THE COLUMN CORRESPONDENCE PRINCIPLE

Suppose $col_j(A) = \sum_{i=1}^{k} c_i \, col_i(A) \quad (\star)$

Also, suppose $rref(A) = \underbrace{E_k \cdots E_2 E_1}_{\text{product of elementary matrices}} A = EA = R$ ← short notation for rref(A).

Multiply $\star$ by E to find

$$E \, col_j(A) = E\left(\sum_{i=1}^{k} c_i \, col_i(A)\right)$$
$$= \sum_{i=1}^{k} c_i \, E \, col_i(A) \quad (\star\star)$$

But, remember $EA = [E\,col_1(A) | \cdots | E\,col_n(A)]$ and notice $col_i(EA) = E\,col_i(A)$

Hence $\underline{col_i(R) = E\,col_i(A)}$. Use this in $(\star\star)$ to get:

$$col_j(R) = \sum_{i=1}^{k} c_i \, col_i(R)$$

$$\boxed{Th^m \text{ (CCP)} \quad col_j(A) = \sum_{i=1}^{k} c_i \, col_i(A) \iff col_j(R) = \sum_{i=1}^{k} c_i \, col_i(R)}$$

__Remark__: the argument can be reversed since $EA = R \Rightarrow A = RE^{-1}$ so we could begin on RHS and go to LHS of $\iff$.

$\boxed{E1}$

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 4 & 2 \\ 3 & 4 & 1 \end{bmatrix} \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \underbrace{\begin{bmatrix} 1 & 4 & 3 \\ 0 & -4 & -4 \\ 0 & -8 & -8 \end{bmatrix}}_{R_1} \xrightarrow[r_3 - 2r_2]{r_1 + r_2} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \end{bmatrix}}_{R_2} \sim \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{R}$$

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ has } \underbrace{col_3(R) = col_2(R) - col_1(R)}_{\text{obvious by inspection of } R.}$$

<u>Every</u> <u>row</u> <u>equivalent</u> matrix to R also has <u>same</u> linear dependence amongst the columns

$$col_3(A) \overset{?}{=} col_2(A) - col_1(A) = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \overset{\checkmark}{=} col_3(A)$$

$$col_3(R_1) \overset{?}{=} col_2(R_1) - col_1(R_1) = \begin{bmatrix} 4 \\ -4 \\ -8 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -8 \end{bmatrix} \overset{\checkmark}{=} col_3(R_1)$$

$$col_3(R_2) \overset{?}{=} col_2(R_2) - col_1(R_2) = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 0 \end{bmatrix} \overset{\checkmark}{=} col_3(R_2)$$

- I focus on comparing A to rref(A) since seeing dependence of columns in rref(A) is totally trivial.

$\boxed{E2}$ If $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 8 \\ 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

① $col_2(A) = 2\,col_1(A)$

② $col_4(A) = 2\,col_3(A) - col_1(A)$

③ $col_6(A) = 8\,col_1(A) + 7\,col_2(A) + 6\,col_3(A)$

<u>How to reverse engineer A</u>

If I give the pivot columns 1, 3, 5 then the non-pivot columns automatically follow by CCP according to ①, ②, ③

$$\left.\begin{aligned} col_1(A) &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ col_3(A) &= \begin{bmatrix} 0 \\ 3 \\ 4 \\ 5 \end{bmatrix} \\ col_5(A) &= \begin{bmatrix} 11 \\ 9 \\ 7 \\ 5 \end{bmatrix} \end{aligned}\right\} \longrightarrow A = \begin{bmatrix} 1 & 2 & 0 & -1 & 11 & 74 \\ 1 & 2 & 3 & 5 & 9 & 83 \\ 1 & 2 & 4 & 7 & 7 & 78 \\ 1 & 2 & 5 & 9 & 5 & 73 \end{bmatrix}$$

$$8\,col_1(A) + 7\,col_2(A) + 6\,col_3(A) = \begin{bmatrix} 8 \\ 8 \\ 8 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 21 \\ 28 \\ 35 \end{bmatrix} + \begin{bmatrix} 66 \\ 54 \\ 42 \\ 30 \end{bmatrix}$$

$\boxed{E3}$

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 7 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{l} col_3(A) = 3col_1(A) + 7col_2(A) \\ col_4(A) = 2col_2(A) - col_1(A) \end{array}$$

If $col_1(A) = \begin{bmatrix} 8 \\ 10 \\ 7 \end{bmatrix}$ and $col_2(A) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ then

$$col_3(A) = 3\begin{bmatrix} 8 \\ 10 \\ 7 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ 37 \\ 21 \end{bmatrix} \text{ and } col_4(A) = 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 8 \\ 10 \\ 7 \end{bmatrix} = \begin{bmatrix} -8 \\ -8 \\ -7 \end{bmatrix}$$

thus $$\underline{A = \begin{bmatrix} 8 & 0 & 24 & -8 \\ 10 & 1 & 37 & -8 \\ 7 & 0 & 21 & -7 \end{bmatrix}}.$$