Differential Forms: (on $\mathbb{R}^n$)

0-forms : function $f: \mathbb{R}^n \to \mathbb{R}$

1-forms : $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$

2-forms $\beta = \beta_1 dy \wedge dz + \beta_2 dz \wedge dx + \beta_3 dx \wedge dy$

3-forms $\gamma = g\, dx \wedge dy \wedge dz$, $g: \mathbb{R}^3 \to \mathbb{R}$

⋮

n-forms

Concept: $\alpha: \mathbb{R}^3 \to T^*\mathbb{R}^3$; $p \mapsto \alpha_p \in \Lambda^1(\mathbb{R}^3) = T_p^*\mathbb{R}$

(dual space to $\mathbb{R}^3$ at p)

$\beta: \mathbb{R}^3 \to \Lambda^2(\mathbb{R}^3)$; $p \to \beta_p \in \Lambda^2(\mathbb{R}^3)_p$

set aside notation, at each p we attach a wedge-prod of dual-vectors

$$\mathbb{R}^n \text{ — } \underset{\text{0-form}}{f}, \ f_1 dx^1 + \dots + f^n dx^n, \ \sum_{i<j} f_{ij}\, dx^i \wedge dx^j, \ \underbrace{g\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n}_{n\text{-form}}$$

Exterior Derivative:

If $\alpha = \sum\limits_{i_1,i_2,\dots,i_p=1} \alpha_{i_1}\alpha_{i_2}\alpha_{i_p}\, dx^{i_1}\wedge dx^{i_2}\wedge\dots\wedge dx^{i_p}$

$$d\alpha = \sum_{i_1 i_2 i_p} d\alpha_{i_1 i_2 \dots i_p} \wedge dx^{i_1}\wedge dx^{i_2}\wedge\dots\wedge dx^{i_p}$$

where $df = \sum\limits_{j=1}^{n} \frac{\partial f}{\partial x^j} dx^j$ (apply this to $f = \alpha_{i_1 i_2 \dots i_p}$)

Ex: Let $\alpha = (x^2+y^2)dz + yz\, dx$

$\beta = (xy)\, dz \wedge dx + z^3 dx \wedge dy$

Brief Overview:

V is a vector space w/ $\beta = \{e_i\} \leftrightarrow V^{**} = \{\hat{v} \mid \hat{v} : V^* \to \mathbb{R}$ linear, $\hat{v}(\alpha) = \alpha(v)$

V^* is dual space w/ $\beta^* = \{e^i\}$

$$V^* = \{ L : V \to \mathbb{R} \mid L \text{ is linear} \}$$

$$\beta^* : e^i(e_j) = \delta_j^i$$

Basic facts:

$x \in V$ then $x = \sum x^i e_i$ and $x^i = e^i(x)$

$\alpha \in V^*$ then $x = \sum x_i e^i$ $\quad x_i = \alpha(e_i)$

Multilinear Maps:

$T \in T_s^r V$ iff $T: \underbrace{V \times V \times \cdots \times V}_{s} \times \underbrace{V^* \times \cdots \times V^*}_{r}$

type $\binom{r}{s}$ tensor if T is linear in each entry.

$T_s^r V$ forms a vector space w/ basis

$$\{ e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes \cdots \otimes e_{i_s} \otimes e^{j_1} \otimes e^{j_2} \otimes e^{j_3} \otimes \cdots \otimes e^{j_r} \}$$

How tensor $\otimes$ works

$(\alpha \otimes \beta)(x, y) = \alpha(x)\beta(y)$ $\quad \forall \alpha, \beta \in V^*, \forall x, y \in V$

$(\hat{x} \otimes \hat{y})(\alpha, \beta) = \alpha(x)\beta(y)$

extend linearly and same rule for addition $\otimes$

$T(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)}) = T(x_1, x_2, \dots, x_k)$ $\forall$ permutation: $\mathbb{N}_k \to \mathbb{N}_k$

$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ even perm.} \\ -1 & \text{if } \sigma \text{ odd perm.} \end{cases}$

→ Symmetric tensor.

→ Antisymmetric tensor

$$(AT^{r_1}_0 V) \otimes (AT^{r_2}_0 V) \neq AT^{r_1+r_2}_0 V$$

$\otimes$ operation does not preserve the antisymmetry between 2 antisymmetric tensor

So we need another operation to preserve this.

Call it "wedge product" ($\wedge$)

<u>Define</u>: $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha \qquad \alpha, \beta \in V^*$

$$\gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_1 \otimes \gamma_2 \otimes \gamma_3 + \gamma_2 \otimes \gamma_3 \otimes \gamma_1 + \gamma_3 \otimes \gamma_1 \otimes \gamma_2 - \gamma_3 \otimes \gamma_2 \otimes \gamma_1 - \gamma_2 \otimes \gamma_1 \otimes \gamma_3 - \gamma_1 \otimes \gamma_3 \otimes \gamma_2$$

$\cancel{3}22$

$$\gamma_{i_1} \wedge \gamma_{i_2} \wedge \gamma_{i_2} \wedge \dots \wedge \gamma_{i_k} = \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma)\, \gamma_{\sigma(i_1)} \otimes \gamma_{\sigma(i_2)} \otimes \dots \otimes \gamma_{\sigma(i_k)}$$

Ex: $V = \mathbb{R}^2$, $V^* = \langle e^1, e^2 \rangle = \operatorname{span}\{e^1, e^2\}$

$$\Lambda^2 V = \langle e^1 \wedge e^2 \rangle$$

$$\Lambda V = \mathbb{R} \oplus \Lambda^1 V \oplus \Lambda^2 V \quad , \mathbb{R} = \Lambda^0(V)$$
$$V^* = \Lambda^1(V)$$

$$\Lambda^3 V = \{0\}$$

$$e^1 \wedge e^2 \wedge e^1 = e^1 \wedge (-e^1 \wedge e^2) = \underbrace{-e^1 \wedge e^1}_{\text{zero}} \wedge e^2$$

$$e^1 \wedge e^1 = -e^1 \wedge e^1 \quad \therefore \quad 2 e^1 \wedge e^1 = 0$$

General Alg. Prop. of $\wedge$

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$$

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \quad \text{where } \alpha \in \Lambda^q V, \beta \in \Lambda^p V$$

Remark: we often ommit $\wedge$ for zero forms

$$\wedge : \Lambda^p V \times \Lambda^q V \to \Lambda^{p+q} V$$

$V = \mathbb{R}^3$:
$$\Lambda^0 \mathbb{R}^3 = \mathbb{R}$$
$$\Lambda^1 \mathbb{R}^3 = \langle e^1, e^2, e^3 \rangle$$
$$\Lambda^2 \mathbb{R}^3 = \langle e^2 \wedge e^3, e^3 \wedge e^1, e^1 \wedge e^2 \rangle$$
$$\Lambda^3 \mathbb{R}^3 = \langle e^1 \wedge e^2 \wedge e^3 \rangle$$
$$\Lambda^4 \mathbb{R}^4 = \langle 0 \rangle$$

Def: $\omega_{\langle F_1,F_2,F_3\rangle} = F_1 e^1 + F_2 e^2 + F_3 e^3$

$\Phi_{\langle F_1,F_2,F_3\rangle} = F_1 e^2 \wedge e^3 + F_2 e^3 \wedge e^1 + F_3 e^1 \wedge e^2$

Identity $\omega_{\vec{A}} \wedge \omega_{\vec{B}} = \Phi_{\vec{A}\times\vec{B}}$

Replace V w/ $T_p\mathbb{R}^n = \text{span}\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}_{i=1}^n$

$= \text{Span}\{(p,v) \mid v\in\mathbb{R}^n\}$

$(p,v) + (p,w) = (p, v+w)$

V^* w/ $T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^* = \text{span}\{d_p x^i\}_{i=1}^n$

$\mathbb{R}^3$) $f:\mathbb{R}^3 \to \mathbb{R}$, 0-form

$\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$, 1-form now $\alpha_i : \mathbb{R}^3\to\mathbb{R}$ for $i=1,2,3$

$\cancel{\alpha(p) = \alpha_1}$

$\beta = \beta_{23}\, dy\wedge dz + \beta_{31}\, dz\wedge dx + \beta_{12}\, dx\wedge dy$

$\gamma = g\, dx\wedge dy\wedge dz$ (Volume for $\mathbb{R}^3$, $g=1$ for example)

$\alpha \in \Omega^s\mathbb{R}^n \Rightarrow \alpha_p \in \Lambda^s(T_p\mathbb{R}^n)$

$\alpha = \frac{1}{s!}\sum_{i_1,i_2,\dots,i_s} \alpha_{i_1 i_2\dots i_s}\, dx^{i_1}\wedge dx^{i_2}\wedge\dots\wedge dx^{i_n} = \sum_{i_1<i_2<\dots<i_s} \alpha_{i_1 i_2 i_3\dots i_s}\, dx^{i_1}\wedge dx\dots$

Integration of forms

A p-form γ must be integrated over p-dim'l space Σ_p

Σ_p has $\Phi : U \subseteq \mathbb{R}^p \to \mathbb{R}^n$ $(\Phi = (x^1, x^2, x^3, \ldots, x^n))$

$$\int_\Sigma \gamma = \int_U \cdots \int \underbrace{\gamma_{i_1 i_2 \cdots i_p}}_{\text{antisymmetric}}(x^1, x^2, \ldots, x^n)\, dx^{i_1} dx^{i_2} \cdots dx^{i_p}$$

Ex $\displaystyle\int_{[0,1]^2} (x^2+y^2)\, dx \wedge dy = \int_0^1 \int_0^1 (x^2+y^2)\, dx\, dy.$

$$\int_S \Phi_{\vec F} = \int_S \vec F \cdot d\vec s \quad \Big| \quad \int_C \omega_{\vec F} = \int_C \vec F \cdot d\vec r \quad \Big| \quad \int_B \mathrm{vol} = \iiint_B dx\,dy\,dz$$

Theorem: $\displaystyle\int_M d\gamma = \int_{\partial M} \gamma$

$d : \Omega^p \longrightarrow \Omega^{p+1}$ (on $\mathbb{R}^n$)

If $\gamma = \sum \gamma_I \, dx^I$ where $I = i_1 i_2 i_3 \cdots i_p$ and $\sum$ is over these indices

$$d\gamma = \sum_I d\gamma_I \wedge dx^I$$

where $d\gamma_I$ is defined as It was in Math 231

$$d\gamma_I = \sum_{j=1}^{r} \frac{\partial \gamma_I}{\partial x^j} \, dx^j$$

Ex: ($\mathbb{R}^n$) $\quad df = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} dx^j = \omega_{\nabla f}$, $\nabla f = \langle \frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n} \rangle$

Ex: ($\mathbb{R}^3$) $\quad d\omega_{\vec{F}} = \Phi_{\nabla \times \vec{F}}$

Ex: $\mathbb{R}^3$ $\quad d\Phi_G = (\nabla \cdot \vec{G}) \, dx \wedge dy \wedge dz$.

laws

1/ $d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^{pq} \alpha_p \wedge d\beta_q$

2/ $d^2 \alpha = 0 \quad \forall \alpha \in \Omega \mathbb{R}^n$.