0.2.8 Definition. *Let $T^{p,q}(M)=\cup_{x\in M}T^{p,q}(T_xM)$. A **1-form** on M is a function α: $M\to T^{0,1}(M)$ such that $\alpha_x\in T^{0,1}(T_xM)$ and (for any $Y\in\Gamma(TM)$) the function $\alpha(Y)$ given by $\alpha(Y)(x)=\alpha_x(Y_x)$ is in $C^\infty(M)$. A **tensor field** of type (p,q) on M is a function S: $M\to T^{p,q}(M)$ such that $S_x\in T^{p,q}(T_xM)$ and (for any 1-forms $\alpha_1,\ldots,\alpha_p$ and vector fields $Y_1,\ldots,Y_q$ on M) the function $S(\alpha_1,\ldots,\alpha_p,Y_1,\ldots,Y_q)$ given by $S(\alpha_1,\ldots,\alpha_p,Y_1,\ldots,Y_q)(x)=S(\alpha_{1x},\ldots,\alpha_{px},Y_{1x},\ldots,Y_{qx})$ is in $C^\infty(M)$. The space of all tensor fields of type (p,q) on M is denoted by $\mathfrak{T}^{p,q}(M)$.*

0.2.9 Definition. *A **k-form** on M is a tensor field $\omega\in\mathfrak{T}^{0,k}(M)$ such that $\omega_x\in\Lambda^k(T_xM)$. The space of k-forms on M is denoted by $\Lambda^k(M)$. For $\alpha\in\Lambda^i(M)$ and $\beta\in\Lambda^j(M)$, we define $\alpha\wedge\beta\in\Lambda^{i+j}(M)$ by $(\alpha\wedge\beta)_x=\alpha_x\wedge\beta_x$. If φ: $U\to\mathbb{R}^n$ is a chart $\varphi=(x^1,\ldots,x^n)$ $(x^i\in C^\infty(U))$, then $dx^1,\ldots,dx^n$ are defined to be those 1-forms on U with $dx^i(\partial_j)=\delta^i_j$. Any $\omega\in\Lambda^k(M)$ can be written on U as*

$$\omega=\frac{1}{k!}\sum\omega_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

where $\omega_{i_1\cdots i_k}=\omega(\partial_{i_1},\ldots,\partial_{i_k})\in C^\infty(U)$.

0.2.10 Definition. *If $f\in C^\infty(M)$, then $df\in\Lambda^1(M)$ is defined by $df(Y)=Y[f]$ for arbitrary $Y\in\Gamma(TM)$. For $\omega\in\Lambda^k(M)$, we define $d\omega$ to be the $(k+1)$-form that when restricted to U (in the notation of 0.2.9) is given by*

$$d\omega=\frac{1}{k!}\sum d(\omega_{i_1\cdots i_k})\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$=\frac{1}{k!}\sum\partial_i\left[\omega_{i_1\cdots i_k}\right]dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

We can prove that $d\omega$, as defined, is independent of the choice of coordinates. In fact, $d\omega$ can be defined (without reference to coordinates) as that $(k+1)$-form such that for any $X_1,\ldots,X_{k+1}\in\Gamma(TM)$ we have

$$d\omega(X_1,\ldots,X_{k+1})=\sum_{i=1}^{k+1}(-1)^{i+1}X_i\left[\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1})\right]$$
$$+\sum_{1\le i<j\le n}(-1)^{i+j}\omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{k+1})$$

*where the circumflex means that the symbol beneath it is to be omitted. The operation d: $\Lambda^k(M)\to\Lambda^{k+1}(M)$ is called **exterior differentiation**. If $\alpha\in\Lambda^i(M)$ and $\beta\in\Lambda^j(M)$, then (from the coordinate definition) we easily obtain $d(\alpha\wedge\beta)=d\alpha\wedge\beta+(-1)^i\alpha\wedge d\beta$; and $d^2\equiv d\circ d=0$.*

0.2.11 Definition. *If f: $M\to N$ is a map and $\omega\in\Lambda^k(N)$, then the **pull-back** $f^*\omega\in\Lambda^k(M)$ is defined by $(f^*\omega)_x(Y_1,\ldots,Y_k)=\omega_{f(x)}(f_{*x}Y_1,\ldots,f_{*x}Y_k)$ for $Y_1,\ldots,Y_k\in T_xM$. When $k=0$, $f^*\omega\equiv\omega\circ f\in C^\infty(M)$. It can be proved that $df^*\omega=f^*d\omega$, $f^*(\alpha\wedge\beta)=f^*\alpha\wedge f^*(\beta)$, and $(f\circ g)^*\omega=g^*f^*\omega$.*

0.2.12 Definition. *In order to integrate forms, we introduce some topological notions. A subset $W\subset M$ is **closed** if its complement $W^c\equiv\{x\in M|x\notin W\}$ is open. The **closure** $\bar{A}$ of an arbitrary subset $A\subset M$ is the intersection of all closed subsets of M that contain A. Note that $\bar{A}$ is the smallest closed set containing A. An **open covering** of $A\subset M$ is a collection $\mathfrak{U}$ of open subsets whose union contains A. A **subcover** $\mathfrak{U}'$ of $\mathfrak{U}$ is a subcollection (i.e., $\mathfrak{U}'\subset\mathfrak{U}$) such that $\mathfrak{U}'$ is an open covering. A subset K of M is **compact** if every open cover of K has a finite subcover. The Hausdorff property of M ensures that a compact subset is closed. If S is a tensor field on M, then the **support** of S, denoted supp S, is the closure of $\{x\in M|S(x)\ne0\}$.*

0.2.13 Definition (Integration). *A nowhere zero n-form ν on an n-manifold M is called an **orientation** for M. The pair (M,ν) is called an **oriented manifold**. Let α be an n-form on the oriented manifold (M,ν) such that $K\equiv\text{supp}\,\alpha$ is compact. The compactness of K ensures that there is a finite number of charts φ_i: $U_i\to M$, $i=1,\ldots,N$, such that $K\subset U_1\cup\cdots\cup U_N$, and $\varphi_i(U_i)\subset\mathbb{R}^n$ is bounded, and (see Kobayashi* *and Nomizu* [1963]) *there exist functions* $\rho_i \in C^\infty(M)$ *such that* $\operatorname{supp}\rho_i \subset U_i$, $0 \leqslant \rho_i \leqslant 1$, *and* $\Sigma_1^N \rho_i(x) = 1$ *for all* $x \in K$. *If* β *is an n-form defined on a bounded open subset* D *of* $\mathbb{R}^n$ *such that* $\operatorname{supp}\beta \subset D$ *is a closed subset of* $\mathbb{R}^n$, *then we define*

$$\int_D \beta = \int_D b$$

where b is the real-valued function defined by $\beta = b\, dx^1 \wedge \cdots \wedge dx^n$. *By an interchange of the components of* φ_i (*if necessary*), *we can assume that* (*on* U_i) $\varphi_i^*(dx^1 \wedge \cdots \wedge dx^n)$ *is a positive multiple of the orientation* ν *for all* $i = 1, \ldots, N$. *Then, we define*

$$\int_M \alpha = \sum_{i=1}^N \int_{\varphi(U_i)} \varphi_i^{-1*}(\rho_i \alpha).$$

We omit the proof that $\int_M \alpha$ *is independent of the choice of coordinate covering, and so on* (*see Spivak* [1971]). *However, the underlying reason is that the component of an n-form* (*in local coordinates*) *changes under a change of coordinates by a factor equal to the Jacobian of the change. Thus, the change of variable formula for integrals applies.*

0.2.14 Theorem (Stokes' Theorem). *Let M be an oriented n-manifold and suppose that* $\alpha \in \Lambda^{n-1}(M)$ *has compact support. Then we have*

$$\int_M d\alpha = 0.$$

Proof. Let $K = \operatorname{supp}\alpha$ and let $\varphi_i\colon U_i \to \mathbb{R}^n$ and $\rho_i \in C^\infty(M)$ ($i = 1, \ldots, N$) be as in 0.2.13. Then $\alpha = \Sigma \rho_i \alpha$, and it follows that

$$\int_M d\alpha = \sum_i \int_M d(\rho_i \alpha) = \sum_i \int_{\varphi(U_i)} \varphi_i^{-1*} d(\rho_i \alpha) = \sum_i \int_{\varphi(U_i)} d(\varphi_i^{-1*} \rho_i \alpha).$$

Each of the integrals in this sum vanishes by the following argument. Let β be a compactly supported $(n-1)$-form on $\mathbb{R}^n$, say $\beta = \Sigma b_j\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$. Then

$$\int_{\mathbb{R}^n} d\beta = \int_{\mathbb{R}^n} \Sigma (-1)^{j-1} \frac{\partial b_j}{\partial x_j} dx^1 \wedge \cdots \wedge dx^n$$

$$= \Sigma (-1)^{j-1} \int_{\mathbb{R}^n} \frac{\partial b_j}{\partial x_j} dx^1 \cdots dx^n = 0,$$

since b_j vanishes outside of a compact subset of $\mathbb{R}^n$. Now, set $\beta = \varphi_i^{-1*}\rho_i \alpha$ on $\varphi(U_i)$ and $\beta \equiv 0$ on $\varphi(U_i)^c$. Then, $0 = \int_{\mathbb{R}^n} d\beta = \int_{\varphi(U_i)} d(\varphi_i^{-1*}\rho_i \alpha)$. ■

0.2.15 Remark. *There is a notion of an n-manifold M with an* $(n-1)$*-manifold* ∂M *for a boundary. The usual version of Stokes' theorem that says* $\int_M d\alpha = \int_{\partial M} \alpha$ *for compactly supported* $(n-1)$*-forms* α (*and where* ∂M *has an orientation induced from that on M*). *We use this result only once* (*in* 10.4.13), *and therefore we omit the proof* (*it can be found in Spivak* [1971], *Sternberg* [1963], *etc.*).

0.2.16 Theorem. *If* $f\colon N \to M$ *is an orientation-preserving diffeomorphism of n-manifolds, then for any compactly supported* $\alpha \in \Lambda^n(M)$ *we have* $\int_N f^*\alpha = \int_M \alpha$.

Proof. Let $K = \operatorname{supp}\alpha$, and let $\varphi_i\colon U_i \to \mathbb{R}^n$ and $\rho_i \in C^\infty(M)$ be as in 0.2.13. Then we have corresponding charts $\psi_i = \varphi_i \circ f\colon f^{-1}(U_i) \to \mathbb{R}^n$ and $\sigma_i = \rho_i \circ f \in C^\infty(N)$, needed to define $\int_N f^*\alpha$. Then

$$\int_N f^*\alpha = \sum_i \int_{\psi_i(f^{-1}(U_i))} \psi_i^{-1*}(\sigma_i f^*\alpha)$$

$$= \sum_i \int_{\varphi_i(U_i)} (f^{-1} \circ \varphi_i^{-1})^*((\rho_i \circ f) f^*\alpha)$$

$$= \sum_i \int_{\varphi_i(U_i)} \varphi_i^{-1*}((\rho_i \circ f \circ f^{-1}) f^{-1*} f^*\alpha)$$

$$= \sum_i \int_{\varphi_i(U_i)} \varphi_i^{-1*}(\rho_i \alpha) = \int_M \alpha. \quad ■$$

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$(\varphi_i(x), \psi_j(y)) \in \mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$. Then $\{\varphi_i \times \psi_j: U_i \times V_j \to \mathbb{R}^{n+m}\}$ will be an atlas (for $M \times N$), making $M \times N$ an $(n+m)$-manifold.

0.2.26 Definition. *Let $f \in C^\infty(M)$ and suppose that $df=0$ at some $x \in M$. then x is called a* **critical point** *of f. The* **Hessian** *of f at such an x is a symmetric bilinear function $(D^2 f)_x$: $T_x M \times T_x M \to \mathbb{R}$ defined by $(D^2 f)_x(Y_x, Z_x) = Y_x[\tilde{Z}[f]]$, where $\tilde{Z} \in \Gamma(TM)$ is such that $\tilde{Z}_x = Z_x$. Suppose $\tilde{Y} \in \Gamma(TM)$ with $\tilde{Y}_x = Y_x$. Then $Z_x[\tilde{Y}[f]] = [\tilde{Z}, \tilde{Y}]_x[f] + Y_x[\tilde{Z}[f]] = Y_x[\tilde{Z}[f]]$, since $df=0$ at x. Thus, $(D^2 f)_x$ is not only independent of the extension $\tilde{Z}$, but also symmetric.*

0.3 LIE GROUPS AND LIE ALGEBRAS

0.3.1 Definition. *Let G be an n-manifold and a group such that the groups operation $G \times G \to G$ given by $(g_1, g_2) \mapsto g_1 g_2$ and the function $G \to G$ given by $g \mapsto g^{-1}$ are (C^∞) maps. Then G is called a* **Lie group**.

0.3.2 Definition. *Let L_g: $G \to G$ be defined by $L_g(g') = gg'$; L_g is a diffeomorphism. Let e be the identity element of G, and let $A \in T_e G$. Define $\bar{A} \in \Gamma(TG)$ by $\bar{A}_g = L_{g*}(A)$; $\bar{A}$ is called the* **left-invariant vector field** *determined by A.*

0.3.3 Definition. *Let $\mathcal{G} = T_e G$, and (for $A, B \in \mathcal{G}$) define $[A, B] \in \mathcal{G}$ by $[A, B] = [\bar{A}, \bar{B}]_e$ (see 0.2.4). Note that $[A, B] = -[B, A]$ and $[A,[B, C]] + [C,[A, B]] + [B,[C, A]] = 0$ (the Jacobi identity). Then $\mathcal{G}$ (together with the bracket operation $[\ ,\ ]$) is called the* **Lie algebra** *of G.*

0.3.4 Definition. *For $A \in \mathcal{G}$, we can prove that $\bar{A}$ is a complete vector field (see 0.2.7). Let $\{\varphi_t\}$ be the one-parameter group of diffeomorphisms generated by $\bar{A} \in \mathcal{G}$. Let γ: $\mathbb{R} \to G$ be the curve through e defined by $\gamma(t) = \varphi_t(e)$. We prove that $\gamma(s+t) = \gamma(s)\gamma(t)$ (group multiplication). Let $s \in \mathbb{R}$ be fixed and let $\gamma_1(t) = \gamma(s+t)$, while $\gamma_2(t) = \gamma(s)\gamma(t)$. Then $\gamma_1'(t) = \gamma'(s+t) = \bar{A}_{\gamma(s+t)}$ and $\gamma_2'(t) = L_{\gamma(s)}{}^*(\gamma'(t)) = L_{\gamma(s)}{}^*(\bar{A}_{\gamma(t)}) = L_{\gamma(s)}{}^*(L_{\gamma(t)}{}^*A) = \bar{A}_{\gamma(s)\gamma(t)}$. Thus, γ_1 and γ_2 are integral curves of the same vector field $\bar{A}$, and (since $\gamma_1(0) = \gamma(s) = \gamma_2(0)$) it follows that $\gamma_1(t) = \gamma_2(t)$ (i.e., $\gamma(s+t) = \gamma(s)\gamma(t)$). Thus, γ: $\mathbb{R} \to G$ is a homomorphism. Conversely, given a curve and homomorphism σ:*

2/13/2

$\mathbb{R} \to G$, then ψ_t: $G \to G$ (defined by $\psi_t(g) = g\sigma(t)$) is a one-parameter group of diffeomorphisms of G such that

$$\bar{B}_g \equiv \frac{d}{dt}\psi_t(g)\Big|_{t=0}$$

defines the left-invariant vector field $\bar{B}$ determined by $B \equiv \bar{B}_e$. Thus, there is a one-to-one correspondence $A \leftrightarrow \gamma$. We define the **exponential map** *exp: $\mathcal{G} \to G$ by $exp(A) = \gamma(1)$. Note that $\gamma(t) = exp(tA)$, and $\varphi_t(g) = g\gamma(t) = g\,exp(tA)$.*

0.3.5 Example. *Let V be a vector space with $\dim V = m < \infty$, and let $GL(V)$ be the group of invertible linear functions F: $V \to V$. By regarding $GL(V)$ as a group of matrices, it is simple to see that $GL(V)$ (an open subset of $\mathbb{R}^{m^2}$) is a Lie group. Let $I \in GL(V)$ be the identity, and denote $T_I(GL(V))$ by $\mathcal{G}\ell(V)$. Note that $\mathcal{G}\ell(V)$ can be identified with the vector space of* **all** *linear functions A: $V \to V$, the correspondence being*

$$A \leftrightarrow \frac{d}{dt}(I + tA)\Big|_{t=0}.$$

For $A \in \mathcal{G}\ell(V)$, let

$$\mathrm{Exp}(A) = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots.$$

It is not hard to prove that the sum converges, and that $Exp((t+s)A) = Exp\,tA\,Exp\,sA$. Thus, $Exp(A)\,Exp(-A) = I$ and so $Exp(A) \in GL(V)$. Note that $t \mapsto Exp(tA)$ is a curve and a homomorphism with

$$\frac{d}{dt}\mathrm{Exp}(tA)\Big|_{t=0} = A.$$

It follows from the discussion in 0.3.4 that Exp is the exponential map for $GL(V)$. In 0.3.10, we will prove that (for $A, B \in \mathcal{G}\ell(V)$) $[A, B] = AB - BA$.

0.3.6 Definition. *A* **Lie subgroup** *of a Lie group G is a submanifold (of G) that is also a subgroup of G. A Lie subgroup H of G is itself a Lie group. Since the homomorphisms γ: $\mathbb{R} \to H$ are also homomor-*

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phisms into G, we have that exp: $\mathcal{H}\to H$ is just exp: $\mathcal{G}\to G$ restricted to $\mathcal{H}$. The next theorem implies that $[\ ,\ ]$ on $\mathcal{H}$ is just $[\ ,\ ]$ on $\mathcal{G}$ restricted to $\mathcal{H}$.

0.3.7 Theorem. *Let G and G′ be Lie groups, and let $F\colon G\to G'$ be a C^∞ homomorphism. Then $F_{*e}\colon \mathcal{G}\to\mathcal{G}'$ is a linear function such that $F_{*e}([A,B])=[F_{*e}A, F_{*e}B]$ (i.e., F_{e*} is a homomorphism of Lie algebras).*

Proof. Note that $F\circ L_g(g')=F(gg')=F(g)F(g')=(L_{F(g)}\circ F)(g')$. Thus, $F_{*g}(\bar{A}_g)=F_{*g}(L_{g*}A)=L_{F(g)*e'}(F_{*e}A)=(\overline{F_{*e}A})_{F(g)}$, and so $F_*(\bar{A})=(\overline{F_{*e}A})$. Using 0.2.5, we obtain $F_{*e}([A,B])=[F_*(\bar{A}), F_*(\bar{B})]_{e'}=[(\overline{F_{*e}A}),(\overline{F_{*e}B})]_{e'}=[F_{*e}A, F_{*e}B]$. ■

0.3.8 Definition. *For $g\in G$, let $\mathrm{Ad}_g\colon G\to G$ be the C^∞ **adjoint** isomorphism given by $\mathrm{Ad}_g(g')=gg'g^{-1}$. We let $\mathfrak{Ad}_g\colon\mathcal{G}\to\mathcal{G}$ be the induced isomorphism of $\mathcal{G}$ provided by 0.3.7 (i.e., $\mathfrak{Ad}_g=\mathfrak{Ad}_{g*e}$). Let $\mathfrak{Ad}\colon G\to GL(\mathcal{G})$ be the homomorphism $g\mapsto\mathfrak{Ad}_g$. Then 0.3.7 gives us an induced homomorphism $\mathfrak{ad}\colon\mathcal{G}\to\mathcal{G}\ell(\mathcal{G})$ (i.e., $\mathfrak{ad}=\mathfrak{Ad}_{*e}$).*

0.3.9 Theorem. *For $A, B\in\mathcal{G}$, we have*

$$\mathfrak{ad}(A)(B)=\frac{\partial^2}{\partial s\,\partial t}(\exp(tA)\exp(sB)\exp(-tA))\bigg|_{s,t=0}=[A,B].$$

Proof. Let $\{\varphi_t\}$ be the one-parameter group generated by $\bar{A}$. By the end of 0.3.4, we have $\varphi_t(g)=g\exp tA$. Using $L_{\bar{A}}\bar{B}=[\bar{A},\bar{B}]$ (see 0.2.7), we have (at $s=t=0$)

$$[A,B]=[\bar{A},\bar{B}]_e=\frac{d}{dt}\varphi_{-t*}\left(\bar{B}_{\varphi_t(e)}\right)$$

$$=\frac{d}{dt}\varphi_{-t*}\left(\frac{d}{ds}\varphi_t(e)\exp(sB)\right)=\frac{d}{dt}\frac{d}{ds}\varphi_{-t}(\varphi_t(e)\exp(sB))$$

$$=\frac{\partial^2}{\partial t\,\partial s}(\exp(tA)\exp(sB)\exp(-tA))$$

$$=\frac{d}{dt}\mathfrak{Ad}(\exp(tA))(B)=\mathfrak{Ad}_{*e}(A)(B)$$

$$=\mathfrak{ad}(A)(B).\ ■$$

0.3.10 Corollary. *If G is any Lie subgroup of $GL(V)$, then the bracket operation on $\mathcal{G}\subset\mathcal{G}\ell(V)$ is given by $[A,B]=AB-BA$.*

Proof. By 0.3.6, it suffices to consider the case in which $G=GL(V)$. Using 0.3.9 with exp = Exp (see 0.3.5), we have

$$[A,B]=\frac{\partial^2}{\partial s\,\partial t}(\mathrm{Exp}(tA)\,\mathrm{Exp}(sB)\,\mathrm{Exp}(-tA))\bigg|_{s,t=0}=AB-BA.\ ■$$

0.3.11 Definition. *Let $e_1,\dots,e_n$ be a basis for the Lie algebra $\mathcal{G}$ of G. The **structure constants** $c_{ij}^k\in\mathbb{R}$ are defined by $[e_i,e_j]=\Sigma c_{ij}^k e_k$. Note that $[e_j,e_i]=-[e_i,e_j]$ implies $c_{ji}^k=-c_{ij}^k$. The Jacobi identity yields $0=[e_i,[e_j,e_k]]+[e_k,[e_i,e_j]]+[e_j,[e_k,e_i]]=[e_i,\Sigma c_{jk}^m e_m]+[e_k,\Sigma c_{ij}^m e_m]+[e_j,\Sigma c_{ki}^m e_m]=\Sigma_{h,m}(c_{im}^h c_{jk}^m+c_{km}^h c_{ij}^m+c_{jm}^h c_{ki}^m)e_h$. Thus, $\Sigma_m c_{im}^h c_{jk}^m+c_{km}^h c_{ij}^m+c_{jm}^h c_{ki}^m=0$ for all h, i, j, k.*

0.3.12 *SU(n)*, the Special Unitary Group. *The computation of the Lie algebra of a Lie group of matrices is illustrated here for the group SU(n), which is frequently used in elementary particle physics. Let $\mathcal{G}\ell(n,\mathbb{C})$ be the space of all $n\times n$ matrices with complex entries. For $A\in\mathcal{G}\ell(n,\mathbb{C})$, let A^* denote the conjugate of the transpose of A. Recall that the unitary group is $U(n)=\{A\in\mathcal{G}\ell(n,\mathbb{C})\,|\,AA^*=I\}$ and $SU(n)=\{A\in U(n)\,|\,\det A=1\}$. If $t\mapsto A(t)$ is a curve in $U(n)$ with $A(0)=I$, then (at $t=0$) we have*

$$0=\frac{d}{dt}(I)=\frac{d}{dt}(A(t)A(t)^*)$$

$$=A'(0)A(0)^*+A(0)A'(0)^*=A'(0)+A'(0)^*.$$

Thus, for $\mathcal{S}=\{B\in\mathcal{G}\ell(n,\mathbb{C})\,|\,B+B^*=0\}$, we have $\mathcal{S}\supset\mathcal{U}(n)\equiv$ the Lie algebra of $U(n)$. Conversely, if $B\in\mathcal{S}$, then $(\mathrm{Exp}\,B)(\mathrm{Exp}\,B)^*=(\mathrm{Exp}\,B)(\mathrm{Exp}\,B^*)=\mathrm{Exp}(B)\,\mathrm{Exp}(-B)=I$, and so $\mathrm{Exp}\,B\in U(n)$. At $t=0$,

$$B=\frac{d}{dt}\mathrm{Exp}\,tB\in\mathcal{U}(n),$$

whence $\mathcal{U}(n)=\mathcal{S}$. The Lie algebra $\mathcal{SU}(n)$ of $SU(n)$ is the subalgebra of $\mathcal{U}(n)$ consisting of matrices with trace 0 (i.e., $\mathcal{SU}(n)=\{B\in\mathcal{U}(n)|$

$\operatorname{tr} B=0\}$). *This follows immediately from the formula* $\det(\operatorname{Exp} B)=e^{\operatorname{tr} B}$, *which is valid for any* $n\times n$ *matrix. We can prove this formula as follows. Let* $f(t)=\det(\operatorname{Exp} tB)$. *At* $h=0$, *we have*

$$\begin{aligned} f'(t) &= \frac{d}{dh}f(t+h)=\frac{d}{dh}\left[\det(\operatorname{Exp} tB)\det(\operatorname{Exp} hB)\right] \\ &= \det(\operatorname{Exp} tB)\frac{d}{dh}\det(I+hB) \\ &= \det(\operatorname{Exp} tB)\operatorname{tr} B=(\operatorname{tr} B)f(t). \end{aligned}$$

Thus, $f(t)=f(0)e^{(\operatorname{tr} B)t}=e^{(\operatorname{tr} B)t}$, *and setting* $t=1$ *yields the result.*

CHAPTER 1

Principal Fiber Bundles and Connections

In the introductions to this and the following chapters, the topics and results to be covered will be outlined, and some motivation will be supplied to whet the grindstone, but no miracles are promised. You may rest assured that you need not comprehend or agree with the introductions in order to understand and accept the proper parts of the chapters.

In this chapter, principal fiber bundles (PFBs) will be defined and some nontrivial examples will be given (i.e., the double covering of the circle and the frame bundle of a manifold). Three ways of defining connections (ie., gauge potentials) will be proved to be equivalent. The connection of a PFB with group $U(1)$ over space-time will be physically identified as the four-dimensional vector potential of electromagnetism.

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2/1/2

SOLUTION The energy stored in the capacitor initially is (see Eq. 26–5)

$$U_1 = \tfrac{1}{2}CV^2,$$

where $C = K\varepsilon_0 A/d$. After the dielectric is removed, the capacitance drops by a factor K but the voltage is increased by a factor K. Hence

$$U_2 = \frac{1}{2}\left(\frac{C}{K}\right)(KV)^2 = K(\tfrac{1}{2}CV^2) = KU_1.$$

So the energy stored in the capacitor has increased by a factor K. Work is therefore required to remove the dielectric and the amount needed (neglecting friction) is

$$W = U_2 - U_1 = \tfrac{1}{2}CV^2(K-1).$$

That work is needed to remove the dielectric can be seen intuitively from the fact that there will be a force of attraction between the induced charge on the dielectric and the charges on the plates (Fig. 26–7c) as it is pulled out. Hence an external force must be exerted to overcome this, and work must be done.

*26–6 Gauss's Law in Dielectrics

We now discuss the use of Gauss's law in a situation when a dielectric is present. Consider a parallel-plate capacitor containing a dielectric that fills the space between the plates as shown in Fig. 26–9. We assume the plates are large (of area A) compared to the separation l so that $\mathbf{E}$ is uniform and perpendicular to the plates. For our gaussian surface, we choose the long rectangular box indicated by the dashed lines in Fig. 26–9, which just barely reaches into the dielectric. The surface encloses both the free charge Q on the conductor, and the induced (bound) charge Q_{ind} on the dielectric, so

$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{Q - Q_{\text{ind}}}{\varepsilon_0}. \qquad (26\text{–}12)$$

Since, from Eq. 26–11b, $Q_{\text{ind}} = Q(1 - 1/K)$, we have $Q - Q_{\text{ind}} = Q(1 - 1 + 1/K) = Q/K$, so

$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{Q}{K\varepsilon_0} = \frac{Q}{\varepsilon}. \qquad (26\text{–}13)$$

This relation (Eq. 26–13), although obtained for a special case, is Gauss's law valid in general when dielectrics are present. Note that Q in Eq. 26–13 is the *free* charge only. The induced bound charge is not included since it is accounted for by the factor K (or ε).

For the surface shown in Fig. 26–9, we have $\mathbf{E} = 0$ within the conductor, so there is no flux through that part of the surface inside the conductor. Also there is essentially no flux through the short sides of the box, since $\mathbf{E}$ is nearly parallel to the sides, and besides the sides are very short so the contribution would be very small anyway. So the only flux is that through the surface within the dielectric. Thus, Gauss's law gives

$$E_d A = \frac{Q}{\varepsilon_0} - \frac{Q_{\text{ind}}}{\varepsilon_0} = \frac{Q}{\varepsilon},$$

where we have used both Eqs. 26–12 and 26–13. E_d represents the field inside the dielectric and is given by either

$$E_d = \frac{Q - Q_{\text{ind}}}{\varepsilon_0 A}$$

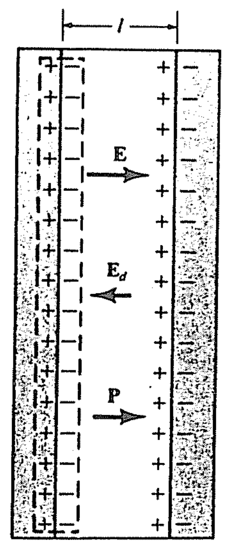

FIGURE 26–9 Gauss's law in a dielectric.

or

$$E_d = \frac{Q}{K\varepsilon_0 A} = \frac{Q}{\varepsilon A} = \frac{\sigma}{\varepsilon}.$$

In this result, as well as in the general form of Gauss's law, Eq. 26–13, we see that our earlier relations for E are altered only by replacing ε_0 by $K\varepsilon_0 = \varepsilon$.

*26–7 The Polarization and Electric Displacement Vectors (P and D)

The rectangular dielectric between the plates of the charged parallel plates in Fig. 26–9 has a dipole moment whose magnitude is

$$Q_{\text{ind}} l$$

where l is the thickness of the dielectric and Q_{ind} is the charge induced on the surface of the dielectric. For any dielectric, we can define a new quantity, the *polarization vector*, **P**, which is the *dipole moment per unit volume*. For a rectangular dielectric of thickness l and whose faces have area A,

$$P = \frac{Q_{\text{ind}} l}{Al} = \frac{Q_{\text{ind}}}{A} = \sigma_{\text{ind}}.$$

Thus the magnitude of the polarization vector, in this case, is equal to the surface charge density induced on the dielectric.†

The polarization vector points from the negative charge sheet on one side of the dielectric to the positive charge sheet on the other (just as a dipole moment does), as shown in Fig. 26–9. For the surface shown, we can write

$$\oint \mathbf{P} \cdot d\mathbf{A} = PA = Q_{\text{ind}} \tag{26–14}$$

since **P** is zero in the conductor and is parallel to the short sides of the rectangular box chosen as our gaussian surface. Equation 26–14 is valid in general and we can combine it with Eq. 26–12 to obtain

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0} - \frac{1}{\varepsilon_0}\oint \mathbf{P} \cdot d\mathbf{A}$$

or

$$\oint (\varepsilon_0 \mathbf{E} + \mathbf{P}) \cdot d\mathbf{A} = Q. \tag{26–15}$$

This is another way to write Gauss's law when dielectrics are present (see also Eqs. 26–12 and 26–13), and it too is a general result. It can be written in terms of a new vector called the *electric displacement*, **D**, defined as

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}. \tag{26–16}$$

Then Gauss's law becomes

$$\oint \mathbf{D} \cdot d\mathbf{A} = Q. \tag{26–17}$$

In the dielectric between the plates of the parallel-plate capacitor of Fig. 26–9, this relation gives

$$D = \frac{Q}{A} \quad \text{[parallel-plate capacitor]} \tag{26–18}$$

where Q is the free charge.

† For more complicated cases, σ is equal to the component of **P** perpendicular to the surface.

There is a simple interpretation of the three vectors $\mathbf{E}$, $\mathbf{D}$, and $\mathbf{P}$. The electric field $\mathbf{E}$ is due to *all* charges, whether free or bound, as implied by Eq. 26–12. The polarization vector $\mathbf{P}$, as can be seen from Eq. 26–14, is connected only with the induced bound charge. Finally, the electric displacement vector $\mathbf{D}$ is associated with the free charge only, as implied by Eqs. 26–17 and 26–18.

The vector $\mathbf{E}$ is, still, the basic electric field vector. The vectors $\mathbf{P}$ and $\mathbf{D}$ are useful auxiliaries for more advanced work, although we will not use them much here.

SUMMARY

A *capacitor* is a device used to store charge and consists of two separated conductors. The two conductors generally carry equal and opposite charges, Q, and the ratio of this charge to the potential difference V between the conductors is called the *capacitance*, C; so $Q = CV$. The capacitance of a parallel-plate capacitor is proportional to the area of each plate and inversely proportional to their separation. The space between the conductors contains a nonconducting material such as air, paper, or plastic. The latter materials are referred to as *dielectrics*, and the capacitance is proportional to a property of dielectrics called the dielectric constant, K (nearly equal to 1 for air).

If two or more capacitors are connected in *parallel*, the equivalent capacitance C of the combination i: sum of the individual capacitances. If several capac are connected in *series*, the reciprocal of the equiv capacitance C is equal to the sum of the reciprocals c individual capacitances.

A charged capacitor stores an amount of electri ergy given by $\frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}Q^2/C$. This energy ca thought of as stored in the electric field between the p In any electric field $\mathbf{E}$ in free space the *energy de* (energy per unit volume) is $\frac{1}{2}\varepsilon_0 E^2$. If a dielectric is pre the energy density is $\frac{1}{2}K\varepsilon_0 E^2 = \frac{1}{2}\varepsilon E^2$, where $\varepsilon = K\varepsilon_0$ i permittivity of the dielectric material.

QUESTIONS

1. Suppose two nearby conductors carry the same negative charge. Can there be a potential difference between them? If so, can the definition of capacitance, $C = Q/V$, be used here?
2. Suppose the separation of plates d in a parallel-plate capacitor is not very small compared to the dimensions of the plates. Would you expect Eq. 26–2 to give an overestimate or underestimate of the true capacitance? Explain.
3. Suppose one of the plates of a parallel-plate capacitor was slid so that the area of overlap was reduced by half, but they are still parallel. How would this affect the capacitance?
4. Suppose one plate of a parallel-plate capacitor is tilted at one end away from the other plate so the separation at that end is $2d$. How would this affect the capacitance?
5. Explain how the relation for the capacitance of a cylindrical capacitor, Example 26–2, makes sense intuitively. Use arguments such as those just after Eq. 26–2.
6. Describe a simple method of measuring ε_0 using a capacitor.
7. When a battery is connected to a capacitor, why do the two plates acquire charges of the same magnitude? Will this be true if the two conductors are different si shapes?
8. A large copper sheet of thickness l is placed betwe parallel plates of a capacitor, but does not touch the p How will this affect the capacitance?
9. Suppose three identical capacitors are connected battery. Will they store more energy if connected in or in parallel?
10. The parallel plates of an isolated capacitor carry op charges, Q. If the separation of the plates is increa a force required? Is the potential difference changed? happens to the work done in the pulling process?
11. How does the energy in a capacitor change if (*a*) th tential difference is doubled, (*b*) the charge on each is doubled, and (*c*) the separation of the plates is do as the capacitor remains connected to a battery?
12. For dielectrics consisting of polar molecules, would you expect the dielectric constant to chang temperature?
13. An isolated charged capacitor has horizontal plate thin dielectric is inserted a short way between the p how will it move when it is then released?

other direction; the net result is a net dipole moment opposing the external field.) Diamagnetism is present in all materials, but is weaker even than paramagnetism and so is overwhelmed by paramagnetic and ferromagnetic effects in materials that display these other forms of magnetism.

*30–9 Magnetization Vector and the Extension of Ampère's Law

The *magnetization vector* $\mathbf{M}$, *defined as the magnetic dipole moment per unit volume* in Section 30–8, is a useful concept for any magnetic material (ferro-, para-, or diamagnetic). It is useful to write the magnetic field in terms of $\mathbf{M}$ when magnetic materials are present, just as we wrote the electric field in terms of the polarization vector $\mathbf{P}$ (Section 26–7).

Let us consider a torus, as in Section 30–7, which is filled with some material (not necessarily ferromagnetic). The total field $\mathbf{B}$ is given by Eq. 30–8:

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_M$$

where $\mathbf{B}_0$ is the field due to the current in the coils and $\mathbf{B}_M$ is the field due to the magnetic material. (B_M is usually small compared to B_0 for para- and diamagnetic materials, but much larger for ferromagnetic materials.) We have seen that B_0 is given by

$$B_0 = \mu_0 n I = \mu_0 \frac{N}{l} I$$

where now N is the total number of coils in length l. The field $\mathbf{B}_M$ due to the material can be imagined as arising from currents within the atoms of the material. The net effect of all these atomic currents can be thought of as a current I_M (the "magnetization current") around the outer surface of the material, Fig. 30–24, in analogy to the induced electric charge on the surface of a dielectric, Fig. 26–7. (In fact one would not be able to measure a current I_M at the surface; the concept of a magnetization current I_M is useful, nonetheless, and to distinguish it from a real conduction current, I, we call the latter a "real current.") In analogy to our relation above for B_0 we write

$$B_M = \mu_0 \frac{N_M I_M}{l},$$

where N_M/l is the effective number of loops per unit length (or, $N_M I_M/l$ is the effective magnetization current per unit length). But the magnetic dipole moment is $N_M I_M A$ (Eq. 29–10), where A is the cross-sectional area of the material, Fig. 30–24. Hence the magnetic dipole moment per unit volume is

$$M = \frac{N_M I_M A}{V} = \frac{N_M I_M A}{Al} = \frac{N_M I_M}{l},$$

where $V = Al$ is the volume around which the total effective current $N_M I_M$ flows. We combine these last two relations and write $\mathbf{B}_M$ in terms of $\mathbf{M}$:

$$\mathbf{B}_M = \mu_0 \mathbf{M}.$$

Then the total field $\mathbf{B}$ is

$$\mathbf{B} = \mathbf{B}_0 + \mu_0 \mathbf{M}. \qquad (30\text{–}12)$$

Ampère's law can be extended to include magnetic materials by including the magnetization current, I_M, on the right side:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 (I + I_M), \qquad (30\text{–}13)$$

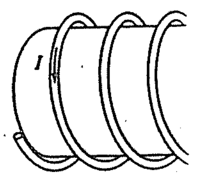

FIGURE 30–24 ... current, I_M, in a ... material in ... (small sectio... current I.

where I is the net real current and I_M is the equivalent magnetization current enclosed by the path of integration. If we now use Eq. 30–12, Ampère's law becomes

$$\oint (\mathbf{B}_0 + \mu_0\mathbf{M}) \cdot d\mathbf{l} = \mu_0(I + I_M).$$

In the absence of any material, $\oint \mathbf{B}_0 \cdot d\mathbf{l} = \mu_0 I$ (remember $\mathbf{B}_0$ refers to $\mathbf{B}$ when no material is present), so from the last equation we obtain

$$\oint \mathbf{M} \cdot d\mathbf{l} = I_M, \tag{30–14}$$

a useful relation between $\mathbf{M}$ and the magnetization current. (This could serve as the definition of I_M.) Again using $\oint \mathbf{B}_0 \cdot d\mathbf{l} = \mu_0 I$, and combining it with Eq. 30–12, we obtain (after dividing through by μ_0):

$$\oint \left(\frac{\mathbf{B} - \mu_0\mathbf{M}}{\mu_0}\right) \cdot d\mathbf{l} = I. \tag{30–15}$$

This result, derived for a special case, is valid in general. It is usually written in terms of a new vector $\mathbf{H}$ defined as

$$\mathbf{H} = \frac{\mathbf{B} - \mu_0\mathbf{M}}{\mu_0}, \tag{30–16a}$$

which can be rewritten

$$\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M}. \tag{30–16b}$$

From Eq. 30–12 we can also write

$$\mathbf{B}_0 = \mu_0\mathbf{H}. \tag{30–17}$$

In terms of $\mathbf{H}$, Eq. 30–15 becomes

$$\oint \mathbf{H} \cdot d\mathbf{l} = I. \tag{30–18}$$

The vector $\mathbf{H}$ is called the **magnetic field strength** and is to be distinguished from $\mathbf{B}$ which is generally referred to as the **magnetic induction**. Equation 30–18 tells us that the line integral of $\mathbf{H}$ around any closed path is equal to the total real current (only) enclosed, even when magnetic materials are present. Thus $\mathbf{H}$ is much like the vector $\mathbf{D}$ in electrostatics which is due only to free charges. The vectors $\mathbf{M}$, $\mathbf{H}$, and $\mathbf{B}$ are the counterparts, respectively, of $\mathbf{P}$, $\mathbf{D}$, and $\mathbf{E}$ for the electric case in dielectrics.

The vector $\mathbf{H}$ is often associated only with free currents, and $\mathbf{M}$ only with "magnetization currents," and $\mathbf{B}$ with all currents. This "association" really refers to the line integrals, Eqs. 30–13, 30–14, and 30–18. It does not mean, for example, that $\mathbf{H}$ is produced only by real currents. For example, at any point in free space (a vacuum, no magnetic materials), we have $B = \mu_0 H$. This is true just outside the pole piece of a permanent magnet, so $H \neq 0$ (since $B \neq 0$ there) even though no real currents are present. From Eq. 30–18, it is easy to see that $\mathbf{H}$ must oppose $\mathbf{B}$ inside the magnet.

This analysis in terms of $\mathbf{H}$, $\mathbf{M}$, and $\mathbf{B}$ is necessary for more advanced treatments; we have included it here simply to give you a little familiarity with these ideas. It is not expected, however, that you will acquire a working knowledge of $\mathbf{M}$ and $\mathbf{H}$ from this brief treatment, and we will not need these ideas for most of what follows in this book. We do mention here, for completeness, the **magnetic susceptibility**, χ_m, defined as:

That is, χ_m is the ratio of magnetization to magnetic field strength. For paramagnetic and diamagnetic materials, χ_m is constant as long as B is not too great; but it is not constant for ferromagnetic materials. From Eqs. 30–16, 30–17, 30–9, and 30–10,

$$\chi_m = K_m - 1$$

and

$$\mu = (1 + \chi_m)\mu_0.$$

In a vacuum, $K_m = 1$, $\chi_m = 0$, and $\mu = \mu_0$.

SUMMARY

Ampère's law states that the line integral of the magnetic field **B** around any closed loop is equal to μ_0 times the total net current I enclosed by the loop:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I.$$

The magnetic field B at a distance r from a long straight wire is directly proportional to the current I in the wire and inversely proportional to r. The magnetic field lines are circles centered at the wire. The magnetic field inside a long tightly wound solenoid is $B = \mu_0 nI$ where n is the number of coils per unit length and I is the current in each coil.

The force that one long current-carrying wire exerts on a second parallel current-carrying wire a distance l away serves as the definition of the ampere unit, and ultimately of the coulomb as well.

The *Biot-Savart law* is useful for determining the magnetic field due to a known arrangement of currents. It states that

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{d\mathbf{l} \times \hat{\mathbf{r}}}{r^2},$$

where $d\mathbf{B}$ is the contribution to the total field at some point P due to a current I along an infinitesimal length $d\mathbf{l}$, and $\hat{\mathbf{r}}$ is the displacement vector from $d\mathbf{l}$ to P. The total field **B** will be the integral over all $d\mathbf{B}$.

Iron and a few other materials can be made into strong permanent magnets. They are said to be *ferromagnetic*. Ferromagnetic materials are made up of tiny *domains*—each a tiny magnet—which are preferentially aligned in a permanent magnet, but randomly aligned in a nonmagnetized sample. When a ferromagnetic material is placed in the magnetic field B_0 due to a current, say inside a solenoid or torus, the material becomes magnetized. When the current is turned off, however, the material remains magnetized, and when the current is increased in the opposite direction (and then again reversed), the total field B does not follow B_0 due to the current; instead the plot of B versus B_0 is a *hysteresis loop*, and the fact that the curves do not retrace themselves is called *hysteresis*.

QUESTIONS

1. The magnetic field due to current in wires in your home can affect a compass. Discuss the problem in terms of currents, depending on whether they are ac or dc, and their distance away.
2. Compare and contrast the magnetic field due to a long straight current and the electric field due to a long straight line of electric charge at rest (Section 23–7).
3. Explain why a field such as that shown in Fig. 30–6b is consistent with Ampère's law. Could the lines curve upward instead of downward?
4. Compare Ampère's law to Gauss's law.
5. (*a*) Write Ampère's law for a path that surrounds both conductors in Fig. 30–4. (*b*) Repeat, assuming the lower current I_2, is in the opposite direction ($I_2 = -I_1$).
6. Can the integral in Ampère's law be carried out over a surface?
7. Suppose the cylindrical conductor of Fig. 30–5a has a concentric cylindrical hollow cavity inside it (so it looks like a pipe). What can you say about **B** in the cavity?
8. What would be the effect on B inside a long solenoid if (*a*) the diameter of the loops was doubled, or (*b*) the spacing between loops was doubled, or (*c*) the solenoid's length was doubled along with a doubling in the total number of loops.
9. A type of magnetic switch similar to a solenoid is a *relay*. A relay is an electromagnet (the iron rod inside the coil doesn't move) which, when activated, attracts a piece of soft iron on a pivot. Design a relay (*a*) to make a doorbell

3/18/2

Definition 2.9 *Assume M is a manifold and that S is a subset of M. If (U, x) is an admissible chart of M which has the submanifold property with respect to S then the chart $(U \cap S, x_S)$ of S defined in the paragraph above is called the chart on S induced by (U, x). We also will refer to x_S as being a chart induced from the manifold structure on M trusting the reader to infer that if we use the notation x_S for a chart on S then it is indeed induced by a chart on M which is denoted x.*

Theorem 2.3 *If M is a manifold and S is a subset of M such that each point of S is in the domain of some admissible chart of M which has the submanifold property relative to S then there is an atlas on S each element of which is an induced chart (U, x_S). There is a unique maximal atlas on S which contains every atlas on S whose members are charts induced by admissible charts of M. It follows that S along with this unique maximal atlas is a manifold.*

Proof. We leave the details of this proof to the reader since they are easy modifications of the proofs of the four Observations given above.

Definition 2.10 *Assume that M is a manifold and that S is a subset of M such that each point of S is in the domain of some admissible chart of M having the submanifold property relative to S. If S is given the differentiable structure which contains all the charts of S induced by admissible charts of M then the manifold S is called a submanifold of M.*

We conclude this section by proving a theorem which provides us with a method of constructing submanifolds S of $\mathbf{R}^n$ without actually having to produce an atlas of charts of S induced by admissible charts of M.

Before getting into the details we need to consider a lemma from matrix theory.

Lemma. Assume A is an $m \times n$ matrix having rank m. Then there exists a nonsingular $m \times m$ matrix E and a nonsingular $n \times n$ matrix

P such that $EAP = [0|I_m]$ where I_m is the $m \times m$ identity and 0 is the

$$m \times (n - m)$$

zero matrix.

Proof. Recall that there is a sequence of row operations which transforms A into its row echelon form. Since A has rank m its row echelon form has m nonzero rows such that for each $1 \leq i \leq m$, row i takes the form $(0, ..., 0, 1, *, *, ..., *)$ where the asterisk denotes unspecified numbers and where the leading nonzero entry of the $(i-1)-th$ row occurs prior to the first nonzero entry of the $i-th$ row. Moreover if a matrix M is transformed to a matrix N by a row transformation then there is an elementary matrix F such that $FM = N$. Consequently there is a matrix E which is a product of elementary matrices such that $EA = R$ where R is the row echelon form of A.In a specific case R thus may assume a form such as

$$R = \begin{pmatrix} 0 & 0 & 1 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{pmatrix}.$$

Since every elementary matrix is nonsingular, E is nonsingular. At this point one may use column transformations to transform each nonzero entry (other than the leading nonzero entry) of the $i-th$ row of R to zero.Thus in the specific example of a row-echelon matrix R given above the matrix R may be transformed to

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

It follows that we have nonsingular matrices E and F such that EAF not only is in row- echelon form but additionally each column contains at most one nonzero entry as in the second example matrix given above. Recall that each row is in fact nonzero since the rank of

EAF is the same as that of A which is m. By interchanging appropriate columns of the matrix EAF one can transform EAF to a matrix of the form $[0|I_m]$ where 0 denotes a $m \times (n-m)$ zero matrix and I_m is the $m \times m$ identity.Recall that if one wishes to interchange the $i-th$ and $j-th$ columns of an $m \times n$ matrix J then one need only multiply J on the left by the matrix P_{ij} obtained from the identity I_n by interchanging the $i-th$ and $j-th$ columns of I_n. Clearly the matrix P_{ij} is nonsingular and since the matrix EAF can be transformed to $[0|I_m]$ via a finite number of column interchanges it is clear that there is a matrix Q which is the product of a finite number of matrices of the form P_{ij} such that $EAFQ = [0|I_m]$.Clearly $P = FQ$ is an $n \times n$ nonsingular matrix such that $EAP = [0|I_m]$ and the lemma follows.

Theorem 2.4 *Assume that $U \subseteq \mathbf{R}^n$ is an open set and that $f : U \to \mathbf{R}^m$ is a class C^r function. If $n > m$ and the Jacobean matrix $J_f(p)$ has rank m for each point p in the level set*

$$S = f^{-1}(0) = \{u : f(u) = 0\}$$

then S is a submanifold of $\mathbf{R}^n$. Moreover the dimension of S is $n-m$.

Proof. By Theorem 2.3 we need only show that each point of S is in the domain of an admissible chart (U, x) of $\mathbf{R}^n$ having the submanifold property relative to S. If we do this each point of S will be in the domain of a chart of S which is induced by an admissible chart of M as is required by Theorem 2.3 . So fix a point p of S and let E and P be nonsingular matrices (whose existence is guaranteed by the Lemma) such that $EJ_f(p)P = [0|I_m]$. Let

$$L_E : \mathbf{R}^m \to \mathbf{R}^m \quad \textit{and} \quad L_P : \mathbf{R}^n \to \mathbf{R}^n$$

be linear maps defined by

$$L_E(v) = vE^t \quad \textit{and} \quad L_P(u) = uP^t$$

for all $v \varepsilon \mathbf{R}^m$ and $u \varepsilon \mathbf{R}^n$.Consider the function $L_E \circ f \circ L_P$. If $q = L_P^{-1}(p)$ then

$$J_{L_E \circ f \circ L_P}(q) = J_{L_E}(f(p))J_f(p)J_{L_P}(q) = EJ_f(p)P = [0|I_m].$$

If $q = (a, b)\varepsilon\mathbf{R}^{n-m} \times \mathbf{R}^m$ then it follows from the remark at the end of Chapter 1 that

$$D_2(L_E \circ f \circ L_P)(q)(k) = (0, k)J_{L_E \circ f \circ L_P}(q)^t$$

and

$$(0, k)J_{L_E \circ f \circ L_P}(q)^t = (0, k)(EJ_f(p)P)^t = (0, k)[0|I_m]^t = k.$$

It follows that $D_2(L_E \circ f \circ L_P)(q) = id_{\mathbf{R}^m}$ and since $(L_E \circ f \circ L_P)(q) = 0$ the hypothesis of the implicit function theorem holds for the function $L_E \circ f \circ L_P$ from the open set $L_P(U) \subseteq \mathbf{R}^{n-m} \times \mathbf{R}^m$ into $\mathbf{R}^m$.It follows that there exists open sets $\tilde{U}$ about $a\varepsilon\mathbf{R}^{n-m}$, $\tilde{V}$ about $b\varepsilon\mathbf{R}^m$ and a mapping $g : \tilde{U} \to \tilde{V}$ such that $\tilde{U} \times \tilde{V} \subseteq L_P^{-1}(U)$, g is of class C^r and

$$(L_E \circ f \circ L_P)(u, v) = 0 \Longleftrightarrow v = g(u).$$

Define $x : \tilde{U} \times \mathbf{R}^m \to \tilde{U} \times \mathbf{R}^m$ by $x(u, v) = (u, v - g(u))$.Observe that x has an inverse with domain $\tilde{U} \times \mathbf{R}^m$ and for $(u, z)\varepsilon\tilde{U} \times \mathbf{R}^m$ $x^{-1}(u, z) = (u, z + g(u))$.Since x and x^{-1} are both continuous and it follows that $x(\tilde{U} \times \tilde{V})$ is open in $\tilde{U} \times \mathbf{R}^m \subseteq \mathbf{R}^n$.It is easy to see that both x and x^{-1} are of class C^r and thus $(\tilde{U} \times \mathbf{R}^m, x)$ is an admissible chart of $\mathbf{R}^n$. If we define $\tilde{S}$ to be $L_P^{-1}(S)$,then $x((\tilde{U} \times \tilde{V}) \cap \tilde{S})) =$
$\{x(u, v) : (u, v)\varepsilon(\tilde{U} \times \tilde{V}) \cap \tilde{S}\} =$
$\{(u, v - g(u)) : (u, v)\varepsilon(\tilde{U} \times \tilde{V}) \cap \tilde{S}\} =$
$\{(u, v - g(u)) : (u, v)\varepsilon(\tilde{U} \times \tilde{V}), (L_E \circ f \circ L_P)(u, v) = 0\} =$
$\{(x, 0) : (u, v)\varepsilon(\tilde{U} \times \tilde{V}\} \cap x(\tilde{U} \times \tilde{V}) =$
$x((\tilde{U} \times \tilde{V})) \cap (\mathbf{R}^{n-m} \times \{0\})$.
It follows that the mapping

$$x \circ L_P^{-1} : L_P(\tilde{U} \times \tilde{V}) \to x(\tilde{U} \times \tilde{V})$$

is an admissible chart of $\mathbf{R}^n$ and that $(x \circ L_P^{-1})(L_P(\tilde{U} \times \tilde{V}) \cap S) =$
$x((\tilde{U} \times \tilde{V}) \cap \tilde{S}) =$
$x(\tilde{U} \times \tilde{V}) \cap (\mathbf{R}^{n-m} \times \{0\}) =$
$(x \circ L_P^{-1})(L_P(\tilde{U} \times \tilde{V})) \cap (\mathbf{R}^{n-m} \times \{0\})$.
Thus the chart $(L_P(\tilde{U} \times \tilde{V}), x \circ L_P^{-1}))$ is admissible and has the submanifold property with respect to S. The theorem follows.