bar{z}(u,f) = (\bar{x}(u), \bar{y}(f) - \bar{y}\varphi(u))$ is similarly defined for charts $\bar{x}, \bar{y}$ then $z, \bar{z}$ are compatible charts.

This exercise shows that $s(U)$ is a submanifold of $U \times F$ in the strongest sense. Note: you can show $z(U_0 \times V_0)$ is open using the inverse function theorem.

Exercise 2.2. Let $E \xrightarrow{\pi} M$ be a fiber bundle with fiber F and assume that $s : U \to E$ is a local section of π defined on an open set $U \subseteq M$. Show that $s(U)$ is a submanifold of the open submanifold $\pi^{-1}(U)$ of E. Also show that $s(U)$ intersects each fiber $\pi^{-1}(p)$, $p \in U$, in precisely one point.

Exercise 2.3 Show that $\Lambda^k M \to M$ is a fiber bundle with fiber $\Lambda^k \mathbb{R}^m$ where $m = \dim M$.

Exercise 2.4: Let $G \in \mathcal{G}\ell(n) \ni G^2 = I$ and $G = G^t$. Define $f : g\ell(n) \to g\ell(n)$ by $f(A) = A^tGA - G$

(a) Let $A \in g\ell(n)$, $B \in T_A g\ell(n)$ then $\exists$ a curve $\lambda \longmapsto A_\lambda \ni$ $A_0 = A$ and $\frac{d}{d\lambda}[A_\lambda]\Big|_{\lambda=0} = B$

$$\therefore\ d_Af(B) = d_{A_0}f\left[\frac{d}{d\lambda}[A_\lambda]\Big|_{\lambda=0}\right] = \frac{d}{d\lambda}\left[f(A_\lambda)\right]\Big|_{\lambda=0}$$

(by the chain rule)

$$= \frac{d}{d\lambda}\left[A_\lambda^t G A_\lambda - G\right]\Big|_{\lambda=0} = \frac{d}{d\lambda}\left[A_\lambda^t G A_\lambda\right]\Big|_{\lambda=0} - \cancelto{0}{\frac{d}{d\lambda}[G]}\Big|_{\lambda=0}$$

$$= \frac{d}{d\lambda}[A_\lambda]^t\Big|_{\lambda=0} GA_\lambda\Big|_{\lambda=0} + A_\lambda^t\Big|_{\lambda=0} \cancelto{0}{\frac{d}{d\lambda}[G]}\Big|_{\lambda=0} A_\lambda\Big|_{\lambda=0} + A_\lambda^t G\Big|_{\lambda=0} \frac{d}{d\lambda}[A_\lambda]\Big|_{\lambda=0}$$

$$= B^t \cancelto{G^t}{G} A + A^tGB = A^tGB + [A^tGB]^t$$ ✓

(b) Let $L : g\ell(n) \longrightarrow g\ell(n)$ be defined by $L(X) = A^tGX$ then

$$L(\alpha X + Y) = A^tG(\alpha X + Y) = \alpha A^tGX + A^tGY$$
$$= \alpha L(X) + L(Y) \quad \text{hence linear.}$$

Let $A \in \mathcal{G}\ell(n)$ then $A^t \in \mathcal{G}\ell(n)$. Consider, $L^{-1}(X) = GA^{t^{-1}}X$ then

$$LL^{-1}(X) = A^tGGA^{t^{-1}}X = A^t\cancelto{I}{G^2}A^{t^{-1}}X = \cancelto{I}{A^tA^{t^{-1}}}X = X$$

and

$$L^{-1}L(X) = G\cancelto{I}{A^{t^{-1}}A^t}GX = GGX = \cancelto{I}{G^2}X = X$$

$\therefore$ L is an isomorphism (has an inverse)

$$L^{-1}\{B \mid B = -B^t\} = \{GA^{t^{-1}}B \mid B = -B^t\} = \{B \mid A^tGB = -(A^tGB)^t\}$$

(because if we have $GA^{t^{-1}}B = C \ni B = -B^t \iff C \ni$ $B = A^tGC$ & $B = -B^t \iff C \ni A^tGC = -(A^tGC)^t$)

but $d_Af(B) = A^tGB + [A^tGB]^t = 0 \iff A^tGB = -[A^tGB]^t$

$\therefore \mathrm{Ker}(d_Af) = L^{-1}\{B \mid B = -B^t\}$

$\therefore \ \forall \ A \in \mathcal{Gl}(n) \ \ Ker(d_A f) = L^{-1}\{B \mid B = -B^t\}$

now L an isomorphism $\Rightarrow L^{-1}$ an isomorphism $\Rightarrow$
$dim(Ker(d_A f)) = dim\{B \mid B = -B^t\} = \frac{n(n-1)}{2}$ (no dependance on A)
(because $E_{ij} - E_{ji}$ for $i<j$ is a basis for $\{B \mid B = -B^t\}$)

(c) Let $A \in gl(n)$ then $f(A)^t = [A^tGA - G]^t = [A^tGA]^t - \overset{G}{\cancel{G^t}}$
$= A^t \overset{G}{\cancel{G^t}} A - G = f(A)$ hence $f(A)$ is symmetric

$\therefore \ f : gl(n) \longrightarrow \{A \in gl(n) \mid A = A^t\}$

Consider $A \in gl(n) \ni f(A) = A^tGA - G = 0 \Rightarrow A^tGA = G \Rightarrow$
$det(A^tGA) = det(A)det(G)det(A) = det(G) \neq 0$
$\Rightarrow det(A) \neq 0 \ \Rightarrow A \in \mathcal{Gl}(n)$

$\therefore \ \forall \ A \in \{A \in gl(n) \mid f(A) = 0\}$ we have $A \in \mathcal{Gl}(n)$ hence

$rank(d_A f) = dim(T_A gl(n)) - dim(ker(d_A f)) =$
$dim(gl(n)) - dim(Ker(d_A f)) = n^2 - \frac{n(n-1)}{2}$

$\therefore \ rank(d_A f) = \frac{2n^2 - n^2 + n}{2} = \frac{n(n+1)}{2} = dim(range(f))$

since $dim\{A \in gl(n) \mid A^t = A\} = \frac{n(n+1)}{2}$ $\therefore$ $d_A f$ has maximal rank $\forall \ A \in gl(n) \ni f(A) = 0$

$\therefore \ \{A \in gl(n) \mid f(A) = 0\}$ is a submanifold of $gl(n)$,

(d) $O(p,k) = \{A \in gl(n) \mid A^tGA = G\} = \{A \in gl(n) \mid f(A) = 0\}$
which is a submanifold if $G = G^t$ and $G^2 = I$ ✓
$\therefore \ O(p,k)$ is a submanifold of $gl(n)$ $(= gl(p+k))$

see back ↪

Yes, $Gl(p+k)$ is an open submanifold of $gl(p+k)$

$\Rightarrow$ $O(p,k) \subseteq Gl(p+k)$ is a submanifold of $G(p+k)$

and $O(p,k)$ is a subgroup of $Gl(p+k)$

$\therefore$ $O(p,k)$ is a Lie subgroup of $Gl(p+k)$

Exercise 2.4. Let $G \in Gl(n)$ such that $G^2 = I_n$, $G^t = G$. Define $f : gl(n) \to gl(n)$ by $f(A) = A^T G A - G$.

(a) Let $A \in gl(n)$ and show that $d_A f(B) = (A^T G B) + (A^T G B)$ for all $B \in T_A(gl(n))$. Hint: For each $B \in T_A\, gl(n)$ choose a curve $\lambda \to A_\lambda$ in $gl(n)$ such that $A_0 = A$, $\frac{d}{d\lambda}(A_\lambda)\big|_{\lambda=0} = B$ then observe that $d_A f(B) = \frac{d}{d\lambda}(f(A_\lambda))\big|_{\lambda=0}$.

(b) Let $L : gl(n) \to gl(n)$ be defined by $L(X) = A^T G X$. Show that L is a vector space isomorphism and that $L^{-1}(\{B \mid B \text{ is skew-symmetric}\}) = \operatorname{Ker}(d_A f)$ for each $A \in gl(n)$. Find $\dim(\operatorname{Ker} d_A f)$ and show that it is independent of A.

(c) If $G^t = G$, show that f maps $gl(n)$ into the set of symmetric matrices. Show that the set of all $A \in gl(n)$ such that $f(A) = 0$ is a submanifold of $gl(n)$. Hint: Recall that if T is a linear transformation on a vector space V then

$$\dim V = \dim(\mathrm{rR}\,T) + \dim(\operatorname{Ker} T).$$

(d) Show that $O(p,k)$ is a submanifold of $gl(p+k)$. Is this enough to say that $O(p,k)$ is a Lie subgroup of $Gl(p+k)$?

<u>Exercise 2.5:</u> Let $SU(2) = \{A \in \mathcal{GL}(2,\mathbb{C}) \mid \bar{A}^t A = I, \det(A) = 1\}$

(a) $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$

$\wp$ $A \in SU(2) \Rightarrow \det(A) = \alpha\delta - \beta\gamma = 1$ &

$\begin{pmatrix} \bar\alpha & \bar\gamma \\ \bar\beta & \bar\delta \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha\bar\alpha + \gamma\bar\gamma & \bar\alpha\beta + \bar\gamma\delta \\ \alpha\bar\beta + \gamma\bar\delta & \beta\bar\beta + \delta\bar\delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\wp$ $\beta \neq 0 \Rightarrow \frac{\alpha\delta - 1}{\beta} = \gamma \Rightarrow \alpha\bar\beta + \left(\frac{\alpha\delta - 1}{\beta}\right)\bar\delta = 0$

✓ $\Rightarrow \alpha\beta\bar\beta + \alpha\delta\bar\delta - \bar\delta = \alpha(\overset{1}{\cancel{\beta\bar\beta + \delta\bar\delta}}) - \bar\delta = 0 \Rightarrow \underline{\underline{\alpha = \bar\delta}}$

$\Rightarrow \alpha\bar\beta + \gamma\bar\delta = \alpha\bar\beta + \gamma\alpha = \alpha(\bar\beta + \gamma) = 0$ $\wp$ $\alpha \neq 0 \Rightarrow \underline{\bar\beta = -\gamma}$

$\wp$ $\alpha = 0 \Rightarrow -\beta\gamma = 1$ and $\gamma\bar\gamma = 1 \Rightarrow \gamma \neq 0$ and $\beta = -\frac{1}{\gamma} = -\bar\gamma$

hence for $\beta \neq 0$ we have $\alpha = \bar\delta$ & $\beta = -\bar\gamma$

$\wp$ $\beta = 0 \Rightarrow \delta\bar\delta = 1$ and $\alpha\delta = 1 \Rightarrow \delta \neq 0$ and $\underline{\alpha = \frac{1}{\delta} = \bar\delta}$

$\Rightarrow \alpha\bar\beta + \gamma\bar\delta = \alpha\bar\beta + \gamma\alpha = \alpha(\bar\beta + \gamma) = 0$ but $\alpha = \bar\delta \neq 0 \Rightarrow$

$\underline{\bar\beta = -\gamma}$ hence for $\beta = 0$ we have $\alpha = \bar\delta$ & $\beta = -\bar\gamma$

$\therefore A = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}$ & $\alpha\delta - \beta\gamma = \alpha\bar\alpha + \beta\bar\beta = 1$ ✓

$\wp$ $A = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}$ & $\alpha\bar\alpha + \beta\bar\beta = 1 \Rightarrow \det(A) = \alpha\bar\alpha + \beta\bar\beta = 1$ ✓

$$\overline{A}^tA = \begin{pmatrix}\overline{\alpha} & -\beta\\ \overline{\beta} & \alpha\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix} = \begin{pmatrix}\alpha\overline{\alpha}+\beta\overline{\beta} & \overline{\alpha}\beta-\overline{\alpha}\beta\\ \alpha\overline{\beta}-\alpha\overline{\beta} & \alpha\overline{\alpha}+\beta\overline{\beta}\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = I$$

$\Rightarrow A \in SU(2)$ $\therefore$ $SU(2) = \left\{\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix} \middle| \alpha,\beta\in\mathbb{C} \ \& \ |\alpha|^2+|\beta|^2=1\right\}$

(b) $SU(2)$ is a Lie-subgroup of $\mathcal{G}\ell(2,\mathbb{C})$ and $su(2)$ (the Lie algebra of $SU(2)$) is a Lie-subalgebra of $g\ell(2,\mathbb{C})$ (the Lie algebra of $\mathcal{G}\ell(2,\mathbb{C})$)

§ $B\in su(2) \Rightarrow \exists$ a curve $A_\lambda \in SU(2) \ni A_0 = I$ &

$$\frac{d}{d\lambda}[A_\lambda]\Big|_{\lambda=0} = B \quad \therefore \quad 0 = \frac{d}{d\lambda}[I]\Big|_{\lambda=0} = \frac{d}{d\lambda}[\overline{A}^tA]\Big|_{\lambda=0}$$

$$= \overline{\frac{d}{d\lambda}[A]\Big|_{\lambda=0}}^t A_0 + \overline{A}_0^t\frac{d}{d\lambda}[A_\lambda]\Big|_{\lambda=0} = \overline{B}^tI + \overline{I}^tB = \overline{B}^t+B$$

$\therefore \underline{\overline{B}^t+B=0}$ ✓

$\exp(B)\in SU(2) \Rightarrow \det\exp(B) = e^{\mathrm{tr}B} = 1 \Rightarrow \underline{\mathrm{tr}B=0}$ ✓

§ $B\in g\ell(2,\mathbb{C}) \ni \overline{B}^t+B=0$ & $\mathrm{tr}B=0$ then we have $\exp(\lambda B)\in\mathcal{G}\ell(2,\mathbb{C})$ defined for some nbhd about zero.

$$\overline{\exp(\lambda B)}^t\exp(\lambda B) = \exp(\overline{\lambda B}^t)\exp(\lambda B) = \exp(\lambda\overline{B}^t)\exp(\lambda B)$$

$\boxed{\overline{B}^t=-B}$ → $\boxed{-\lambda B \ \& \ \lambda B \text{ commute}}$

$$= \exp(-\lambda B)\exp(\lambda B) \overset{\downarrow}{=} \exp(-\lambda B+\lambda B) = \exp(0) = I$$

also $\det\exp(\lambda B) = e^{\mathrm{tr}\lambda B} = e^{\lambda\cancel{\mathrm{tr}B}^{\,0}} = e^0 = 1 \Rightarrow \exp(\lambda B)\in SU(2)$
$\Rightarrow B\in su(2)$

$$\therefore \ su(2) = \left\{B\in g\ell(2,\mathbb{C}) \,\middle|\, \overline{B}^t+B=0 \ \& \ \mathrm{tr}B=0\right\}$$

$B=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \ni \alpha,\beta,\gamma,\delta\in\mathbb{C}$ $\operatorname{tr}B=0 \Rightarrow \underline{\alpha=-\delta}$

and $\bar{B}^t+B=\begin{pmatrix}\bar{\alpha} & \bar{\gamma}\\ \bar{\beta} & \bar{\delta}\end{pmatrix}+\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}\alpha+\bar{\alpha} & \bar{\gamma}+\beta\\ \bar{\beta}+\gamma & \bar{\delta}+\delta\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

$\Rightarrow \alpha+\bar{\alpha}=0$ ie α is pure imaginary & $\beta=-\bar{\gamma}$

Let $\alpha=x_3 i$, $\beta=-x_2+x_1 i$ where $(x_1,x_2,x_3)\in\mathbb{R}^3$ then

$B=\begin{pmatrix}x_3 i & -x_2+x_1 i\\ x_2+x_1 i & -x_3 i\end{pmatrix}$ Conversely, if we have such a B

$\operatorname{tr}B=x_3 i-x_3 i=0$ & $\bar{B}^t+B=\begin{pmatrix}-x_3 i & x_2-x_1 i\\ -x_2-x_1 i & x_3 i\end{pmatrix}+\begin{pmatrix}x_3 i & -x_2+x_1 i\\ x_2+x_1 i & -x_3 i\end{pmatrix}$

$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \Rightarrow B\in su(2)$

$\therefore su(2)=\left\{\begin{pmatrix}x_3 i & -x_2+x_1 i\\ x_2+x_1 i & -x_3 i\end{pmatrix} \,\middle|\, (x_1,x_2,x_3)\in\mathbb{R}^3\right\}$

define: $\varphi:\mathbb{R}^3\to su(2)$ by $\varphi(x_1,x_2,x_3)=\begin{pmatrix}x_3 i & -x_2+x_1 i\\ x_2+x_1 i & -x_3 i\end{pmatrix}=\hat{x}$

$\varphi(cx_1+y_1, cx_2+y_2, cx_3+y_3)=\begin{pmatrix}(cx_3+y_3)i & -(cx_2+y_2)+(cx_1+y_1)i\\ (cx_2+y_2)+(cx_1+y_1)i & -(cx_3+y_3)i\end{pmatrix}$

$=c\begin{pmatrix}x_3 i & -x_2+x_1 i\\ x_2+x_1 i & -x_3 i\end{pmatrix}+\begin{pmatrix}y_3 i & -y_2+y_1 i\\ y_2+y_1 i & -y_3 i\end{pmatrix}=c\varphi(x_1,x_2,x_3)+\varphi(y_1,y_2,y_3)$

$\ni \varphi(x_1,x_2,x_3)=\begin{pmatrix}x_3 i & -x_2+x_1 i\\ x_2+x_1 i & -x_3 i\end{pmatrix}=0 \Rightarrow x_3=0$ &

$\begin{matrix}-x_2+x_1 i=0\\ x_2+x_1 i=0\end{matrix} \Rightarrow \begin{matrix}2x_1 i=0\\ -2x_2 i=0\end{matrix} \Rightarrow \begin{matrix}x_1=0\\ x_2=0\end{matrix} \quad \therefore (x_1,x_2,x_3)=0$

φ obviously onto. $\Rightarrow$ isomorphism

(c) $-\frac{1}{2}\operatorname{tr}(\hat{x}\hat{y}) = -\frac{1}{2}\operatorname{tr}\left[\begin{pmatrix} x_3 i & -x_2+x_1 i \\ x_2+x_1 i & -x_3 i \end{pmatrix}\begin{pmatrix} y_3 i & -y_2+y_1 i \\ y_2+y_1 i & -y_3 i \end{pmatrix}\right]$

$= -\frac{1}{2}\operatorname{tr}\begin{bmatrix} -x_3y_3 - x_2y_2 - x_1y_1 - \underline{x_2y_1}i + x_1y_2 i & * \\ * & -y_2x_2 - y_1x_1 - y_2x_1 i + y_1\underline{x_2} i - x_3y_3 \end{bmatrix}$

$= -\frac{1}{2}(-2x_3y_3 - 2x_2y_2 - 2x_1y_1) = x_1y_1 + x_2y_2 + x_3y_3 = \langle x, y\rangle$

✓ and $\|x\|^2 = x_1^2 + x_2^2 + x_3^2$ $\quad \det(\hat{x}) = x_3^2 - (-x_2^2 - x_1^2 + x_2x_1 i - x_2x_1 i)$

$= x_1^2 + x_2^2 + x_3^2$ $\therefore$ $\det(\hat{x}) = \|x\|^2$ & $\langle x, y\rangle = -\frac{1}{2}\operatorname{tr}(\hat{x}\hat{y})$

Define $\langle \hat{x}, \hat{y}\rangle_0 = -\frac{1}{2}\operatorname{tr}(\hat{x}\hat{y})$ for $\hat{x}, \hat{y} \in su(2)$

$\langle , \rangle_0$ is an inner-product because $\langle , \rangle$ is and we have $\langle \hat{x}, \hat{y}\rangle_0 = \langle x, y\rangle$ where $x \longmapsto \hat{x}$ is an isomorphism.

$\Rightarrow$ $x \longmapsto \hat{x}$ is an isometry & $\|\hat{x}\|_0^2 = \|x\|^2 = \det(\hat{x})$

(d) Let $\tilde{\varphi} : SU(2) \longrightarrow End(su(2))$ be defined by
$\tilde{\varphi}(A)(X) = AXA^{-1}$ for $X \in su(2)$

✓ (*) $\overline{AXA^{-1}}^{\,t} + AXA^{-1} = (\bar{A}^t)^{-1}\bar{X}^t\bar{A}^t + AXA^{-1}$ but $A \in SU(2) \Rightarrow$

$\bar{A}^tA = I$ hence $\bar{A}^t = A^{-1}$ & $(\bar{A}^t)^{-1} = A$ $\therefore$ (*) $= A\bar{X}^tA^{-1} + AXA^{-1}$

$= A(\cancel{\bar{X}^t + X})A^{-1} = 0$ & $\operatorname{tr}(A(XA^{-1})) = \operatorname{tr}((X\cancel{A^{-1})A}) = \operatorname{tr}(X) = 0$

$\Rightarrow$ $\tilde{\varphi}(A)(X) \in su(2)$ we can see that $\tilde{\varphi}(A)$ is obviously linear $\Rightarrow$ $\tilde{\varphi}(A) \in End(su(2))$ ✓

$$\langle \widetilde{\varphi}(A)(\mathbb{X}), \widetilde{\varphi}(A)(\mathbb{Y})\rangle_0 = -\tfrac{1}{2}\mathrm{tr}(A\mathbb{X}\overset{I}{\cancel{A^{-1}A}}\mathbb{Y}A^{-1}) = -\tfrac{1}{2}\mathrm{tr}(A(\mathbb{X}\mathbb{Y}A^{-1}))$$

$$= -\tfrac{1}{2}\mathrm{tr}((\mathbb{X}\mathbb{Y}\cancel{A^{-1}})\overset{I}{\cancel{A}}) = -\tfrac{1}{2}\mathrm{tr}(\mathbb{X}\mathbb{Y}) = \langle \mathbb{X}, \mathbb{Y}\rangle_0 \checkmark$$

(e) Let $\psi : \mathbb{R}^3 \longrightarrow su(2)$ be defined by $\psi(x) = \hat{x}$

recall ψ is an isometry.

Define $\varphi : SU(2) \longrightarrow Gl(3,\mathbb{R})$ by $\varphi(A) = \psi^{-1} \circ \widetilde{\varphi}(A) \circ \psi$

$\hookrightarrow$ which means $\widehat{\varphi(A)(x)} = \psi(\varphi(A)(x)) = \widetilde{\varphi}(A)(\psi(x)) = \widetilde{\varphi}(A)(\hat{x})$

(note: all is well defined because ψ is a bijection)

Now, ψ (and thus ψ^{-1}) is linear, so is $\widetilde{\varphi}(A) \Rightarrow \varphi(A)$ is linear (composition of linear maps is linear) $\Rightarrow \varphi(A) \in gl(3,\mathbb{R})$

& $A \in SU(2) \Rightarrow \det(A) = 1 \Rightarrow \det(A^{-1}) = 1$ also $\bar{A}^t A = I \therefore$

$(\overline{A^{-1}})^t(A^{-1}) = (\bar{A}^t)^{-1}A^{-1} = (A\bar{A}^t)^{-1} = [\overline{(\bar{A}^tA)^t}]^{-1} = [\overline{(I)^t}]^{-1} = I \Rightarrow A^{-1} \in SU(2)$

$\widetilde{\varphi}(A^{-1})\widetilde{\varphi}(A)(\mathbb{X}) = \cancel{A^{-1}}(\cancel{A}\mathbb{X}\cancel{A^{-1}})A = \mathbb{X} \Rightarrow \widetilde{\varphi}(A^{-1})\widetilde{\varphi}(A) = I$ likewise

$\widetilde{\varphi}(A)\widetilde{\varphi}(A^{-1}) = I \therefore \widetilde{\varphi}(A)^{-1} = \widetilde{\varphi}(A^{-1})$ hence $\widetilde{\varphi}(A)$ is injective

ψ, ψ^{-1} are injective $\Rightarrow \varphi(A) = \psi^{-1} \circ \widetilde{\varphi}(A) \circ \psi$ is injective

(composition of injective maps is injective) $\Rightarrow \varphi(A) \in Gl(3,\mathbb{R})$

Hence φ is well defined. ✓

(f) ψ is an isometry $\Rightarrow \psi^{-1}$ is an isometry.

$\langle \varphi(A)(x), \varphi(A)(y)\rangle = \langle \psi^{-1} \circ \widetilde{\varphi}(A) \circ \psi(x), \psi^{-1} \circ \widetilde{\varphi}(A) \circ \psi(y)\rangle$

$= \langle \widetilde{\varphi}(A)(\psi(x)), \widetilde{\varphi}(A)(\psi(y))\rangle_0$ (b/c ψ^{-1} is an isometry)

$= \langle \psi(x), \psi(y)\rangle_0$ (by Part (d))

$= \langle x, y\rangle$ (b/c ψ is an isometry) $\forall A \in SU(2), x, y \in \mathbb{R}^3$

$\therefore \varphi(A) \in O(3) \quad \forall A \in SU(2)$

Let $\{e_i\}$ be the standard basis for $\mathbb{R}^3$ then $\varphi(A)(e_i) = \varphi(A)^j_i e_j$ for some $\varphi(A)^j_i \in \mathbb{R}$. $\widetilde{\varphi}(A)(\hat{e}_i) = \widetilde{\varphi}(A)(\psi(e_i)) = \psi(\varphi(A)(e_i))$

$= \psi(\varphi(A)^j_i e_j) = \varphi(A)^j_i \psi(e_j) = \varphi(A)^j_i \hat{e}_j$

$\Longrightarrow A \longmapsto \det(\varphi(A)^j_i)$ is continuous.

Exercise 2.6: $\varphi : SU(2) \longrightarrow SO(3)$ is onto & $Ker(\varphi) = \{\pm I\}$

recall $\psi : \mathbb{R}^3 \rightarrow su(2)$ is an isometry. & $\varphi(A) = \psi^{-1} \circ \tilde{\varphi}(A) \circ \psi$

$\therefore \varphi(AB) = \psi^{-1} \circ \tilde{\varphi}(AB) \circ \psi = \psi^{-1} \circ \tilde{\varphi}(A)\tilde{\varphi}(B) \circ \psi = \psi^{-1} \circ \tilde{\varphi}(A) \circ \psi \circ \psi^{-1}$
$\circ \tilde{\varphi}(B) \circ \psi = \varphi(A)\varphi(B)$ (φ is group homomorphism – this is why we can speak of its kernel)

note: $\tilde{\varphi}(AB)(X) = AB\,X(AB)^{-1} = A(BXB^{-1})A^{-1} = \hat{\varphi}(A)\hat{\varphi}(B)(X)$

$$R_x^\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix},\ R_y^\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \in SO(3)$$

Nice

$$R_x^\theta R_y^\phi(0,0,1) = R_x^\theta(\sin\phi, 0, \cos\phi) = (\sin\phi, \sin\theta\cos\phi, \cos\theta\cos\phi)$$

$\Rightarrow$ by choosing appropriate angles we can rotate to anywhere on the sphere (S^2)

define an action of $SU(2)$ on S^2 ($= \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$) by $A \cdot x = \varphi(A)(x)$

note: $(AB) \cdot x = \varphi(AB)(x) = \varphi(A)\varphi(B)(x) = \varphi(A)(B \cdot x) = A \cdot (B \cdot x)$
and $I \cdot x = \varphi(I)(x) = I(x) = x$ $\therefore$ this is an action ✓

(a) Let $x \in S^2$ then $\exists\, R_x^\theta, R_y^\phi \in SO(3) \ni R_x^\theta R_y^\phi(0,0,1) = x$
φ is onto & $R_x^\theta R_y^\phi \in SO(3) \Rightarrow \exists\, A \in SU(2) \ni \varphi(A) = R_x^\theta R_y^\phi$

$\Rightarrow A \cdot (0,0,1) = \varphi(A)(0,0,1) = R_x^\theta R_y^\phi(0,0,1) = x$
$\therefore SU(2) \cdot (0,0,1) = S^2$ hence the action is transitive.

(b) Let $\{e_i\}$ be the standard basis for $\mathbb{R}^3$

$$e_3 = (0,0,1) \longmapsto \hat{e}_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

stabilizer of e_3

$$\$\ A \in SU(2)_{e_3} \Leftrightarrow A \cdot e_3 = e_3 \Leftrightarrow \varphi(A)(e_3) = e_3$$

$$\Leftrightarrow \psi^{-1} \circ \tilde{\varphi}(A) \circ \psi(e_3) = e_3 \Leftrightarrow \tilde{\varphi}(A) \circ \psi(e_3) = \psi(e_3)$$

rewrite: $\tilde{\varphi}(A)(\hat{e}_3) = \hat{e}_3 \Leftrightarrow A\hat{e}_3 A^{-1} = \hat{e}_3$ ✓

$\$\ A\hat{e}_3A^{-1} = \hat{e}_3$. Let $A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ $(|\alpha|^2 + |\beta|^2 = 1)$

$\therefore A^{-1} = \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}$ hence $A\hat{e}_3A^{-1} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}$

$$= \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} \bar{\alpha}i & -\beta i \\ -\bar{\beta}i & -\alpha i \end{pmatrix} = \begin{pmatrix} (|\alpha|^2 - |\beta|^2)i & -2\alpha\beta i \\ -2\bar{\alpha}\bar{\beta}i & -(|\alpha|^2 - |\beta|^2)i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$\Rightarrow -2\alpha\beta i = 0$ hence either $\alpha = 0$ or $\beta = 0$

$\$\ \alpha = 0$ then $-|\beta|^2 i = i \Rightarrow |\beta|^2 = -1 \rightarrow\leftarrow \therefore \beta = 0$

hence $A = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$ and $|\alpha|^2 + \cancel{|\beta|^2}^{\,0} = |\alpha|^2 = 1$ ✓

$\$\ A = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$ & $|\alpha|^2 = 1 \Rightarrow A^{-1} = \begin{pmatrix} \bar{\alpha} & 0 \\ 0 & \alpha \end{pmatrix}$ and

$$A\hat{e}_3A^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} \bar{\alpha} & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}\begin{pmatrix} \bar{\alpha}i & 0 \\ 0 & -\alpha i \end{pmatrix} =$$

$$\begin{pmatrix} |\alpha|^2 i & 0 \\ 0 & -|\alpha|^2 i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \hat{e}_3$$

$$\therefore SU(2)_{e_3} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \middle|\ \alpha \in \mathbb{C}\ \&\ |\alpha|^2 = 1 \right\}$$

(c) $SU(2)_{e_3} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \middle| \, |\alpha| = 1 \right\}$ is obviously a closed subgroup of $SU(2)$

∴ We get a P.F.B. $SU(2) \longrightarrow SU(2)/SU(2)_{e_3}$
with fiber $SU(2)_{e_3}$

transitive

but $SU(2)/SU(2)_{e_3} \cong SU(2) \cdot e_3 = S^2$ (orbit/stabilizer thm)

and $\Theta : SU(2)_{e_3} \longrightarrow U(1)$ defined by $\Theta\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} = \alpha$
is an isomorphism (obviously smooth)

- $\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \in SU(2)_{e_3} \Rightarrow |\alpha| = 1 \Rightarrow \alpha \in U(1)$ ✓
- $\wp$ $\Theta\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} = 0 \Rightarrow \alpha = 0 \Rightarrow \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
- $\wp$ $\alpha \in U(1) \Rightarrow |\alpha| = 1 \Rightarrow \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \in SU(2)_{e_3}$ & $\Theta\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} = \alpha$

  ∴ onto
- $\Theta\left(\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}\begin{pmatrix} \beta & 0 \\ 0 & \bar{\beta} \end{pmatrix}\right) = \Theta\begin{pmatrix} \alpha\beta & 0 \\ 0 & \overline{\alpha\beta} \end{pmatrix} = \alpha\beta = \Theta\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}\Theta\begin{pmatrix} \beta & 0 \\ 0 & \bar{\beta} \end{pmatrix}$

∴ $SU(2)_{e_3} \cong U(1)$

Hence we have a P.F.B. $\begin{tikzcd} SU(2) \arrow[d, "\pi"] \\ S^2 \end{tikzcd}$ with fiber $U(1)$

(d) $\wp$ $SU(2) = S^2 \times U(1) \Rightarrow \pi_1(SU(2)) \overset{\pi_1(S^2 \times U(1))}{=} \pi_1(S^2) \times \pi_1(U(1))$
but $U(1) \cong S^1$ ∴ $\pi_1(U(1)) = \mathbb{Z}$ & $\pi_1(S^2) = \{0\}$
but $SU(2) \cong S^3$ ∴ $\pi_1(SU(2)) = \pi_1(S^3) = \{0\} \ncong \mathbb{Z} \times \{0\}$

$\Rightarrow$ $SU(2)$ is not a trivial bundle. //

Exercise 2.5 Let $SU(2)$ denote the set of all matrices $A \in Gl(2,\mathbb{C})$ such that $\bar{A}^t A = I$, $\det A = 1$

(a) Show that $A \in SU(2)$ iff $A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ for $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.

(b) The Lie algebra $su(2)$ of $SU(2)$ is a sub-Lie algeb of the Lie algebra $gl(2,\mathbb{C})$ of $Gl(2,\mathbb{C})$. Show that $B \in su(2)$ iff $\bar{B}^t + B = 0$ and $\mathrm{Tr}\, B = 0$, then show that this is true iff $B = \hat{x}$ for some $x \in \mathbb{R}^3$. Here $\hat{x}$ is defined by

$$\hat{x} = \begin{pmatrix} i x_3 & -x_2 + i x_1 \\ x_2 + i x_1 & -i x_3 \end{pmatrix}$$

Show that the mapping from $\mathbb{R}^3$ to $su(2)$ defined by $x \to \hat{x}$ is a vector space isomorphism.

Standard dot product

(c) For $x, y \in \mathbb{R}^3$ show that $\langle x, y \rangle = -\frac{1}{2} \mathrm{Tr}(\hat{x}\hat{y})$ and that $\|x\|^2 = \det(\hat{x})$. It follows that we have an inner product $\langle\ ,\ \rangle_0$ defined on $su(2)$ by $\langle \hat{x}, \hat{y} \rangle_0 \equiv -\frac{1}{2} \mathrm{Tr}(\hat{x}\hat{y})$ relative to which $x \to \hat{x}$ is an isometry and $\|\hat{x}\|_0^2 = \det \hat{x}$,

(d) Let $\varphi : SU(2) \to \mathrm{End}(su(2))$ be defined by $\varphi(A)(X) = A X A^{-1}$ for $X \in su(2)$. Explain why $A X A^{-1} \in su(2)$ for $X \in su(2)$ and show that

$$\langle \tilde{\varphi}(A)(\mathbf{X}), \tilde{\varphi}(A)(\mathbf{Y}) \rangle_0 = \langle \mathbf{X}, \mathbf{Y} \rangle_0$$

for all $\mathbf{X}, \mathbf{Y} \in su(2)$.

(e) Define $\varphi : SU(2) \to Gl(3, \mathbb{R})$ by

$$\widehat{\varphi(A)(x)} = \tilde{\varphi}(A)(\hat{x})$$

for all $x \in \mathbb{R}^3$. Show that this is well-defined by showing that $\varphi(A) : \mathbb{R}^3 \to \mathbb{R}^3$ is invertible. This should follow from the fact that $\tilde{\varphi}(A)$ is invertible which in turn should be a consequence of the identity at the top of the page.

(f) Show that $\varphi(A) \in O(3)$ $\forall\, A \in SU(2)$ by showing that $\langle \varphi(A)(x), \varphi(A)(y) \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^3$. If $\{e_i\}$ is the standard basis of $\mathbb{R}^3$ write $\varphi(A)(e_i) = \varphi(A)^j_i\, e_j$ so that $A \to (\varphi(A)^j_i)$ is a map from $SU(2)$ into 3×3 matrices. Show that the matrix of $\tilde{\varphi}(A)$ relative to $\{\hat{e}_i\}$ satisfies $\tilde{\varphi}(A)(\hat{e}_i) = \varphi(A)^j_i\, \hat{e}_j$. Thus $A \to \det(\varphi(A)^j_i)$ is continuous. Use this to show that $\varphi(SU(2)) \subseteq SO(3)$.

Exercise 2.6

It may be shown that $\varphi : SU(2) \to SO(3)$ of Exercise 2.5 is "onto" and that $\operatorname{Ker}\varphi = \{\pm I\}$. Define an action of $SU(2)$ on the 2-sphere

$$S^2 = \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \}$$

by $(A, x) \longrightarrow \varphi(A)x$, $A \in SU(2), x \in S^2$.

(a) Show that the action $SU(2) \times S^2 \to S^2$ is transitive.

(b) If $\{e_i\}$ is the standard basis of $\mathbb{R}^3$ show that $A \in SU(2)$ is in the isotropy subgroup of $e_3 \in \mathbb{R}^3$ iff $A \hat{e}_3 A^{-1} = \hat{e}_3$ which is true iff

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & +\bar{\alpha} \end{pmatrix} \qquad \alpha \in \mathbb{C}, \ |\alpha| = 1.$$

see notes

(c) Show that $SU(2) \to S^2$ is a principal fiber bundle with fiber a Lie subgroup of $SU(2)$ which is isomorphic to $U(1) \cong \{ z \in \mathbb{C} \mid |z| = 1 \}$

(d) It is known that the fundamental group of a product of two spaces is the product of the fundamental groups:

$$\pi_1(P \times Q) \cong \pi_1(P) \times \pi_1(Q)$$

Use this fact to show that the principal fiber bundle $SU(2) \longrightarrow \underset{\uparrow}{S^2}$ is not trivial

simply connected

$\pi_1(S^1) = \mathbb{Z}$

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4/28/2
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Exercises 2.7 & 2.8

Let $P \xrightarrow{\pi} M$ be a principal fiber bundle with structure group G. Let F be a manifold. $\ni$ G acts on the left of F.

Define $\sigma : (P \times F) \times G \longrightarrow P \times F$ by $\sigma((u,z),g) = (u \cdot g, g^{-1} \cdot z)$
σ is smooth because the actions & inversion in G are smooth. ✓

Exercise 2.7:

(1) $\sigma((u,z), g_1 g_2) = (u \cdot (g_1 g_2), \overset{(g_2^{-1} g_1^{-1})}{\overset{\|}{(g_1 g_2)^{-1}}} \cdot z) = ((u \cdot g_1) \cdot g_2, g_2^{-1} \cdot (g_1^{-1} \cdot z))$

✓ $= \sigma(((u \cdot g_1), (g_1^{-1} \cdot z)), g_2) = \sigma(\sigma((u,z), g_1), g_2)$ ✓

$\sigma((u,z), e) = (u \cdot e, \overset{e}{\cancel{e^{-1}}} \cdot z) = (u,z)$ ✓
$\Rightarrow \sigma$ is a right action

(2) Let $\tau : P \times F \longrightarrow M$ be defined by $\tau(u,z) = \pi(u)$
$\tau((u,z) \cdot g) = \tau(u \cdot g, g^{-1} \cdot z) = \pi(u \cdot g) = \pi(u)$
(the group action moves us up & down the fiber)
✓ $\Rightarrow \tau((u,z) \cdot g) = \pi(u) = \tau(u,z) \quad \forall (u,z) \in P \times F, \ g \in G$ ✓

Let $(u,z)_G = \{ (u,z) \cdot g \mid g \in G \}$ (the G-orbit of (u,z))

$\tilde{\tau} : P \times F / G \longrightarrow M$ defined by $\tilde{\tau}((u,z)_G) \equiv \tau(u,z) = \pi(u)$
✓ is well defined.
$\ni (v,y) \in (u,z)_G \Rightarrow \exists g \in G \ni (v,y) = (u,z) \cdot g$
$\therefore \tilde{\tau}((u,z)_G) = \tau((u,z) \cdot g) = \tau(u,z) = \pi(u)$ ✓

(3) Let $x \in M$ and $u_0 \in \pi^{-1}\{x\}$

$\ni (u,z)_G \in \tilde{\tau}^{-1}(x) \Rightarrow \tilde{\tau}(u,z)_G = \pi(u) = x \ \therefore u \in \pi^{-1}(x)$
hence $\exists g \in G \ni u_0 \cdot g = u$ (action is transitive on fibers)
$(u_0, g \cdot z) \cdot g = (u_0 \cdot g, \cancel{g^{-1} g} \cdot z) = (u,z) \Rightarrow$

$(u,z)_G = (u_0, g\cdot z)_G \in \{(u_0,z)_G \mid z\in F\}$

If $z\in F$ then $\tilde{\tau}((u_0,z)_G) = \pi(u_0) = x \Rightarrow (u_0,z)_G \in \tilde{\tau}^{-1}\{x\}$

$\therefore \tilde{\tau}^{-1}\{x\} = \{(u_0,z)_G \mid z\in F\}$ $\left(\subseteq {}^{P\times F}/_{G}\right)$

$\Rightarrow \omega : F \longrightarrow \tilde{\tau}^{-1}(x)$ def. by $\omega(z) = (u_0,z)_G$ is onto

If $\omega(z_1) = \omega(z_2) \Rightarrow (u_0,z_1)_G = (u_0,z_2)_G$

$\therefore (u_0,z_1) = (u_0,z_2)\cdot g$ for some $g\in G$

$\Rightarrow (u_0,z_1) = (u_0\cdot g, g^{-1}\cdot z_2) \Rightarrow u_0 = u_0\cdot g \Rightarrow g = e$
(because the action is free)

$\therefore z_1 = \overset{e}{\cancel{e^{-1}}}\cdot z_2 = z_2 \Rightarrow \omega$ is 1-1 $\therefore z \longmapsto (u_0,z)_G$ is a bijection

(4) Let $s : U \longrightarrow P$ be a local section of π also define $\Phi_U : U\times F \longrightarrow \tilde{\tau}^{-1}(U)$ by $\Phi_U(x,z) = (s(x),z)_G$

We know $\tilde{\tau}^{-1}(U) = \bigcup_{x\in U}\tilde{\tau}^{-1}\{x\} = \bigcup_{x\in U}\{(u_x,z)_G \mid z\in F\}$

$= \{(u_x,z)_G \mid x\in U, z\in F\}$ where u_x is a fixed element of $\pi^{-1}\{x\}$ we can choose $\underline{s(x) = u_x}$.

$\Rightarrow$ $w\in\tilde{\tau}(U)$ then $w = (s(x),z)_G$ for some $x\in U$, $z\in F$
$\Rightarrow \Phi_U(x,z) = (s(x),z)_G = w$ hence Φ_U is onto.

✓

If $\Phi_u(x,z) = \Phi_u(x_1,z_1) \Rightarrow (s(x),z)_G = (s(x_1),z_1)_G$
hence $x = \pi(s(x)) = \tau((s(x),z)_G) = \tau((s(x_1),z_1)_G) = \pi(s(x_1) = x_1$
but $z \longmapsto (u_0,z)_G$ is a bijection hence $(s(x),z)_G = (s(x),z_1)_G$
$\Rightarrow z = z_1 \therefore (x,z) = (x_1,z_1) \Rightarrow \Phi_u$ is a bijection ✓

$\mathcal{F}M \xrightarrow{\pi} M$ be the frame bundle
and denote an action of $\mathcal{G}l(n)$ on $(\mathbb{R}^n)^*$, $g \circ \alpha$
where $(g \circ \alpha)(x) = \alpha(g^{-1}x)\ \forall x \in \mathbb{R}^n$
define
$\varrho : \mathcal{F}M \times (\mathbb{R}^n)^* \longrightarrow T^*M$ by $\varrho((p,\{e_i\}),\alpha) = \alpha_i e^i$
(where $\alpha \in (\mathbb{R}^n)^* \Rightarrow \alpha = \alpha_i r^i$, $\{r_i\}$ & $\{r^i\}$ is the standard basis and its dual) e^i is a basis for $(T_pM)^* \Rightarrow \alpha_i e^i \in (T_pM)^*$
Now let $g \in \mathcal{G}l(n)$ then $(g^{-1} \circ \alpha)(r_j) = \alpha(g r_j) = \alpha(g^i_j r_i)$
$= g^i_j \alpha(r_i) = g^i_j \alpha_i$
$\therefore ((p,\{e_i\}),\alpha) \circ g = ((p,\{e_i\}) \circ g, g^{-1} \circ \alpha) = ((p,\{e_j g^j_i\}), g^i_j \alpha_i r^j)$.
hence $\varrho(((p,\{e_i\}),\alpha) \circ g) = g^i_j \alpha_i g^{-1j}_{\ \ l} e^l = g^i_j g^{-1j}_{\ \ l} \alpha_i e^l = \delta^i_l \alpha_i e^l = \alpha_i e^i$
$= \varrho(((p,\{e_i\}),\alpha))$ ✓ hence constant on orbits
So we can induce a mapping $\tilde{\varrho} : \dfrac{\mathcal{F}M \times (\mathbb{R}^n)^*}{\mathcal{G}l(n)} \longrightarrow T^*M$

Now construct an inverse by, $\varsigma(x,\gamma) = ((x,\{e_i\}),\gamma(e_i) r^i)_{\mathcal{G}l(n)}$
where $\{e_i\}$ is any basis of T_xM. Let $\{f_i\}$ be any other basis.
then $f_i = g^j_i e_j$ for some $g = (g^i_j) \in \mathcal{G}l(n)$
$((x,\{f_i\}),\gamma(f_i) r^i) = ((x,\{g^j_i e_j\}), g^j_i \gamma(e_j) r^i) = ((x,\{e_j\}) \circ g, g^{-1} \circ (\gamma(e_i) r^i)$
$= ((x,\{f_i\}),\gamma(e_i) r^i) \circ g \Rightarrow$ They are in the same orbit
$\Rightarrow \varsigma$ is well defined. ✓

$\varsigma \circ \tilde{\varrho}(((p,\{e_i\}),\alpha) \circ \mathcal{G}l(n)) = \varsigma(p,\alpha_i e^i) = ((p,\{e_i\}), \alpha_i \overset{\delta^i_j}{\cancel{e^i(e_j)}} r^j)$
$= ((p,\{e_i\}),\alpha_j r^j) = ((p,\{e_i\}),\alpha)$
$\tilde{\varrho} \circ \varsigma(p,\gamma) = \tilde{\varrho}(((p,\{e_i\}),\gamma(e_j) r^j) \circ \mathcal{G}l(n)) = (p,\gamma(e_j) e^j)$
$= (p,\gamma) \quad \therefore \tilde{\varrho}^{-1} = \varsigma$ ✓

ρ is smooth & since it is constant on orbits it induces a smooth map $\tilde{\rho}$.

Take a chart on T^*M say (T^*U, T^*x)
where $(p,\alpha) \in T^*U$ then $T^*x(x,\alpha) = (x(p), (\alpha_i(p)))$
where $\alpha = \alpha_i(p)\, d_p x^i$
And a chart on $\Finv M$ say $(\Finv U, \Finv x)$
where $(p,\{e_i\}) \in \Finv U$ then $\Finv x(p,\{e_i\}) = (x(p), (A^i_j))$
where $e_i = A^j_i \partial/\partial x_j|_p$. let $\tilde{x} = \widetilde{\Finv x \times y}$ $\quad \underset{\mathbin{/\!/}\rho}{y}(\alpha_i r^i) = (\alpha_i)$

$$\tilde{x} \circ \xi \circ T^*x^{-1}((c_i),(\alpha_i)) \qquad \tilde{x} \circ \xi(x^{-1}(c_i), \alpha_i d_p x^i)$$

$$= \tilde{x}\Big(((x^{-1}(c_i), \{\partial/\partial x^i|_p\}), \alpha_i r^i) \cdot Gl(n)\Big)$$

$$= (x \circ x^{-1}(c_i), I) \times (\alpha_i) = ((c_i), I, (\alpha_i))$$ which is smooth ✓
$\Rightarrow \xi$ is smooth

Thus we have a bundle isomorphism:

$$\begin{tikzcd}
T^*M \arrow[rr, "\xi"] \arrow[dr] & & (\Finv M \times (\mathbb{R}^n)^*)/Gl(n) \arrow[dl] \\
& M &
\end{tikzcd}$$

Exercise 2.8: $\mathcal{F}M \xrightarrow{\pi} M$ the frame bundle and denote an action of $\mathcal{G}l(n)$ on $T^1_1\mathbb{R}^n$ by $g\cdot\lambda$ where
$(g\cdot\lambda)(x,\alpha) = \lambda(g^{-1}\cdot x, g^{-1}\cdot\alpha)$ $\forall\, x\in\mathbb{R}^n, \alpha\in(\mathbb{R}^n)^*, \lambda\in T^1_1\mathbb{R}^n, g\in\mathcal{G}l(n)$
define
$\rho : \mathcal{F}M\times T^1_1\mathbb{R}^n \longrightarrow T^1_1M$ by $\rho((p,\{e_i\}),\lambda) = (p, \lambda^i_j e^j\otimes e_i)$
(where $\lambda\in T^1_1\mathbb{R}^n \Rightarrow \lambda^i_j r^j\otimes r_i$, $\{r_i\}$ & $\{r_j\}$ is the standard basis and its dual) e_i is a basis for T_pM, e^i is a basis for $(T_pM)^* \Rightarrow e^i\otimes e_j$ is a basis for $(T^1_1M)_p \Rightarrow (p,\lambda^i_j e^j\otimes e_i)\in T^1_1M$ ✓
Now let $g\in\mathcal{G}l(n)$ then as before $(g^{-1}\cdot\alpha)(r_j) = g^i_j\alpha_i$ and
$g^{-1}\cdot x(r^j) = \overset{-1}{g}{}^j_i x^i \Rightarrow (g\cdot\lambda)(r_i, r^j) = \lambda(\overset{-1}{g}{}^l_i r_l, g^j_k r^k) =$
$\overset{-1}{g}{}^l_i g^j_k \lambda(r_l, r^k) = \overset{-1}{g}{}^l_i g^j_k \lambda^k_l$
$\Rightarrow ((p,\{e_i\}),\lambda)\cdot g = ((p,\{e_i\})\cdot g, g^{-1}\cdot\lambda)$
$= ((p,\{e_j g^j_i\}), g^l_i \overset{-1}{g}{}^j_k \lambda^k_l r^i\otimes r_j)$
$\therefore \rho(((p,\{e_i\}),\lambda)\cdot g) = \rho((p,\{e_j g^j_i\}), g^l_i \overset{-1}{g}{}^j_k \lambda^k_l r^i\otimes r_j)$
$= g^l_i \overset{-1}{g}{}^j_k \lambda^k_l (e^a \overset{-1}{g}{}^i_a)\otimes(e_b g^b_j) = \overset{-1}{g}{}^i_a g^l_i \overset{-1}{g}{}^j_k g^b_j \lambda^k_l e^a\otimes e_b$
$= \delta^l_a \delta^b_k \lambda^k_l e^a\otimes e_b = \lambda^b_a e^a\otimes e_b = \rho((p,\{e_i\}),\lambda)$
$\therefore \rho$ is constant on the orbits.
hence we get an induced map $\tilde\rho : \mathcal{F}M\times T^1_1\mathbb{R}^n / \mathcal{G}l(n) \longrightarrow T^1_1M$

As before we now construct the inverse: $\varsigma : T^1_1M \longrightarrow \mathcal{F}M\times T^1_1\mathbb{R}^n / \mathcal{G}l(n)$
$\varsigma(x,\lambda) = ((x,\{e_i\}), \lambda(e_i,e^j) r^i\otimes r_j)_{\mathcal{G}l(n)}$ where $\{e_i\}$ is any basis for T_xM and $\{e^i\}$ is its dual. Let $\{f_i\}$ be any other basis for T_xM (with dual $\{f^i\}$) Then $f_i = g^j_i e_j$ & $f^i = \overset{-1}{g}{}^i_j e^j$ for some $g = (g^i_j)\in\mathcal{G}l(n)$. $((x,\{f_l\}), \lambda(f_i,f^j) r^i\otimes r_j) =$
$((x,\{g^k_l e_k\}), \lambda(g^a_i e_a, \overset{-1}{g}{}^j_b e^b) r^i\otimes r_j) = ((x,\{e_k\})\cdot g, g^a_i \overset{-1}{g}{}^j_b \lambda(e_a,e^b) r^i\otimes r_j)$
$= ((x,\{e_k\})\cdot g, g^{-1}\cdot(\lambda(e_a,e^b) r^a\otimes r_b)) = ((x,\{e_k\}), \lambda(e_a,e^b) r^a\otimes r_b)\cdot g$
$\Rightarrow$ they are in the same orbit $\therefore$ choice of basis doesn't matter
$\Rightarrow \varsigma$ is well defined ✓

$(\lambda = \lambda^i_j r^j\otimes r_i)$

$\varsigma\circ\tilde\rho((p,\{e_i\}),\lambda)_{\mathcal{G}l(n)}) = \varsigma(p, \lambda^i_j e^j\otimes e_i) = ((p,\{e_i\}),$ back $\hookrightarrow$

$\lambda^i_j e^j \otimes e_i (e_a, e^b) n^a \otimes n_b)_{\mathfrak{gl}(n)} = ((p,\{e_i\}), \lambda^i_j \delta^j_a \delta^b_i n^a \otimes n_b)_{\mathfrak{gl}(n)}$

$= ((p,\{e_i\}), \lambda^b_a n^a \otimes n_b)_{\mathfrak{gl}(n)} = ((p,\{e_i\}), \lambda)_{\mathfrak{gl}(n)}$ ✓

$\tilde{\rho} \circ \xi(p, \lambda) = \tilde{\rho}(((p,\{e_\ell\}), \lambda(e_i, e^j) n^i \otimes n_j)_{\mathfrak{gl}(n)})$

$= (p, \lambda(e_i, e^j) e^i \otimes e_j) = (p, \lambda)$ ✓

$\therefore\ \hat{\rho} = \xi^{-1}$ hence ξ is 1-1 / onto

$\xi^{-1} = \bar{\rho}$ is smooth as in Exercise 2.7 ✓

Take a chart on $T^1_1 M$ say $(T^1_1 U, T^1_1 x)$
where $(p,\lambda) \in T^1_1 U$ then $T^1_1 x(p,\lambda) = (x(p), (\lambda^j_i))$
where $\lambda = \lambda^j_i\, d_p x^i \otimes \frac{\partial}{\partial x^j}\big|_p$
And take a chart on $\mathcal{F}M$ say $(\mathcal{F}U, \mathcal{F}x)$
where $(p,\{e_i\}) \in \mathcal{F}U$ then $\mathcal{F}x(p,\{e_i\}) = (x(p), (A^i_j))$
where $e_i = A^j_i \frac{\partial}{\partial x^j}\big|_p$ let $\tilde{x} = \widetilde{\mathcal{F}x \times y}$ $\quad y(\lambda^i_j n^j \otimes n_i) = (\lambda^i_j)$

$\tilde{x} \circ \xi \circ (T^1_1 x)^{-1}((c_i), (\lambda^j_i)) = \tilde{x} \circ \xi(\overset{p}{\overbrace{x^{-1}(c_i)}}, \lambda^j_i\, d_p x^i \otimes \frac{\partial}{\partial x^j}\big|_p)$

$= \tilde{x}(((x^{-1}(c_i), \{\frac{\partial}{\partial x^\ell}\big|_p\}), \lambda^j_i n^i \otimes n_j)_{\mathfrak{gl}(n)})$

$= (x \circ x^{-1}(c_i), I) \times (\lambda^j_i) = (c_i, I, \lambda^j_i) \Rightarrow \xi$ is smooth

Thus we have a bundle isomorphism:

$$\begin{tikzcd}
T^1_1 M \arrow[rr, "\xi"] \arrow[dr] & & (\mathcal{F}M \times T^1_1 \mathbb{R}^n)/\mathfrak{gl}(n) \arrow[dl] \\
& M &
\end{tikzcd}$$

<u>Due</u>: 4/26

Assume that $P \xrightarrow{\pi} M$ is a PFB with structure group G and that G acts on the left of some manifold F. Define a mapping σ from $(P\times F)\times G \to P\times F$ by $((u,\xi),g) \to (ug, g^{-1}\xi)$. The mapping is clearly smooth.

Exercise 2.7

(1) Show that σ is a right action of G on $P\times F$.

(2) Let $\tau : P\times F \Longrightarrow M$ be defined by $\tau(u,\xi) = \pi(u)$. Show that $\tau((u,\xi)g) = \tau(u,\xi)$ $\forall (u,\xi)\in P\times F$, $g\in G$ and that one obtains a well-defined function $\hat{\tau} : P\times F/G \longrightarrow M$ defined by $\tilde{\tau}((u,\xi)_G) = \tau(u,\xi) = \pi(u)$. Where $(u,\xi)_G = \{(u,\xi)g \mid g\in G\}$ is the G orbit of (u,ξ) in $P\times G$.

(3) Let $x\in M$ and $u_0 \in \pi^{-1}(x)$. Write $w_0 = (u_0,\xi_0)_G$ for some arbitrary $\xi_0\in F$. Show that

$$\hat{\tau}^{-1}(x) = \{(u_0,\xi)_G \mid \xi\in F\}$$

and that $\xi \mapsto (u_0,\xi)_G$ is a bijection from F onto $\hat{\tau}^{-1}(x)$.

(4) Let $s : U \to P$ be a local section of π and show that $\Phi_U : U\times F \to \hat{\tau}^{-1}(U)$ defined by $\Phi_U(x,\xi) = (s(x),\xi)_G$ is a bijection.

Let $P = \mathcal{F}M \xrightarrow{\pi} M$ denote the frame bundle. $Gl(n)$ acts on $(\mathbb{R}^n)^*$ via $(g,\alpha) \to g\cdot\alpha$ where $(g\cdot\alpha)(x) = \alpha(g^{-1}x)$ $\forall x \in \mathbb{R}^n$.

Consider the bundle $(\mathcal{F}M \times (\mathbb{R}^n)^*)/Gl(n)$. We show that it is really T^*M. To show this first define a mapping ρ from $\mathcal{F}M \times (\mathbb{R}^n)^* \longrightarrow T^*M$ as follows:

$$\rho((p,e_i),\alpha) = \alpha_i e^i.$$

Notice that $\alpha \in (\mathbb{R}^n)^*$ implies $\alpha = \alpha_i r^i$ where $\{r_i\}$ is the standard basis of $\mathbb{R}^n$ and $\{r^i\}$ is the dual basis. So the α_i are components of α relative to $\{r^i\} \subseteq (\mathbb{R}^n)^*$. But $\{e_i\}$ is a basis of T_pM and so $\{e^i\}$ is the dual basis in T_p^*M. Thus $\alpha_i e^i \neq \alpha$ as it is in T_p^*M. We show ρ is constant on orbits of the action of G on $\mathcal{F}M \times (\mathbb{R}^n)^*$. This action is defined by $((p,e_i),\alpha)\cdot g = ((p,e_i)\cdot g, g^{-1}\alpha)$

But $(p,\{e_i\})\cdot g = (p,\{e_j g^j_i\})$ and $(g^{-1}\alpha) = g^i_j \alpha_i r^j$ since

$$(g^{-1}\alpha)_i = (g^{-1}\alpha)(r_i) = \alpha(g r_i) = \alpha(g_i^j r_j) = g_i^j \alpha(r_j) = g_i^j \alpha_j$$

But if $f_i = e_j g_i^j$ then the dual basis is $f^k = g^{-1}{}_r^k e^r$

Thus

$$((p,e_i),\alpha)\cdot g = ((p,e_i)\cdot g,\ g^{-1}\cdot\alpha)$$

$$= ((p,\{e_j g_i^j\}),\ \underline{g_i^j \alpha_j r^i}) \longleftarrow$$

ρ replaces r^i with dual to $\{e_j g_i^j\}$

$$\rho(\underline{((p,e_i),\alpha)}\cdot g) = \rho\big(((p,\{e_j g_i^j\}),\ g_i^j \alpha_j r^i)\big)$$

$$= (g_i^j\alpha_j)(g^{-1}{}_r^i e^r) = \delta_r^j \alpha_j e^r$$

$$= \alpha_j e^j = \rho(\underline{((p,e_i),\alpha)})$$

So ρ is constant on $((p,e_i),\alpha)\cdot Gl(n)$. Thus ρ induces a mapping $\tilde{\rho} : [FM\times(\mathbb{R}^n)^*]/Gl(n) \longrightarrow T^*M$

We Construct an inverse to $\tilde{\rho}$. Let $\varsigma : T^*M \to FM\times(\mathbb{R}^n)^*/Gl(n)$ be defined by $\varsigma(x,\gamma) = ((x,\{e_i\}),\gamma(e_i)r^i)_{Gl(n)}$ where $\{e_i\}$ is *any* basis of T_xM. We must show that ς is well-defined by showing that it is independent of the choice $\{e_i\}$. Let $\{f_i\}$ be any other basis of T_xM and write $f_i = g_i^j e_j$. Clearly $(g_i^j)\in Gl(n)$.

Notice ~~that~~

$$((x,\{f_i\}),\gamma(f_i)r^i) = ((x,\{g_i^j e_j\}),\ g_i^j \gamma(e_j) r^i)$$

$$= ((x,\{e_i\})g,\ g^{-1}\cdot(\gamma(e_i)r^i))$$

(see calculation at top of Page 1)

$$= ((x,\{e_i\}), \gamma(e_i)r^i)\cdot g$$

Thus $((x,\{f_i\}), \gamma(f_i)r^i)$ and $((x,\{e_i\}), \gamma(e_i)r^i)$ are on the same $Gl(n)$ orbit of $\mathcal{F}M\times(\mathbb{R}^n)^*$ and $((x,f_i), \gamma(f_i)r_i)_{Gl(n)} = ((x,e_i), \gamma(e_i)r^i)_{Gl(n)}$ and so ς is well-defined. It is easy to check that ς is the inverse of $\hat{\rho}$. It is trivial to show that ρ is smooth and since it is constant on orbits it induces a smooth map $\hat{\rho}$. Smoothness of the inverse requires more effort which we leave to the reader. Thus we have a bundle isomorphism

$$\begin{tikzcd}
T^*M \arrow[rr, "\varsigma"] \arrow[dr] & & (\mathcal{F}M\times(\mathbb{R}^n)^*)/Gl(n) \arrow[dl] \\
& M &
\end{tikzcd}$$

Exercise 2.8 Consider the action of $Gl(n)$ on $T^1_1\mathbb{R}^n$ defined by $(g\cdot\lambda)(x,\alpha) = \lambda(g^{-1}\cdot x, g^{-1}\cdot\alpha)$. Show that one has a bundle isomorphism $\varsigma : T^1_1 M \to (\mathcal{F}M\times T^1_1\mathbb{R}^n)/Gl(n)$

2.7 extra credit
2.8 ✓ $\frac{18}{18}$

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Exercises 2.7 & 2.8

Problem 1: Let $\omega: T(\mathcal{F}(TM)) \longrightarrow gl(n)$ be a connection on the P.F.B. $\mathcal{F}(TM) \xrightarrow{\pi} M$ where $\dim M = n$.
Let $(U, x = (x^\mu))$ and $(\bar{U}, \bar{x} = (\bar{x}^\nu))$ be overlapping charts on M. Also let $s: U \longrightarrow \mathcal{F}(TM)$ and $\bar{s}: \bar{U} \longrightarrow \mathcal{F}(TM)$ be local sections of π, defined by $s(p) = (p, \{\frac{\partial}{\partial x^\mu}|_p\})$, $\bar{s}(p) = (p, \{\frac{\partial}{\partial \bar{x}}|_p\})$

Finally, let $\Gamma_\mu = s^*\omega(\frac{\partial}{\partial x^\mu})$ and $\bar{\Gamma}_\nu = \bar{s}^*\omega(\frac{\partial}{\partial x^\nu})$.
by the Thm on page 264 $\exists!$ mapping $g: U \cap \bar{U} \longrightarrow G \ni$
✓ $\bar{s}(p) = s(p)g(p)$ and $(\bar{s}^*\omega)_p = Ad(g(p)^{-1})(s^*\omega)_p + d_{g(p)} l_{g(p)}^{-1} \circ d_p g$

$\bar{s}(p) = (p, \{\frac{\partial}{\partial \bar{x}^\nu}|_p\}) = (p, \{\frac{\partial x^\mu}{\partial \bar{x}^\nu} \frac{\partial}{\partial x^\mu}|_p\}) = (p, \{\frac{\partial}{\partial x^\mu}|_p\}) \circ (\frac{\partial x^\mu}{\partial \bar{x}^\nu}|_p)$
✓ $= s(p) \circ g(p)$ but g is unique hence $g(p) = (\frac{\partial x^\mu}{\partial \bar{x}^\nu}|_p)$

Now let $\gamma(t) \in M$ be a curve $\ni \gamma(0) = p$ & $\gamma'(0) = \frac{\partial}{\partial x^\nu}|_p$ $(= \partial_\nu(p))$

$\therefore$ $d_{g(p)} l_{g(p)}^{-1} \circ d_p g(\partial_\nu(p)) = d_{g(p)} l_{g(p)^{-1}} \circ d_p g\left(\frac{d}{dt}[\gamma(t)]\Big|_{t=0}\right)$ use chain rule

$= \frac{d}{dt}\left[l_{g(p)^{-1}}(g(\gamma(t)))\right]\Big|_{t=0} = \frac{d}{dt}\left[g(p)^{-1} g(\gamma(t))\right]\Big|_{t=0}$

$= \cancelto{0}{\frac{d}{dt}\left[g(p)^{-1}\right]}\Big|_{t=0} g(\gamma(0)) + g(p)^{-1} \frac{d}{dt}\left[g(\gamma(t))\right]\Big|_{t=0} = g(p)^{-1} d_{\gamma(0)} g(\gamma'(0))$

$= g(p)^{-1} d_p g(\partial_\nu(p)) = g(p)^{-1} (\partial_\nu g)(p)$ ✓

$\therefore \bar{\Gamma}_\nu(p) = (\bar{s}^*\omega)_p(\partial_\nu) = Ad(g(p)^{-1})(s^*\omega)_p(\partial_\nu) + d_{g(p)} l_{g(p)}^{-1} \circ d_p g(\partial_\nu)$
$= Ad(g(p)^{-1}) \Gamma_\nu(p) + g(p)^{-1} (\partial_\nu g)(p)$

$\therefore \underline{\bar{\Gamma}_\nu(p) = g(p)^{-1} \Gamma_\nu(p) g(p) + g(p)^{-1} (\partial_\nu g)(p)}$ ✓

back

$$\overline{\Gamma}_{\mu\delta}^{\ \ e}(p) = \left[g(p)^{-1}\Gamma_{\mu}(p)\,g(p) + g(p)^{-1}(\partial_{\mu}g)(p)\right]^{e}_{\ \delta}$$

but $g(p)^{e}_{\ \delta} = \frac{\partial x^{e}}{\partial \overline{x}^{\delta}}\Big|_{p}$ and $\frac{\partial x^{e}}{\partial \overline{x}^{\delta}}\frac{\partial \overline{x}^{\delta}}{\partial x^{\mu}}\Big|_{p} = \delta^{e}_{\mu} \Rightarrow g(p)^{-1\,e}_{\ \ \ \delta} = \frac{\partial \overline{x}^{e}}{\partial x^{\delta}}\Big|_{p}$

$$\therefore \overline{\Gamma}_{\mu\delta}^{\ \ e}(p) = \frac{\partial \overline{x}^{e}}{\partial x^{k}}\Big|_{p}\Gamma_{\mu\lambda}^{\ \ k}(p)\frac{\partial x^{\lambda}}{\partial \overline{x}^{\delta}}\Big|_{p} + \frac{\partial \overline{x}^{e}}{\partial x^{k}}\Big|_{p}\partial_{\mu}\left(\frac{\partial x^{k}}{\partial \overline{x}^{\delta}}\Big|_{p}\right)$$

$$\Rightarrow \underline{\overline{\Gamma}_{\mu\delta}^{\ \ e} = \Gamma_{\mu\lambda}^{\ \ k}\frac{\partial \overline{x}^{e}}{\partial x^{k}}\frac{\partial x^{\lambda}}{\partial \overline{x}^{\delta}} + \frac{\partial^{2} x^{k}}{\partial x^{\mu}\partial \overline{x}^{\delta}}\Big|_{p}\frac{\partial \overline{x}^{e}}{\partial x^{k}}\Big|_{p}} \quad \checkmark$$

Problem 2: Make appropriate identifications:

$$W^\mu(p) = (s(p), r^\mu)G \cong d_p x^\mu$$

where $s(p) = (p, \{\partial/\partial x^\mu|_p\})$ also, $\alpha(p) = (s(p), \hat{\alpha}(s(p)))G$

$$= \hat{\alpha}_\mu(s(p))(s(p), r^\mu)G = \hat{\alpha}_\mu(s(p))W^\mu(p) \cong \hat{\alpha}_\mu(s(p))\, d_p x^\mu$$

So, $\alpha \cong \hat{\alpha}_\mu(s(p))\, d_p x^\mu = \alpha_\mu(p)\, d_p x^\mu$

Using the equation from class we get,

$$(\nabla_{\partial_\mu}\alpha)(p) = \left(d\alpha_\nu(\partial_\mu(p)) + [(s^*\omega)(\partial_\mu(p))\circ\alpha(p)]_\nu\right)W^\nu(p)$$

$$= \left((\partial_\mu\alpha_\nu)(p) + [\Gamma_\mu(p)\cdot\alpha(p)]_\nu\right)W^\nu(p)$$

$$\cong \left((\partial_\mu\alpha_\nu)(p) + [\Gamma_\mu(p)\circ\alpha(p)]_\nu\right)d_p x^\nu$$

$$(\Gamma_\mu\circ\alpha)(p) = \Gamma_\mu(p)\cdot\alpha(p) = \Gamma_\mu(p)\circ(s(p), \hat{\alpha}(s(p)))G$$

$$= -\Gamma^\nu_{\mu\lambda}(p)\alpha_\nu(p)W^\lambda(p) \cong -\Gamma^\nu_{\mu\lambda}(p)\alpha_\nu(p)\,d_p x^\lambda$$

$$\Rightarrow \boxed{\nabla_{\partial_\mu}\alpha \cong \left[\partial_\mu\alpha_\lambda - \Gamma^\nu_{\mu\lambda}\alpha_\nu\right]dx^\lambda}$$

handout: 4/24/2

Final Homework

Let $P \xrightarrow{\pi} M$ be a principal fiber bundle with structure group G. Assume there is given a left action $G\times V \xrightarrow{\sigma} V$ on a finite dimensional vector space V and assume that for each $g \in G$ the mapping $v \longrightarrow \sigma(g,v) = g\cdot v$, $v\in V$, is linear. Define a mapping $\hat{\sigma} : \mathfrak{g}\times V \to V$ by

$$\hat{\sigma}(\xi, v) = \frac{d}{dt}[\sigma(\exp(t\xi), v)]\Big|_{t=0}$$

for $(\xi,v)\in \mathfrak{g}\times V$. Observe that for each $\xi\in\mathfrak{g}$ the mapping $v \longmapsto \hat{\sigma}(\xi,v) \equiv \xi\cdot v$, $v\in V$, is also linear.

Let $\mathrm{Hom}_G(P,V)$ denote the linear space of all equivariant mappings from P to V and let $\Gamma_M(P\times_G V)$ denote the set of all sections of $P\times_G V \xrightarrow{\tau} M$.

Theorem There is a vector space isomorphism Ξ from $\mathrm{Hom}_G(P,V)$ onto $\Gamma_M(P\times_G V)$ defined by

$$\Xi(\hat{\varphi}) = \varphi.$$

The mappings $\hat{\varphi}$ and φ are related by the identities: $\varphi(x) = (u_x, \hat{\varphi}(u_x))G$ where the latter denotes the G orbit of $(u_x,\hat{\varphi}(u_x))\in P\times V$ for $u_x\in\pi^{-1}(x)$ and $\hat{\varphi}(u) = \hat{u}^{-1}(\varphi(\pi(u)))$ for $u\in P$ and where $\hat{u}$ is the mapping from

$\downarrow$ V into $\tau^{-1}(\pi(u)) \subseteq P\times_G V$ defined by $\hat{u}(\xi) = (u,\xi)G$.

Definition Given a PFB $\pi : P \to M$ and a principal connection $\omega : TP \to \mathfrak{g}$ defined on P define the horizontal lift of a vector field X on M to be the vector field $\tilde{X}$ on P such that

$$d_u\pi(\tilde{X}_u) = X_{\pi(u)} \qquad \omega_u(\tilde{X}_u) = 0$$

for all $u \in P$. It may be shown that $\tilde{X}$ exists and is unique given a vector field X on M.

Proposition Let $P \xrightarrow{\pi} M$ be a PFB and ω a principal connection on P. Let X be a vector field defined on an open subset $U \subseteq M$ which is a chart domain and on which a local section $s : U \to P$ is defined. Then, for each $p \in U$

$$\boxed{\tilde{X}(s(p)) = d_p s(X(p)) - \delta_{(s^*\omega)(X_p)}(s(p))\,.}$$

In particular if (x^μ) is a chart of M defined on U then for $A_\mu(p) = (s^*\omega)(\partial_\mu)$,

$$\boxed{\tilde{\partial}_\mu(s(p)) = (\partial_\mu s)(p) - \delta_{A_\mu(p)}(s(p))}$$

Proof Clearly $d_p s(X_p) \in T_{s(p)}(s(U)) \subseteq T_{s(p)}P$ and

$$d_{s(p)}\pi\left(d_p s(X_p)\right) = d_p(\pi\circ s)(X_p) = X_p = d_{s(p)}\pi\left(\tilde{X}_{s(p)}\right)$$

Thus $d_{s(p)}\pi\left(\tilde{X}_{s(p)} - d_p s(X_p)\right) = 0$ and $\tilde{X}_{s(p)} - d_p s(X_p)$ is vertical. Since $\delta_{s(p)} : \quad \to VT_pP$ is an isomorphism there exists $\xi \in \mathfrak{g}$ such that $\tilde{X}_{s(p)} - d_p s(X_p) = \delta_\xi(s(p))$ and

$$\tilde{X}_{s(p)} = d_p s(X_p) + \delta_\xi(s(p))$$

But $0 = \omega(\tilde{X}_{s(p)}) = \omega(d_p s(X_p)) + \omega(\delta_\xi(s(p)))$

and $\xi = -\omega(d_p s(X_p)) = -(s^*\omega)(X_p)$.

Thus $\tilde{X}_{s(p)} = d_p s(X_p) - \delta_{(s^*\omega)(X_p)}(s(p))$. When $X = \partial_\mu$,

$$\tilde{\partial}_\mu(s(p)) = d_p s(\partial_\mu) - \delta_{(s^*\omega)(\partial_\mu)}(s(p)) = (\partial_\mu s)(p) - \delta_{A_\mu(p)}(s(p)),$$

Proposition Given $P \xrightarrow{\pi} M$ a PFB, ω a principal connection on P, $G \times V \to V$ a left action on the finite dimensional vector space V and a vector field X on M it follows that $\tilde{X}\hat{\varphi} \in \mathrm{Hom}_G(P,V)$ for every $\hat{\varphi} \in \mathrm{Hom}_G(P,V)$.

Proof $\tilde{X}(\hat{\varphi})(ua) = d_{ua}\hat{\varphi}(\tilde{X}(ua)) = d_{ua}\hat{\varphi}\left(d_u r_a(\tilde{X}(u))\right)$

$$= d_u(\hat{\varphi}\circ r_a)(\tilde{X}(u)) = d_u(a^{-1}\hat{\varphi})(\tilde{X}(u))$$

$$= a^{-1} d_u\hat{\varphi}(\tilde{\underline{X}}(u)) = a^{-1} \tilde{\underline{X}}(\hat{\varphi})(u),$$

or each $a \in G$, $u \in P$. Here we have used the identity $d_u r_a(\tilde{\underline{X}}(u)) = \tilde{\underline{X}}(ua)$. This follows from the fact that $\tilde{\underline{X}}(ua) \in T_{ua}P$ is the unique horizontal vector such that $d_{ua}\pi(\tilde{\underline{X}}(ua)) = \underline{X}_{\pi(ua)}$, whereas one also has ~~that~~ $d_u r_a(\tilde{\underline{X}}(u)) \in d_u r_a(H(u)) = H(ua)$ is horizontal and $d_{ua}\pi(d_u r_a(\tilde{\underline{X}}(u))) = d_u(\pi \circ r_a)(\tilde{\underline{X}}(u)) = d_u\pi(\tilde{\underline{X}}(u)) = \underline{X}_{\pi(u)} = \underline{X}_{\pi(ua)}$. Uniquess implies $\tilde{\underline{X}}(ua) = d_u r_a(\tilde{\underline{X}}(u))$ as we require.

Definition Let $\pi : P \to M$ be a PFB with structure group G, $\omega : TP \to \mathfrak{g}$ a principal connection and $\hat{\varphi} \in \mathrm{Hom}_G(P, V)$ where V is a finite dimensional vector space on which there is a left (linear) action $G \times V \to V$. The *covariant derivative of $\hat{\varphi}$* is the mapping $D\hat{\varphi} : TP \to \mathfrak{g}$ defined by $(D\hat{\varphi})_u(\underline{Y}) = d\hat{\varphi}((h\underline{Y})_u)$ for $u \in P$, $\underline{Y} \in T_uP$.

$\uparrow$ **Theorem** Given $P \xrightarrow{\pi} M$, $\omega : TP \to \mathfrak{g}$, $G \times V \to V$ as above it follows that

$$\boxed{D\hat{\varphi} = d\hat{\varphi} + \omega \cdot \hat{\varphi}},$$

i.e., if $u \in P$ and $Y \in T_u P$,

$$(D\hat{\varphi})_u(Y) = d_u\hat{\varphi}(Y) + \omega_u(Y) \cdot \hat{\varphi}(u).$$

Proof Let $Y \in T_u P$ and observe that $(vY)_u \in VT_uP$ and so $(vY)_u = \delta_{A(u)}(u)$ for some $A(u) \in \mathfrak{g}$.

But
$$\omega(Y_u) = \omega((hY)_u) + \omega((vY)_u) = 0 + \omega(\delta_{A(u)}(u)) = A(u)$$

Thus
$$(vY)_u = \delta_{\omega(Y_u)}(u) = \frac{d}{dt}\left[u \cdot \exp(t\,\omega(Y_u))\right]\Big|_{t=}$$

$$* \left\{ \begin{aligned} \text{and } d_u\hat{\varphi}((vY)_u) &= d_u\hat{\varphi}\, \frac{d}{dt}\left[u \exp t(\omega(Y_u))\right]\Big|_{t=0} \\ &= \frac{d}{dt}\left[\hat{\varphi}(u \exp(t\,\omega(Y_u)))\right]\Big|_{t=0} \\ &= \frac{d}{dt}\left[\exp(-t\,\omega(Y_u))\, \hat{\varphi}(Y_u)\right]\Big|_{t=} \\ &= -\omega(Y_u) \cdot \hat{\varphi}(Y_u) \end{aligned} \right.$$

it follows that

$$\begin{aligned} (D\hat{\varphi})_u(Y_u) &= (d\hat{\varphi})_u((hY)_u) \\ &= (d\hat{\varphi})_u(Y_u - (vY)_u) \\ &= (d\hat{\varphi})_u(Y_u) - d_u\hat{\varphi}((vY)_u) \end{aligned}$$

$$= (d\hat{\varphi})_u(Y_u) - \left(-\omega(Y_u)\cdot\hat{\varphi}(Y_u)\right)$$

$$= (d\hat{\varphi})_u(Y_u) + \omega(Y_u)\cdot\hat{\varphi}(Y_u).$$

Definition If $\varphi \in \Gamma_M(P\times_G V)$ and X is a vector field on M then $\nabla_X\varphi$, the covariant derivative of φ in the direction X, is that element of $\mathrm{Hom}_G(P,V)$ such that

$$\nabla_X\varphi = \Xi(\tilde{X}\hat{\varphi})$$

Where $\varphi = \Xi(\hat{\varphi})$. Thus

$$(\nabla_X\varphi)(x) = [u_x, \tilde{X}(\hat{\varphi})(u_x)G$$

for $u_x \in \pi^{-1}(x)$, $x \in M$.

Theorem Assume that $\varphi \in \Gamma_M(P\times_G V)$ and that $s: U \to P\times_G V$ is a local section of $P\times_G V \xrightarrow{\tau} M$. Let $\{e_b\}$ be a basis of $\mathfrak{g}$ and $\{V_a\}$ a basis of V, thus $W_a(x) = (s(x), V_a)G$ is a basis of $\tau^{-1}(x)$ for each $x \in U$, if $\varphi(x) = \varphi^a(x)W_a(x)$ $\forall x \in U$ and $\hat{\varphi}(u) = \hat{\varphi}^a(u)V_a$ for all $u \in P$, then $\varphi^a(x) = \hat{\varphi}^a(s(x))$ for all $x \in U$. Moreover if $(\nabla_X\varphi)(x) = (\nabla_X\varphi)^a(x)\,W_a(x)$ then

$$\boxed{(\nabla_X\varphi)^a(x) = d\varphi^a(X_x) + h^a_{bc}A^b(x)\varphi^c(x)}$$

for all $x \in U$. Here $\{h^a_{bc}\}$ are ~~the~~ numbers defined by the action of $\mathfrak{g}$ on $\mathbb{V}$, i.e.

$$e_b \cdot v_c = h^a_{bc} v_a .$$

~~Proof~~ Since $\varphi(x) = (s(x), \hat\varphi(s(x)))G = (s(x), \hat\varphi^a(x) v_a)G$ $= \hat\varphi^a(x)(s(x), v_a)G = \hat\varphi^a(x) W_a(x)$ we see that $\varphi^a(x) = \hat\varphi^a(s(x))$

Now

$$\begin{aligned}(\nabla_{\underline{X}}\varphi)(x) &= (s(x), \widetilde{X}(\hat\varphi)(s(x)))G \\ &= (s(x), d\hat\varphi(\widetilde{X}(s(x))))G \\ &= (s(x), d\hat\varphi^a(\widetilde{X}(s(x)))v_a)G \\ &= d\hat\varphi^a(\widetilde{X}(s(x)))W_a(x)\end{aligned}$$

and also since $\widetilde{X}(s(x)) = d_x s(\underline{X}_x) - \delta_{(s^*\omega)(\underline{X}_x)}(s(x))$

$$\begin{aligned}d\hat\varphi(\widetilde{X}(s(x))) &= d\hat\varphi(d_x s(\underline{X}_x)) - d\hat\varphi(\delta_{(s^*\omega)(\underline{X}_x)}(s(x))) \qquad \textbf{Use (*)} \\ &= d\hat\varphi(d_x s(\underline{X}_x)) + (s^*\omega)(\underline{X}_x)\cdot\hat\varphi\, s(x)) \\ &= d\hat\varphi^c(d_x s(\underline{X}_x)) + (s^*\omega)^b(\underline{X}_x)\,\hat\varphi^a(s(x))\,(e_b \cdot v_a) \\ &= \Big[d\hat\varphi^c(d_x s(\underline{X}_x)) + h^c_{ba}(s^*\omega)^b(\underline{X}_x)\,\hat\varphi^a(s(x))\Big] v_c \\ &= \Big[d_x(\hat\varphi^c \circ s)(\underline{X}_x) + h^c_{ba} A^b(\underline{X}_x)(\hat\varphi^a \circ s)(x)\Big] v_c \\ &= \Big[d_x\varphi^c(\underline{X}_x) + h^c_{ba} A^b(\underline{X}_x)\varphi^a(x)\Big] v_c .\end{aligned}$$

So $(\nabla_{\underline{X}}\varphi)(x) = d\hat\varphi^a(\widetilde{X}(s(x)))W_a(x) = \Big[d_x\varphi^c(\underline{X}_x) + h^c_{ba} A^b(\underline{X}_x)\hat\varphi^a(x)\Big]$

Remark Notice that in the proof of the ast Theorem we have shown that

$$(\nabla_X \varphi)(x) = (s(x), d\hat{\varphi}(\tilde{X}(s(x))))\, G$$

and that

$$d\hat{\varphi}(\tilde{X}(s(x))) = d\hat{\varphi}(d_x s(X_x)) + (s^*\omega)(X_x) \cdot \hat{\varphi}(s(x)).$$

Moreover, $\varphi = \hat{\varphi} \circ s$ so that

$$\begin{aligned}(\nabla_X \varphi)(x) &= \left(s(x),\ d_x\varphi(X_x) + (s^*\omega)(X_x) \cdot \varphi(x)\right) G \\ &= d_x\varphi^a(X_x) + \left((s^*\omega)(X_x) \cdot \varphi(x)\right)^a (s(x), v_a)\, G \\ &= \left[(d_x\varphi^a)(X_x) + \left[(s^*\omega)(X_x) \cdot \varphi(x)\right]^a\right] W_a(x)\end{aligned}$$

It follows that the components of $\nabla_X \varphi$ relative to the local basis of sections $\{W_a\}$ are

$$\boxed{d_x\varphi^a(X_x) + \left[(s^*\omega)(X_x) \cdot \varphi(x)\right]^a.}$$

(1)

Problem 1 Consider the PFB $\mathcal{F}(TM) \xrightarrow{\pi} M$ and let $\omega : T(\mathcal{F}(TM)) \to gl(n)$ be a principal connection ($n = \dim M$). Choose a chart (x^μ) on M and define a local section s of π by $s(p) = (p, \frac{\partial}{\partial x^\mu}|_p)$ Let $\Gamma_\mu = (s^*\omega)(\partial_\mu)$. Use the Theorem on page 264 to prove that if $\bar{s}(p) = (p, \frac{\partial}{\partial \bar{x}^\nu})_p$ is another section defined by a chart $(\bar{x}^\nu)$ then $\bar{\Gamma}_\nu = (\bar{s}^*\omega)(\partial_\nu)$ is related to Γ_μ by

$$\bar{\Gamma}_\nu(x) = g(x)^{-1}\Gamma_\nu(x)\, g(x) + g^{-1}(x)(\partial_\nu g)(x)$$

where $g(x)$ is the matrix $\left(\frac{\partial x^\nu}{\partial \bar{x}^\mu}(x)\right) \in Gl(n)$.

Use this fact to show that

$$\bar{\Gamma}^{\rho}_{\mu\delta} = \Gamma^{k}_{\mu\lambda}\left(\frac{\partial \bar{x}^\rho}{\partial x^k}\right)\left(\frac{\partial x^\lambda}{\partial \bar{x}^\delta}\right) + \left(\frac{\partial^2 x^k}{\partial \bar{x}^\delta \partial \bar{x}^\mu}\right)\frac{\partial \bar{x}^\rho}{\partial x^k}$$

Problem 2 Recall that T^*M is bundle isomorphic to $\mathcal{F}(TM) \times_G (\mathbb{R}^n)^*$ where $G = Gl(n)$ and the action of G on $(\mathbb{R}^n)^*$ is given by $(A \cdot r^i)(r_k) = r^i(A^{-1} \cdot r_k) = r^i(A^{-1\,l}_{\ \ k}\, r_l)$

$= A^{-1\,i}_{\ \ k} = (A^{-1})^i_{\ p}\, r^p(r_k)$ so that

$$\boxed{A \cdot r^i = A^{-1\,i}_{\ \ p}\, r^p}$$

Here $\{r_i\}$ is the standard basis of $\mathbb{R}^n$ and $\{r^j\}$ is the basis of $(\mathbb{R}^n)^*$ dual to $\{r_i\}$. This isomorphism sends

$$\overline{W}^{\mu}(x) = (s(x), r^{\mu})G$$

to dx^{μ} if the section s is defined by $s(p) = (p, \frac{\partial}{\partial x^{\mu}}|_p)$. Since a section α of $\mathcal{F}(TM)\times_G (\mathbb{R}^n)^* \longrightarrow M$ may be written as

$$\alpha(p) = (s(p), \hat{\alpha}(s(p)))G = \hat{\alpha}_{\mu}(s(p))(s(p), r^{\mu})G$$

and $\overline{W}^{\mu}(p) = (s(p), r^{\mu})G$ is identified with dx^{μ}_p we see that α is identified with the 1-form $\hat{\alpha}_{\mu}(s(p))\,dx^{\mu} \equiv \alpha_{\mu}(p)\,dx^{\mu}$.

② Problem 2 Use the Remark above to show that $\nabla_{\partial_{\mu}}\alpha$ may be identified with

$$[\partial_{\mu}\alpha_{\nu} + (\Gamma_{\mu}\cdot\alpha)_{\nu}]\,dx^{\nu}$$

and that

$$(\Gamma_{\mu}\cdot\alpha)(x) = \Gamma_{\mu}(x)\cdot\alpha(x) = \Gamma_{\mu}(x)(s(x), \hat{\alpha}(s(x)))G$$

$$= -\Gamma^{\nu}_{\mu\lambda}(x)\,\alpha_{\nu}(x)\,\overline{W}^{\lambda}(x)$$

$$\simeq -\Gamma^{\nu}_{\mu\lambda}(x)\,\alpha_{\nu}(x)\,dx^{\lambda},$$

So $\nabla_{\partial_{\mu}}\alpha = [\partial_{\mu}\alpha_{\lambda} - \Gamma^{\nu}_{\mu\lambda}\,\alpha_{\nu}]\,dx^{\lambda}$.