

**Sign of derivative test: when does a function decreases or increases:**

- If  $\frac{df}{dx} > 0$  for all  $x \in (a, b)$  then  $f$  is increasing on  $(a, b)$ .
- If  $\frac{df}{dx} < 0$  for all  $x \in (a, b)$  then  $f$  is decreasing on  $(a, b)$ .
- If  $f'(x) = 0$  then the function is not increasing or decreasing at  $x$ .

We defined increasing and decreasing in Definition 2.3.2. In words, a function is increasing if as  $x$  increases the values of the function likewise increase. Let's examine a few examples of this test in action:

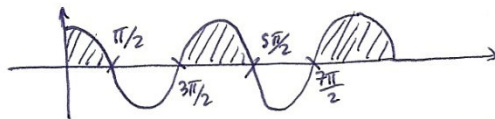
**Example 5.3.7 and 5.3.8:**

**[E7]** Let  $f(x) = e^x$  then  $f'(x) = e^x > 0$  for  $x \in \mathbb{R}$  so  $f(x) = e^x$  is increasing on all of  $\mathbb{R}$ .

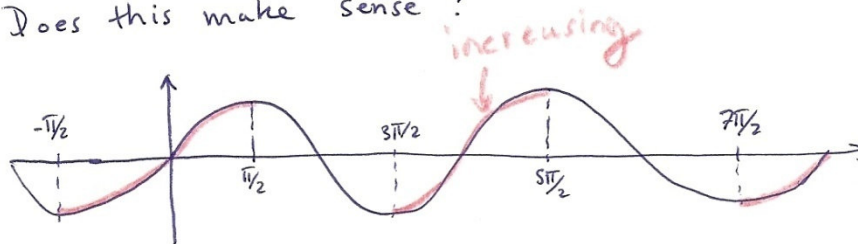
**[E8]** Let  $f(x) = x^2 - 2x + 1$  then  $f'(x) = 2x - 2 = 2(x-1)$  so  $f'(x) > 0$  when  $x > 1$  and  $f'(x) < 0$  when  $x < 1$ . Thus  $f$  is increasing on  $(1, \infty)$  and  $f$  is decreasing on  $(-\infty, 1)$ .

**Example 5.3.9:**

**[E9]** Let  $f(x) = \sin(x)$  then  $f'(x) = \cos(x)$  so we draw a graph



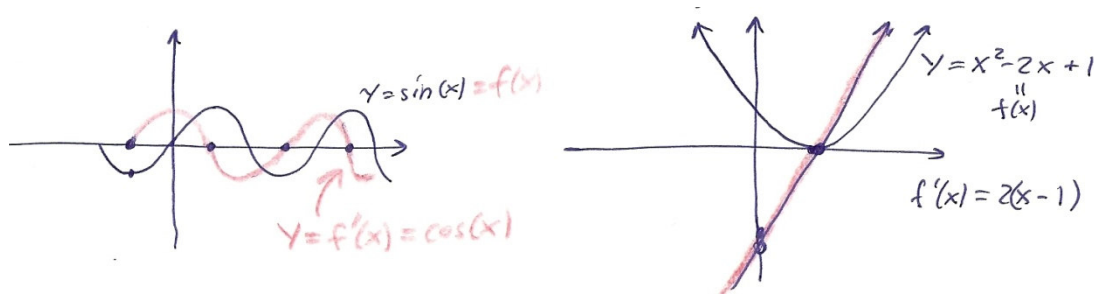
$f'(x) > 0$  when  $x$  is in  $(-\pi/2, \pi/2), (3\pi/2, 5\pi/2), (7\pi/2, 9\pi/2), \dots$   
 $f'(x) < 0$  when  $x$  is in  $(\pi/2, 3\pi/2), (5\pi/2, 7\pi/2), (9\pi/2, 11\pi/2), \dots$   
Does this make sense?



the graph above is of  $y = \sin(x)$  and the red illustrates  $\cos(x) > 0$  when the sine function is increasing.

Discussion: What does the sign of the derivative changing suggest?

If the derivative  $df/dx$  is a continuous function then we can conclude that the derivative must be zero on some interval if it changes sign on that interval. (Recall that the Intermediate Value Theorem helped us find zeros of functions by examining if the function changed signs on some interval.) Notice that when the derivative is zero at some point that means the function has a horizontal tangent line at that point. Such points are local extrema, it will be the point on the graph which is the highest or lowest point in an open neighborhood around the point. For examples:



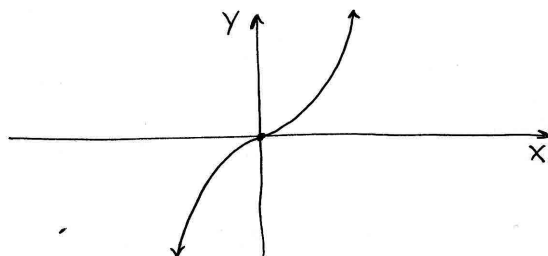
The derivatives are graphed in red and you can see that they are zero where the function is minimized or maximized. This motivates (but does not prove!) the First Derivative Test:

**Theorem 5.3.3:** (First Derivative Test) If we are given function  $f$  which is continuous on an open interval containing a critical number  $c$  then:

1. If  $df/dx$  changes signs from positive to negative at  $c$  then  $f(c)$  is a local max.
2. If  $df/dx$  changes signs from negative to positive at  $c$  then  $f(c)$  is a local min.
3. If  $df/dx$  does not change signs at  $c$  then  $f(c)$  is not a local extrema.

**Example 5.3.10a:** (this one illustrates case 3.)

Example of C.) Let  $f(x) = x^3$  then  $f'(x) = 3x^2$   
 thus  $c = 0$  is a critical # for  $f$  but  
 $f'(x) = 3x^2 > 0$   
 So  $f'(x)$  does not change signs at  $c = 0$ . Why?



The example above is missing something I'd like to put on each example as a rule. I think we should use a ***sign chart*** to organize information about signs. Unfortunately, the homework solutions do not have this organizational aid for the most part. Look ahead to Example 5.3.12 for examples of "sign charts". Logically we don't really need them, but I think I'm going to make their use a requirement for our class. This is just so we can be on the same page when organizing our ideas. I will add many of these sign charts in lecture, so it is important to take notes on this point.

Notice that Examples 5.3.7, 5.3.8 and 5.3.9 all illustrate cases 1 and 2 of the First Derivative Test. Let me give one more examples before we go on to the Second Derivative Test.

***Example 5.3.10b:***

Let  $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x$ . Find all critical numbers and classify the critical points as local maximums, minimums, or neither. Observe,

$$f'(x) = x^2 + x - 6 = (x - 2)(x + 3).$$

We have two critical numbers;  $c = 2$  and  $c = -3$ . Lets draw the sign chart,



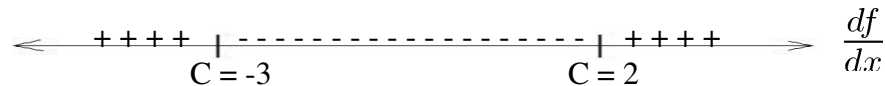
Then we test a point somewhere in the interior of each region,

$$f'(-4) = (-4 - 2)(-2 + 3) = 8 > 0$$

$$f'(0) = (-2)(3) = -6 < 0$$

$$f'(3) = (3 - 2)(3 + 3) = 6 > 0$$

Which suggests we fill in the sign chart as follows:



By the First Derivative Test we conclude,

$$f(-3) = -27/3 + 9/2 - 6(-3) = 27/2 \text{ is a local maximum (case 1),}$$

$$f(2) = 8/3 + 4/2 - 6(2) = -22/3 \text{ is a local minimum (case 2).}$$

***Question:*** Observe that  $f(x) = x \frac{\sqrt{(x-1)^2}}{x-1}$  for  $x \neq 1$  and  $f(1) = 1$  has critical number  $c = 1$ . Moreover, the derivative changes sign from  $df/dx = -1$  for  $x < 1$  to  $df/dx = 1$  for  $x > 1$ . Is it in fact the case that  $f$  obtains the local maximum  $f(1) = 1$  at  $(1,1)$  ? Does this contradict the First Derivative Test ? Explain.

### **Concave-up and Concave-down:**

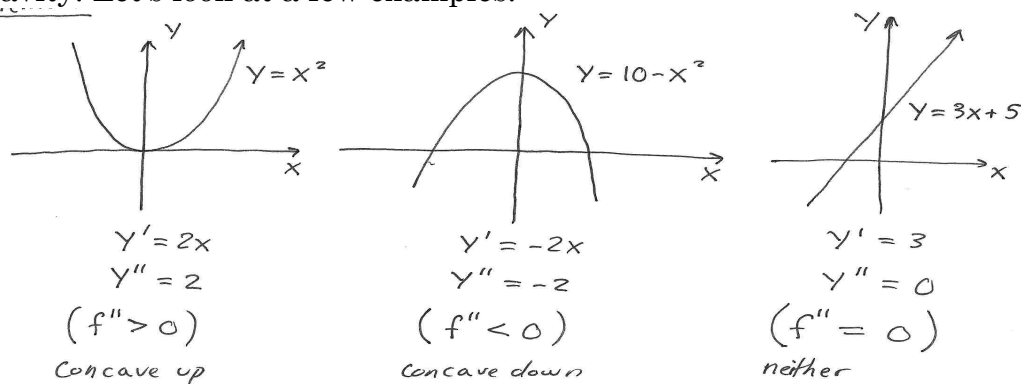
The change in a function is captured by the first derivative; increasing functions have positive derivative functions while decreasing functions have negative derivative functions. It turns out that the *concavity* of a function is captured by the second derivative. A function is *concave up* on an interval  $J$  if the function has the shape of a bowl which is right-side up over that interval  $J$ . A function is *concave down* on an interval  $J$  if the function has the shape of a bowl which is up-side down over that interval  $J$ . I personally find these sort-of geometric definitions rather unsatisfying. I challenge you to find a clear definition stated in Stewart's text. That said, I say we take the following as the real definition for concave up or down.

**Definition 5.3.4:** Let  $f$  be a function with  $f'(x)$  and  $f''(x)$  well-defined for each  $x \in J$  then we say that

- $f(x)$  is concave-up on  $J$  if  $f''(x) > 0$  for each  $x \in J$
- $f(x)$  is concave-down on  $J$  if  $f''(x) < 0$  for each  $x \in J$

A function is concave up(or down) at a point if there exists a neighborhood about the point for which the function is concave up(or down). We say the function changes concavity at a point  $c$  if the function is concave-up(down) to the left of the point and concave-down(up) to the right of the point  $(c, f(c))$ . If the function changes concavity at a point  $(c, f(c))$  then we say that  $(c, f(c))$  is a point of inflection.

Consider this, if a function has the shape of a bowl right side up then the slopes of the tangent lines will increase as we increase  $x$ . On the other hand, if a function has the shape of a bowl upside down then the slopes of the tangent lines will decrease as  $x$  increases. In other words, the derivative is an increasing function where the function is concave-up and the derivative is a decreasing function where the function is concave-down. This proves the definition given above is equivalent to the geometric bowl-based definition for concavity. Let's look at a few examples:



- The line is an exceptional case, then thing between concave up and down.

(by the way, the term "convex" used to be used for concave down, this term is still used in physics particularly in the study of optics)

**Concavity test: when is a function concave up or down:**

- If  $f''(x) > 0$  for all  $x \in (a, b)$  then  $f$  is concave-up on  $(a, b)$ .
- If  $f''(x) < 0$  for all  $x \in (a, b)$  then  $f$  is concave-down on  $(a, b)$ .
- If  $f''(c) = 0$  then the function might have an inflection point at  $c$ .

I emphasize that when the second derivative is zero we might find an inflection point, but it doesn't have to be the case (see picture on last page for example). Also, it could be that we find an inflection point where the second derivative does not exist. There are many possibilities. The same is true for critical points. When a critical point is not at a local max or min it could be an inflection point, but it might be something else, there are countless other options. **Bonus Point:** find me an example of a continuous function which has a nonzero derivative and a critical point which is neither at a local maximum, minimum or inflection point.

**Theorem 5.3.3:** (Second Derivative Test) If a function  $f$  has  $\frac{df}{dx}$  continuous on an open neighborhood containing  $c$  such that  $f'(c) = 0$  then:

1. If  $f''(c) < 0$  then  $f(c)$  is a local maximum.
2. If  $f''(c) > 0$  then  $f(c)$  is a local minimum.
3. If  $f''(c) = 0$  then the Second Derivative Test is inconclusive.

*Proof:* Suppose that  $f''(c) < 0$  and  $f'(c) = 0$  then  $(c, f(c))$  has a horizontal tangent line and the function is upside-down bowl-shaped near the point, hence it is geometrically clear that the point is a local maximum. Likewise if  $f''(c) > 0$  and  $f'(c) = 0$  then  $(c, f(c))$  has a horizontal tangent line and the function is rightside-up bowl-shaped near the point, hence it is geometrically clear that the point is a local minimum. Technically, this proof leaves something to be desired, but this is the heart of it.

**[EII]** Consider  $f(x) = x^3 - 12x - 5$

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$

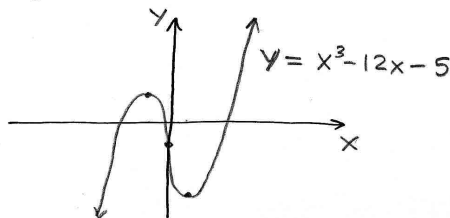
$$f''(x) = 6x$$

Critical #'s are  $x = \pm 2$  where  $f'(\pm 2) = 0$ . Then

$$f''(2) = 6(2) = 12 \quad \therefore f(2) = 8 - 24 - 5 = -21 \quad \text{local min.}$$

$$f''(-2) = 6(-2) = -12 \quad \therefore f(-2) = -8 + 24 - 5 = 11 \quad \text{local max}$$

We can additionally note  $x = 0$  is an inflection point. Sketching  $f$ ,



### Example 5.3.12:

**E12** We will study a # of questions about  $f(x) = \frac{x}{(1+x)^2}$ .  
I will introduce some notational helps to keep track of signs of  $f$ ,  $f'$  and  $f''$ . If you have your own system for finding zeroes etc... from precalculus then feel free to use it. Find the following,

- CRITICAL POINTS
- intervals on which  $f$  is decreasing/increasing
- local maximums/minimums
- intervals on which  $f$  is concave up/down
- inflection points
- zeroes of function (precalculus here)
- graph  $f$  carefully using  $a \rightarrow f$ .

Lets begin by calculating  $f'$  and  $f''$ ,

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(1+x)^2 - x \frac{d}{dx}(1+x)^2}{(1+x)^4} \\ &= \frac{1+2x + x^2 - 2x - 2x^2}{(1+x)^4} \\ &= \frac{1-x^2}{(1+x)^4} \quad : 1-x^2 = (1+x)(1-x) \text{ thus,} \\ &= \boxed{\frac{1-x}{(1+x)^3} = f'(x)} \end{aligned}$$

- a.) Notice  $f'(1) = 0$  and  $f'(-1)$  d.n.e thus the critical #'s of  $f$  are  $C = \pm 1$ . Now we know that  $f'$  can only change sign at critical #'s thus we can just check a pt. from each region to completely describe  $f'$  positive/neg.



- b.) from the sign-chart of  $f'$  we can read off that

$f$  is decreasing on  $(-\infty, -1)$  and  $(1, \infty)$   
 $f$  is increasing on  $(-1, 1)$

**Example 5.3.12 continued:**

c.) We use the 1<sup>st</sup> derivative test to conclude that since  $f'$  changes from (+) to (-) at  $x = 1$  the value  $f(1) = \frac{1}{2^2} = \frac{1}{4}$  is a local max of  $f$  at  $x = 1$ . ( $x = -1$  is a vertical asymptote so although it's a critical point  $f(-1)$  can't be a local max, it d.n.e.)

d.) We need to find  $f''$  and how it behaves,

$$f''(x) = \frac{d}{dx} \left( \frac{1-x}{(1+x)^3} \right) = \frac{2(x-2)}{(x+1)^4}$$

Notice that  $f''$  can only change sign at its zeroes and vertical asymptotes which occur at  $x = 2$  and  $x = -1$ . We need to check 3 points to generate the following,

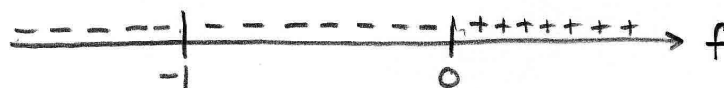


We can read off the sign chart for  $f''$  that

$f$  is concave down on  $(-\infty, 2)$   
 $f$  is concave up on  $(2, \infty)$

e.)  $f$  changes concavity at  $x = 2$  which is the only inflection point.

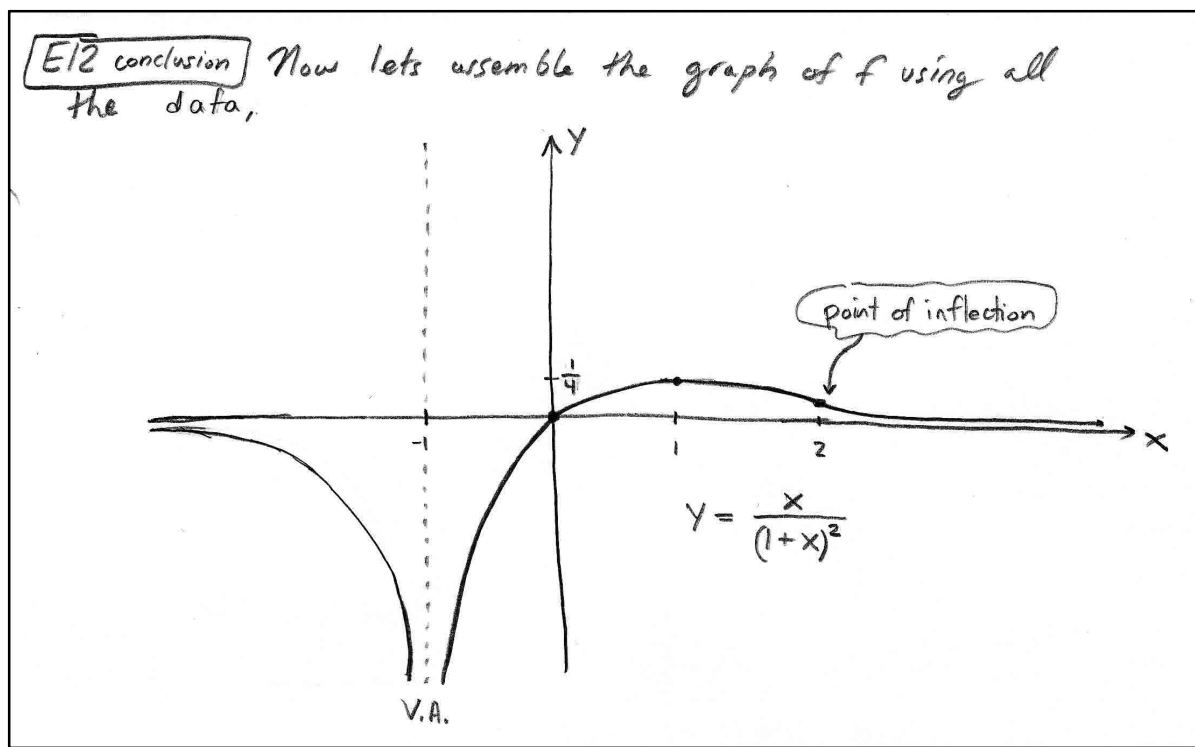
f.) arguably I should've put this 1<sup>st</sup>, oh well anyway  $f(0) = 0$  and  $f(-1)$  d.n.e.,  $f$  must change signs either 0 or -1 so we again just need to check 3 test pts.



ok, I only asked for the zeroes of  $f$  but this will help us graph  $f$  next

**Remark:** a function which is continuous everywhere except at vertical asymptotes can only change sign at its zeroes and vertical asymptotes. This idea applies to the function, its derivative and its second derivative. This is a consequence of the Intermediate Value Theorem and plain-old common sense.

**Example 5.3.12 conclusion:**



**Remark on case 3 of 2<sup>nd</sup> Derivative Test:** Maybe you are wondering, what is an example of a function which falls into case 3 of the derivative test? One simple example is a line  $f(x) = mx + b$  which has  $f'(x) = m$ . Clearly  $df/dx$  is continuous everywhere. Notice  $f''(x) = 0$  for each  $x \in \mathbb{R}$ . There are two cases:

1.  $m = 0$  thus  $f(x) = b$  and  $y = b$  is the maximum and minimum value of the function at all points.
2.  $m \neq 0$  then  $f(x) = mx + b$  and the function has no local or global extreme values. If we restrict attention to a closed interval then the line takes its extreme values with respect to that interval on the endpoints.



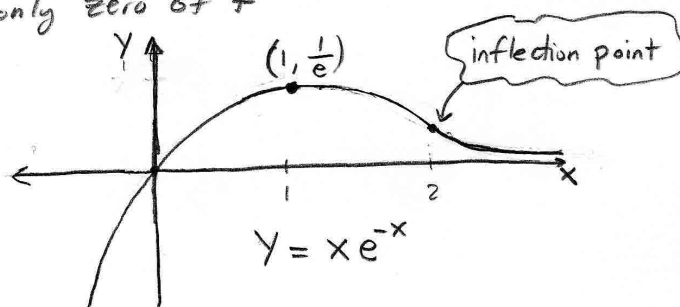
**Example 5.3.13:**

**E13** Do the same as **E12** for  $xe^{-x}$ ,

$$\begin{aligned} f(x) &= xe^{-x} \Rightarrow \text{sign chart for } f \\ f'(x) &= e^{-x}(1-x) \Rightarrow \text{sign chart for } f' \\ f''(x) &= e^{-x}(x-2) \Rightarrow \text{sign chart for } f'' \end{aligned}$$

Therefore,

- critical point is  $x=1$
- $f$  increases on  $(-\infty, 1)$  and decreases on  $(1, \infty)$
- $f$  has a local max of  $f(1) = \frac{1}{e}$  at  $x=1$  by 1<sup>st</sup> Der. Test.
- $f$  concave up on  $(2, \infty)$  and concave down on  $(-\infty, 2)$ .
- $x=2$  is an inflection point of  $f$ .
- $x=0$  is the only zero of  $f$
- graph



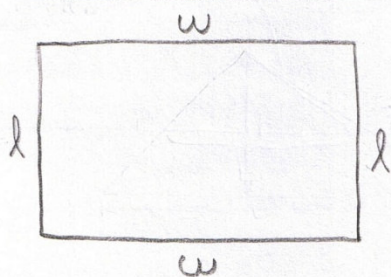
We have at our disposal all the tools we need to figure out what a function looks like locally. Given the formula we simply take a derivative or two and think. The global picture of the function requires one more piece of information: the asymptotic behavior of the function. Such behavior is captured by the limits at  $\pm\infty$ . We will discuss this last topic on graphing in the section 5.5.

## 5.4. OPTIMIZATION

This section seeks to use calculus to answer questions like: what is the biggest, smallest, cheapest longest, shortest, hottest, coldest, best, worst, etc ... Given some word problem or equation which models a particular physical problem what are the possibilities, what are the extremes? The analysis developed and discussed in the preceding section allows us to tackle such questions in a way that was unavailable before the advent of calculus.

### Example 5.4.1:

E1: Given 400ft of fencing what dimensions should you give a rectangular pen as to maximize the area?



$$2l + 2w = 400$$

$$A = lw$$

$$\text{Notice that } l = 200 - w$$

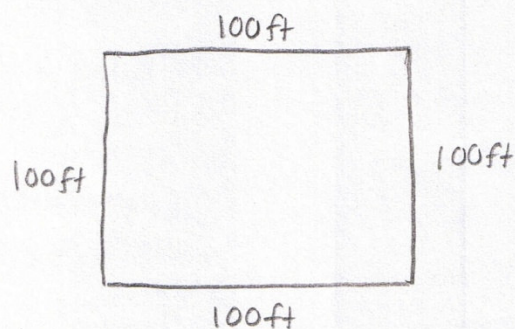
$$\Rightarrow A = (200 - w)w = 200w - w^2$$

Maximize  $A$  as a function of  $w$  then,

$$\frac{dA}{dw} = 200 - 2w \Rightarrow \frac{dA}{dw} = 0 \text{ when } w = 100$$

Then note  $\frac{d^2A}{dw^2} = -2 < 0 \Rightarrow A(100)$  is a maximum.

Thus by 2<sup>nd</sup> derivative test  $w = 100$  gives max. area. Thus

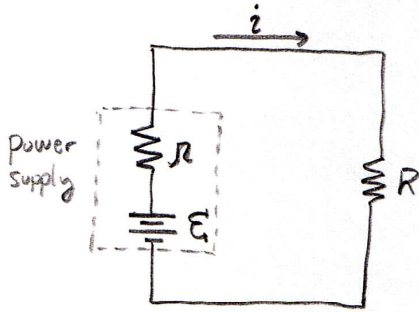


$$l = 200 - 100 = 100$$

These dimensions maximize the area of the pen.

**Example 5.4.2:**

**E2** Given a power supply with fixed voltage  $\mathcal{E}$  and internal resistance  $r$  what load  $R$  will maximize the power delivered to the load?



$$\mathcal{E} = ir + iR \quad : \text{Kirchhoff's Voltage Law}$$

$$P = i^2 R \quad : \text{Power delivered to } R \text{ by current } i$$

Solve to find  $i = \frac{\mathcal{E}}{r+R}$  then  
 substitute into  $P = i^2 R = \frac{\mathcal{E}^2 R}{(r+R)^2}$

Now do calculus on  $P$  as a function of  $R$ ,

$$\frac{dP}{dR} = \mathcal{E}^2 \frac{d}{dR} \left( \frac{R}{(r+R)^2} \right)$$

$$= \mathcal{E}^2 \left[ \frac{(r+R)^2 - 2(r+R)R}{(r+R)^4} \right]$$

$$= \frac{\mathcal{E}^2}{(r+R)^4} [r^2 + \cancel{2rR} + R^2 - \cancel{2rR} - 2R^2]$$

$$= \frac{\mathcal{E}^2}{(r+R)^4} [r^2 - R^2]$$

$$= \frac{\mathcal{E}^2}{(r+R)^2} [(r-R)(r+R)] \quad : \quad r, R > 0 \text{ for physical reasons so } R=r \text{ is only interesting critical point.}$$



Thus by 1<sup>st</sup> derivative test the max power is delivered to  $R$  when  $R = r$ . This is a simple case of the

"Max Power Transfer Th<sup>m</sup>"

Notice it implies that the Max Efficiency of a simple power supply is 50%. Electrical engineers can build more efficient power supplies called "switching" power supplies, in those supplies the output is automatically tailored as to be best for the load. It's a tricky business.

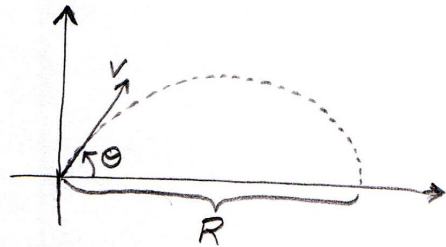


Example 5.4.3:

**E3** The range of a projectile fired at angle  $\Theta$  with velocity  $V$  can be shown to be  $R = \frac{V \sin(2\Theta)}{g}$  assuming no friction and  $g = 9.8 \text{ m/s}^2$ . What angle  $\Theta$  maximizes the range. What is  $R_{\max}$ ? 78

We notice that  $V$  and  $g$  are constants,  $R$  is the dependent (like  $y$ ) variable and  $\Theta$  is the independent variable (like  $x$ ).

$$\begin{aligned}\frac{dR}{d\Theta} &= \frac{d}{d\Theta} \left( \frac{V}{g} \sin(2\Theta) \right) \\ &= \frac{V}{g} \frac{d}{d\Theta} (\sin(2\Theta)) \\ &= \frac{2V}{g} \cos(2\Theta)\end{aligned}$$



For physical reasons we restrict  $\Theta$  to  $0 \leq \Theta \leq \pi/2$ .

$$\cos(2\Theta) = 0 \quad \text{for} \quad 2\Theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$\Rightarrow \Theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \dots$$

only  $\Theta = \pi/4$  is a physically interesting critical value.

$$\text{Calculate } \frac{d^2R}{d\Theta^2} = -\frac{4V}{g} \sin(2\Theta) \Rightarrow \frac{d^2R}{d\Theta^2}(\Theta = \frac{\pi}{4}) = -\frac{4V}{g} \sin(\frac{\pi}{2})$$

Thus we note  $\frac{d^2R}{d\Theta^2}(\Theta = \frac{\pi}{4}) = -\frac{4V}{g} < 0$  hence by 2<sup>nd</sup> derivative test we have that  $R(\pi/4)$  is the max. range.  $\Theta = \pi/4$

$$R_{\max} = \frac{V \sin(2 \cdot \frac{\pi}{4})}{g} = \boxed{\frac{V}{g} = R_{\max}}$$

**Example 5.4.4:**

**E4** The height of a projectile launched straight up with velocity  $V_0$  is given by:

$$Y = V_0 t - \frac{1}{2} g t^2$$

What is the maximum height and at what time is it reached?

Notice that  $V_0$  and  $g$  are just numbers so,

$$\frac{dy}{dt} = \frac{d}{dt} \left( V_0 t - \frac{1}{2} g t^2 \right) = V_0 - g t \quad (\text{the velocity})$$

$$\frac{d^2 y}{dt^2} = \frac{d}{dt} (V_0 - g t) = -g \quad (\text{the acceleration})$$

Now  $\frac{dy}{dt} = 0 = V_0 - g t \Rightarrow t_{\max} = V_0/g$  is critical time.

We know that  $Y(t_{\max}) = Y_{\max} = V_0 \left( \frac{V_0}{g} \right) - \frac{1}{2} g \left( \frac{V_0}{g} \right)^2 = \frac{V_0^2}{2g} = Y_{\max}$   
is indeed the max height because  $Y'' = -g < 0$ . (2<sup>nd</sup> der. test.)

**Example 5.4.5:**

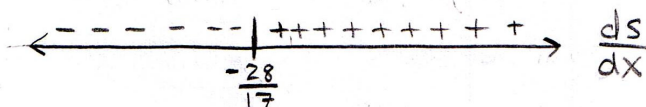
**E5** Find the point on the line  $Y = 4x + 7$  that is closest to the origin  $(0,0)$ .

We want to minimize the distance  $S = \sqrt{x^2 + y^2}$   
where  $Y = 4x + 7$  so consider

$$\frac{d}{dx} \left( \sqrt{x^2 + (4x+7)^2} \right) = \frac{2x + 8(4x+7)}{2\sqrt{x^2 + (4x+7)^2}} = \frac{ds}{dx}$$

So then  $\frac{ds}{dx} = 0 \Leftrightarrow 2x + 8(4x+7) = 0 \Leftrightarrow 34x = -56$

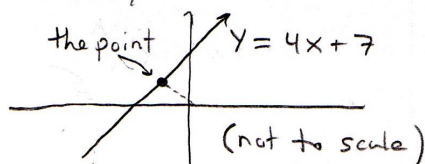
Thus  $x = -28/17$  is the only critical point.



Thus  $s(-28/17 = x)$  is the min. distance to  $Y = 4x + 7$   
from the origin by 1<sup>st</sup> derivative test. The  $Y$ -coordinate is

$$Y\left(-\frac{28}{17}\right) = 4\left(-\frac{28}{17}\right) + 7 = \frac{7}{17}$$

So the closest point is in fact  $\left(-\frac{28}{17}, \frac{7}{17}\right)$



**Example 5.4.6:**

What lengths  $x_{min}$  and  $x_{max}$  give a mass-spring system obeying Hooke's Law  $F = -kx$  the minimum and maximum Kinetic energy  $K$ ? Recall that energy is conserved such a system and in particular we have that  $K = \frac{1}{2}mv^2$  and  $U = \frac{1}{2}kx^2$  with  $K + U = E_o$  (a positive constant).

**Solution:** Solve for  $K$ ,

$$K = E_o - \frac{1}{2}kx^2$$

Look for critical numbers:

$$\frac{dK}{dx} = 0 - \frac{1}{2}k(2x) = -kx \implies x = 0 \text{ only critical number}$$

Furthermore, notice

$$\frac{d^2K}{dx^2} = -k < 0 \text{ the spring constant } k > 0 \text{ by convention}$$

Which shows that  $x_{max} = 0$  yields the maximum Kinetic energy of  $K = E_o$ . How did I miss  $x_{min}$ ? What am I not paying attention to about the Kinetic energy function? Let me give a colossal hint: what is the domain for  $K$ ? We know that mass is positive and so is  $v^2$  so it stands to reason that  $K \geq 0$  hence  $E_o - \frac{1}{2}kx^2 \geq 0$  or simply  $E_o \geq \frac{1}{2}kx^2$  thus  $x^2 \leq 2E_o/k$  which indicates that  $-\sqrt{2E_o/k} \leq x \leq \sqrt{2E_o/k}$ . We need to use the closed interval method for this problem. Perhaps that was not obvious from the start. Lets check  $K$  on the endpoints,

$$K(x = -\sqrt{2E_o/k}) = E_o - \frac{1}{2}k(-\sqrt{2E_o/k})^2 = 0$$

$$K(x = \sqrt{2E_o/k}) = E_o - \frac{1}{2}k(\sqrt{2E_o/k})^2 = 0$$

Thus  $x_{min} = \pm\sqrt{2E_o/k}$  where the Kinetic energy is zero.

**Moral of story:** domains matter, its not "just" a math thing. It can happen that the interesting case is at an endpoint and not at a critical point.

**Example 5.4.7:**

**E7** Find two positive numbers whose product is 100 and whose sum is a minimum

Let  $m$  and  $n$  be the numbers. We have that

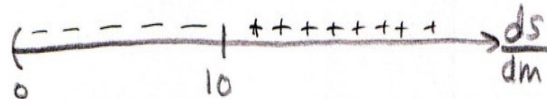
$$mn = 100$$

$$S = m + n = m + \frac{100}{m}$$

Now minimize  $S$  as a function of  $m$ ,

$$\frac{dS}{dm} = 1 - \frac{100}{m^2} = 0 \Rightarrow m = \pm 10 \text{ are critical values of } m.$$

Notice that  $m=0$  is also a critical value. However only  $m=10$  matters because we're looking for positive #'s.



Thus  $S$  is minimized for  $m=10$ . Thus the numbers are  $m=10$  and  $n=10$ .

**Example 5.4.8:** when you take differential equations you will learn how to solve Newton's equations in the case of a velocity dependent friction force. In the following example we analyze the solution to see how the spring vibrates.

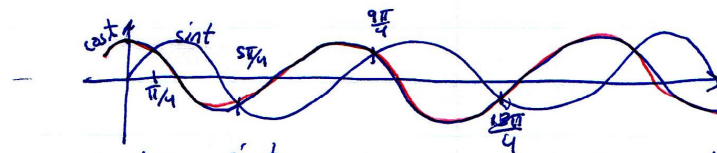
E8

Newton's 2<sup>nd</sup> Law.  $ma = F_{\text{spring}} + F_{\text{friction}}(\text{velocity})$  has sol<sup>n</sup> below,

$$y = A \sin(t) e^{-t}$$

$$\begin{aligned} y' &= (A \cos t) e^{-t} + (A \sin t) (-e^{-t}) \\ &= A e^{-t} (\cos t - \sin t) \end{aligned}$$

$$\begin{aligned} y'(t) = 0 &\Rightarrow \cos t - \sin t = 0 \Rightarrow \tan(t) = 1 \\ t &= \pi/4, 5\pi/4, 9\pi/4, 13\pi/4, \dots \end{aligned}$$



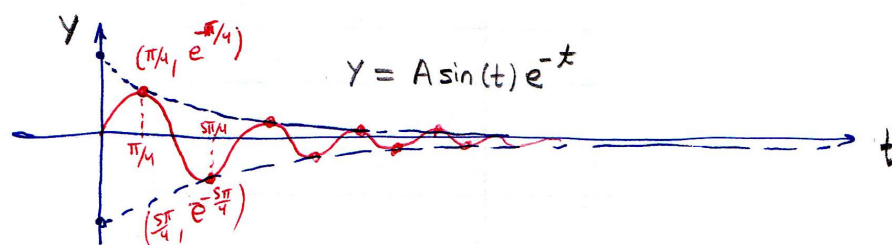
$$\tan(t) = \frac{\sin t}{\cos t} = 1 \text{ when graphs match.}$$

$$\begin{aligned} y'' &= -A e^{-t} (\cos t - \sin t) + A e^{-t} (-\sin t - \cos t) \\ &= A e^{-t} (\sin t - \cos t - \sin t - \cos t) \\ &= A e^{-t} (-2 \cos t) \\ &= -A e^{-t} \cos t \end{aligned}$$

$$y''(t) < 0 \text{ for } t = \pi/4, 9\pi/4, \dots \text{ (Maximums)}$$

$$y''(t) > 0 \text{ for } t = 5\pi/4, 13\pi/4, \dots \text{ (Minimums)}$$

So  $y(\pi/4)$  is the absolute maximum and  $y(5\pi/4)$  is the absolute minimum. All subsequent points get smaller & smaller thanks to  $e^{-t}$ .





## 5.5. LIMITS AT INFINITY

The behavior a function at very very very... very big values of  $x$  is captured by the limit of the function at  $\infty$ . Now infinity is not a number so such a limit has not yet been discussed. In Chapter 3 of these notes we learned how to calculate the limit of a function at some finite point. Sometimes the output of the limit turned out to be  $\pm\infty$ , now we turn our attention to the case that the argument of the function tends toward a very big input or a very large negative input.

**Definition 5.5.1:**

The limit at  $\infty$  for a function  $f$  is  $L \in \mathbb{R}$  if the values  $f(x)$  get arbitrarily close to  $L$  as the input  $x$  gets arbitrarily large. We write

$$\lim_{x \rightarrow \infty} f(x) = L$$

in this case. To be more precise we should say that  $\lim_{x \rightarrow \infty} f(x) = L$  if for each  $\epsilon > 0$  there exists  $N > 0$  such that if  $x > N$  then  $|f(x) - L| < \epsilon$ .

The precise definition essentially says that if we pick a band of width  $2\epsilon$  about the line  $y = L$  then for points to the right of  $x = N$  the graph fits inside the band. Stewart has nice pictures of this, go look at them in section 4.4 pg. 238.

**Example 5.5.1:**

Let  $f(x) = \frac{1}{x}$ . Calculate the limit of  $f(x)$  at  $\infty$ . Observe that,

$$f(10) = 0.1, \quad f(100) = 0.01, \quad f(1000) = 0.001.$$

We see that the values of the function are getting closer and closer to zero as  $x$  gets larger and larger. This leads us to conclude,

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

In other words, if we divide something nonzero by a very big number then we get something very small. This sort of limit is not ambiguous, to determine the answer we either need to think about a table of values or perhaps a graph.

Or if you want to be picky you can argue as follows: Let  $\epsilon > 0$  choose  $N = 1/\epsilon$  and observe that for  $x > N = 1/\epsilon$  we have that  $1/x < \epsilon$ .

Consequently,  $|f(x) - 0| = |1/x| = 1/x < \epsilon$ . Hence by the precise definition

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

We should also define limits at minus infinity.

**Definition 5.5.2:**

The limit at  $-\infty$  for a function  $f$  is  $L \in \mathbb{R}$  if the values  $f(x)$  get arbitrarily close to  $L$  as the input  $x$  gets arbitrarily negative and large. We write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

in this case. To be more precise we should say that  $\lim_{x \rightarrow -\infty} f(x) = L$  if for each  $\epsilon > 0$  there exists  $M < 0$  such that if  $x < M$  then  $|f(x) - L| < \epsilon$ .

Let's look at a simple example:

**Example 5.5.2:**

Let  $f(x) = \frac{1}{x}$ . Calculate the limit of  $f(x)$  at  $-\infty$ . Observe that,

$$f(-10) = -0.1, \quad f(-100) = -0.01, \quad f(-1000) = -0.001.$$

We see that the values of the function are getting closer and closer to zero as  $x$  gets larger and negative. This leads us to conclude,

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

In other words, if we divide something nonzero by a very big negative number then we get something very small and negative. This sort of limit is not ambiguous, to determine the answer we either need to think about a table of values or perhaps a graph.

Or if you want to be picky you can argue as follows: Let  $\epsilon > 0$  choose  $M = -1/\epsilon$  and observe that for  $x < M = -1/\epsilon$  we have that  $-1/x < \epsilon$  and clearly  $x < 0$  since  $x < -1/\epsilon < 0$  so  $|x| = -x$  thus:

$$|f(x) - 0| = |1/x| = -1/x < \epsilon$$

Hence by the precise definition

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

I do not require you to be proficient in picky  $\epsilon$  type arguments, I include them here just in case the common sense arguments fail to convince you. Let us consider an example which we will rely on in later examples quite frequently.

**Example 5.5.3:**

Let  $f(x) = 1/x^n$  where  $n > 0$ . Calculate the limit of  $f(x)$  at  $\infty$ . Observe that,

$$f(10) = 1/10^n, \quad f(100) = 1/100^n, \quad f(1000) = 1/1000^n.$$

We see that the values of the function are getting closer and closer to zero as  $x$  gets larger and larger. This leads us to conclude,

$$\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0$$

In other words, if we divide something nonzero by a very big number then we get something very small. This sort of limit is not ambiguous, to determine the answer we either need to think about a table of values or perhaps a graph.

Let  $\epsilon > 0$  choose  $N = 1/\epsilon^{\frac{1}{n}}$ . Suppose  $x > N = 1/\epsilon^{\frac{1}{n}}$  thus  $1/x < \epsilon^{\frac{1}{n}}$  which implies  $1/x^n < (\epsilon^{\frac{1}{n}})^n = \epsilon$ . Consider then, for  $x > N$

$$|f(x) - 0| = |1/x^n| = 1/x^n < \epsilon$$

Therefore by the precise definition for limits at infinity,

$$\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0.$$

The graphical significance of all three examples thus far considered is that the function has a horizontal asymptote of  $y = 0$  at infinity or minus infinity.

**Definition 5.5.3:**

If  $\lim_{x \rightarrow \infty} f(x) = L \in \mathbb{R}$  if then the function is said to have a horizontal asymptote of  $y = L$ . Likewise, if  $\lim_{x \rightarrow -\infty} f(x) = L \in \mathbb{R}$  if then the function is said to have a horizontal asymptote of  $y = L$ .

**Example 5.5.4:**

Let  $f(x) = \tan^{-1}(x)$ . We saw back in section 2.4.5 page 17 of these notes that the inverse tangent has horizontal asymptotes of  $y = \pi/2$  and  $y = -\pi/2$ . (Those facts follow from the graph which can be obtained from flipping  $y = \tan(x)$  about the line  $y = x$ ). We find that

$$\lim_{x \rightarrow \infty} (\tan^{-1}(x)) = \frac{\pi}{2} \quad \lim_{x \rightarrow -\infty} (\tan^{-1}(x)) = \frac{-\pi}{2}.$$

Vertical asymptotes of the function correspond to horizontal asymptotes for the inverse function. Challenge: what are the horizontal asymptotes of  $y = \tan^{-1}(3x)$ ? We can also discuss limits which go to infinity at infinity.

**Definition 5.5.4:**

The limit at  $\infty$  for a function  $f$  is  $\infty$  if the values  $f(x)$  get arbitrarily large as the input  $x$  gets arbitrarily large. We write

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

in this case. Likewise, the limit at  $-\infty$  for a function  $f$  is  $\infty$  if the values  $f(x)$  get arbitrarily large as the input  $x$  gets arbitrarily large and negative. We write

$$\lim_{x \rightarrow -\infty} f(x) = \infty$$

Finally, if the function takes on increasingly large negative values as the input becomes increasing large or negative then we write

$$\lim_{x \rightarrow -\infty} f(x) = \infty \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = -\infty$$

These definitions are very natural, let's examine a number of easy cases. I will just state the answers in this example since they are derived from plain-old common sense for the most part. As before we will have to show more work for the limits which are indeterminate, we discuss those trickier examples later in this section.

**Example 5.5.5:** the last three columns we dealt with earlier but I thought it might be instructive to include those here as well just to make sure you understand the difference between limits which go to infinity and those which are taken at infinity.

$$\begin{array}{ccccc} \lim_{x \rightarrow \infty} \frac{1}{x} = 0 & \lim_{x \rightarrow -\infty} \frac{1}{x} = 0 & \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty & \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty & \lim_{x \rightarrow 0} \frac{1}{x} = dne \\ \lim_{x \rightarrow \infty} \frac{1}{x^2} = 0 & \lim_{x \rightarrow -\infty} \frac{1}{x^2} = 0 & \lim_{x \rightarrow 0^+} \frac{1}{x^2} = \infty & \lim_{x \rightarrow 0^-} \frac{1}{x^2} = \infty & \lim_{x \rightarrow 0} \frac{1}{x} = \infty \\ \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0 & \lim_{x \rightarrow -\infty} \frac{1}{\sqrt{x}} = ? & \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{x}} = \infty & \lim_{x \rightarrow 0^-} \frac{1}{\sqrt{x}} = ? & \lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} = dne \\ \lim_{x \rightarrow \infty} \sqrt{x} = \infty & \lim_{x \rightarrow -\infty} \sqrt{x} = ? & \lim_{x \rightarrow 0^+} \sqrt{x} = 0 & \lim_{x \rightarrow 0^-} \sqrt{x} = ? & \lim_{x \rightarrow 0} \sqrt{x} = dne \\ \lim_{x \rightarrow \infty} x^2 = \infty & \lim_{x \rightarrow -\infty} x^2 = \infty & \lim_{x \rightarrow 0^+} x^2 = 0 & \lim_{x \rightarrow 0^-} x^2 = 0 & \lim_{x \rightarrow 0} x^2 = 0 \\ \lim_{x \rightarrow \infty} x^3 = \infty & \lim_{x \rightarrow -\infty} x^3 = -\infty & \lim_{x \rightarrow 0^+} x^3 = 0 & \lim_{x \rightarrow 0^-} x^3 = 0 & \lim_{x \rightarrow 0} x^3 = 0 \end{array}$$

I have used “?” instead of d.n.e. in a few places just to make it fit. Those limits are taken at a limit point which is not in the domain of the function, in some cases not even on the boundary of the function. If we can't take values close to the limit point then by default the limit is said to not exist, in which case we use d.n.e. as a shorthand (or just for this example “?”)

We can also have limits which fail to exist at plus or minus infinity due to oscillation. All of the functions in the next example fall into that category.

**Example 5.5.6:** the following limits all involve cyclic functions. They never settle down to one value for large positive or negative input values so the limits d.n.e.

$$\begin{array}{lll} \lim_{x \rightarrow \infty} \sin(x) = d.n.e. & \lim_{x \rightarrow \infty} \cos(x) = d.n.e. & \lim_{x \rightarrow \infty} \tan(x) = d.n.e. \\ \lim_{x \rightarrow -\infty} \sin(x) = d.n.e. & \lim_{x \rightarrow -\infty} \cos(x) = d.n.e. & \lim_{x \rightarrow -\infty} \tan(x) = d.n.e. \\ \lim_{x \rightarrow \infty} \csc(x) = d.n.e. & \lim_{x \rightarrow \infty} \sec(x) = d.n.e. & \lim_{x \rightarrow \infty} \sec(x) = d.n.e. \\ \lim_{x \rightarrow -\infty} \csc(x) = d.n.e. & \lim_{x \rightarrow -\infty} \sec(x) = d.n.e. & \lim_{x \rightarrow -\infty} \sec(x) = d.n.e. \end{array}$$

I should probably not neglect the other common elementary functions which we encounter often in examples. These limits I would consider basic, I recall these on the basis of the graph of the function and a little common sense.

**Example 5.5.7:** return to Chapter 2 if you are rusty on the graphs and basic properties of these functions. The interplay between a function and its inverse is especially enlightening for  $\ln(x)$ ,  $\sin^{-1}(x)$ ,  $\cos^{-1}(x)$ .

$$\begin{array}{lll} \lim_{x \rightarrow \infty} \sin^{-1}(x) = d.n.e. & \lim_{x \rightarrow \infty} \cos^{-1}(x) = d.n.e. & \lim_{x \rightarrow \infty} \tan^{-1}(x) = \pi/2 \\ \lim_{x \rightarrow -\infty} \sin^{-1}(x) = d.n.e. & \lim_{x \rightarrow -\infty} \cos^{-1}(x) = d.n.e. & \lim_{x \rightarrow -\infty} \tan^{-1}(x) = -\pi/2 \\ \lim_{x \rightarrow \infty} e^x = \infty & \lim_{x \rightarrow \infty} e^{-x} = 0 & \lim_{x \rightarrow \infty} (1/2)^x = 0 \\ \lim_{x \rightarrow -\infty} e^x = 0 & \lim_{x \rightarrow -\infty} e^{-x} = \infty & \lim_{x \rightarrow -\infty} (1/2)^x = \infty \end{array}$$

The domain of  $\sin^{-1}(x)$  and  $\cos^{-1}(x)$  will be the range of sine and cosine respectively; that is  $\text{dom}(\sin^{-1}(x)) = [-1, 1]$  and  $\text{dom}(\cos^{-1}(x)) = [-1, 1]$  so clearly the limits at plus and minus infinity are not sensible as  $\sin^{-1}$  and  $\cos^{-1}$  are not even defined at  $\pm 2$ . In contrast the range of the exponential function is all positive real numbers and  $\ln(x)$  is the inverse function of  $e^x$  thus

$$\lim_{x \rightarrow -\infty} \ln(x) = d.n.e. \quad \lim_{x \rightarrow 0^+} \ln(x) = -\infty \quad \lim_{x \rightarrow \infty} \ln(x) = \infty$$

For  $x < 0$  the  $\ln(x)$  is not real, the middle limit you should have thought about in the earlier discussion of limits. The last one is true although an uncritical appraisal of the graph  $y = \ln(x)$  gives the appearance of a horizontal asymptote, but appearances can be deceiving.

I don't believe we defined limits going to infinity precisely before, but it is a simple matter. We say that  $\lim_{x \rightarrow a} f(x) = \infty$  if for each  $M > 0$  there exists  $\delta > 0$  such that if  $0 < |x - a| < \delta$  then  $f(x) \geq M$ . What then does  $\lim_{x \rightarrow \infty} f(x) = \infty$  mean precisely? It

means that for each  $M > 0$  there exists a  $N > 0$  such that for  $x > N$  we find  $f(x) > M$ . Take  $f(x) = \ln(x)$  for example. Let  $M > 0$  and choose  $N = e^M > 0$ . Suppose  $x > N = e^M$  then  $\ln(x) > \ln(e^M) = M$ . Hence the natural log goes to infinity at infinity. The fact that natural log didn't mess up the inequality follows from the observation that  $\frac{d}{dx} \ln(x) = \frac{1}{x} > 0$  so  $\ln(x)$  is an increasing function on  $(0, \infty)$ . **The analysis in this paragraph is not a required topic. I have included it just in case you were curious how we can make these arguments rigorous.**

## Indeterminant Limits at Infinity

Up to this point I have attempted to catalogue the basic results. I'm sure I forgot something important, but I hope these examples give you enough of a basis to do those limits which are unambiguous at plus or minus infinity. There is another category of problems where the limits which are given are not obvious, there is some form of indeterminacy. All the same indeterminant forms (see section 3.3 page 33) arise again and most of the algebraic techniques we used back in section 3.5 will arise again here although perhaps in a slightly altered form.

The good news is that limits at infinity enjoy all the same properties as limits which are taken at a finite limit point, at least in as much as the properties make sense. Of course we can only apply the limit properties when the values of the limit are finite. For example,

$$\lim_{x \rightarrow \infty} (x - 2x) = \lim_{x \rightarrow \infty} (x) + \lim_{x \rightarrow \infty} (-2x) = \infty - \infty$$

is not valid because you might be tempted to cancel and find  $\lim_{x \rightarrow \infty} (x - 2x) = 0$  yet  $\lim_{x \rightarrow \infty} (x - 2x) = \lim_{x \rightarrow \infty} (-x) = -\infty$  is the correct result. So we should only split limits by the limit laws when the subsequent limits are finite. That said, I do admit there are certain cases it doesn't hurt to apply the limit laws even though the limits are infinite. In particular if  $\lim f = \infty$  then  $\lim cf = c \lim f = c\infty$  and if we agree to understand that  $c\infty = \infty$  for  $c > 0$  whereas  $c\infty = -\infty$  if  $c < 0$ . Such statements are dangerous because the reader may be tempted to apply laws of arithmetic to expressions involving  $\infty$  and it's just not that simple.

**Example 5.5.8:** this one is of type  $\infty/\infty$  to begin.

$$\begin{aligned} \lim_{x \rightarrow \infty} \left( \frac{x+3}{x-2} \right) &= \lim_{x \rightarrow \infty} \left( \frac{\frac{x}{x} + \frac{3}{x}}{\frac{x}{x} - \frac{2}{x}} \right) && \text{divided top and bottom by } x \\ &= \lim_{x \rightarrow \infty} \left( \frac{1+0}{1-0} \right) && c/x \rightarrow 0 \text{ as } x \rightarrow \infty \\ &= 1. \end{aligned}$$

Here is another example of the same type.

**Example 5.5.9:** this one is of type  $\infty/\infty$  to begin.

$$\begin{aligned}\lim_{x \rightarrow \infty} \left( \frac{x^3 + 3x - 2}{x^4 - 2x + 1} \right) &= \lim_{x \rightarrow \infty} \left( \frac{\frac{1}{x} + \frac{3}{x^3} - \frac{2}{x^4}}{1 - \frac{2}{x^3} + \frac{1}{x^4}} \right) && \text{divided top and bottom by } x^4 \\ &= \lim_{x \rightarrow \infty} \left( \frac{0 + 0 - 0}{1 - 0 + 0} \right) && \text{for } n = 1, 2, 4, c/x^n \rightarrow 0 \text{ as } x \rightarrow \infty \\ &= 0.\end{aligned}$$

Please sir may I have another? Yes.

**Example 5.5.10:** this one is of type  $\infty/\infty$  to begin.

$$\begin{aligned}\lim_{x \rightarrow -\infty} \left( \frac{x^3 + 3x - 2}{x^2 - x + 7} \right) &= \lim_{x \rightarrow -\infty} \left( \frac{x + \frac{3}{x} - \frac{2}{x^2}}{1 - \frac{2}{x} + \frac{7}{x^2}} \right) && \text{divided top and bottom by } x^4 \\ &= \lim_{x \rightarrow -\infty} \left( \frac{x}{1} \right) && \text{for } n = 1, 2, c/x^n \rightarrow 0 \text{ as } x \rightarrow -\infty \\ &= -\infty.\end{aligned}$$

Another way of thinking about this one is to put in very big negative values of  $x$ . For example, when  $x = -1000$  we find

$$\frac{x^3 + 3x - 2}{x^2 - x + 7} = \frac{1000^3 + 3000 - 2}{1000^2 - 1000 - 2} \approx \frac{1000^3}{1000^2} = 1000 = x$$

This sort of reasoning is a good method to try if you are lost as to what algebraic step to apply. There are problems which no amount of algebra will fix, sometimes considering numerical evidence is the best way to figure out a limit.

**Example 5.5.11:** this one is of type  $\infty/\infty$  to begin.

$$\begin{aligned}\lim_{x \rightarrow \infty} \left( \frac{\sqrt{2x^4 + 3x - 2}}{x^2 - x + 7} \right) &= \lim_{x \rightarrow \infty} \left( \frac{\frac{1}{x^2} \sqrt{2x^4 + 3x - 2}}{\frac{1}{x^2} (x^2 - x + 7)} \right) \\ &= \lim_{x \rightarrow \infty} \left( \frac{\sqrt{\frac{2x^4 + 3x - 2}{x^4}}}{1 - \frac{1}{x} + \frac{7}{x^2}} \right) \\ &= \lim_{x \rightarrow \infty} \left( \frac{\sqrt{2 + \frac{3}{x^3} - \frac{2}{x^4}}}{1 - \frac{1}{x} + \frac{7}{x^2}} \right) \\ &= \sqrt{2}\end{aligned}$$

**Example 5.5.12:** this one is of type  $0 \cdot \infty$  to begin.

$$\begin{aligned}\lim_{x \rightarrow \infty} (e^{-x} 2^x) &= \lim_{x \rightarrow \infty} (e^{\ln(2^x)} e^{-x}) && \text{sneaky step} \\ &= \lim_{x \rightarrow \infty} (e^{x \ln(2)} e^{-x}) \\ &= \lim_{x \rightarrow \infty} (e^{x(\ln(2)-1)}) \\ &= 0.\end{aligned}$$

In the last step I noticed  $\ln(2) - 1 \approx 0.692 - 1 < 0$  thus the limit amounts to the exponential function evaluated at ever increasing large negative values, this gives zero. This example really belongs in the section with L'Hopital's Rule, I

We see that limits of type  $\infty/\infty$  can result in many different final answers depending on how the indeterminacy is resolved. The next example is more general, I think it is healthy to think about something a little more abstract from time to time. The strategy used is essentially identical other examples'.



**Example 5.5.13:** let  $P$  be a polynomial of degree  $p$  and let  $Q$  be a polynomial of degree  $q$ . This means there exist real coefficients  $a_p, a_{p-1}, \dots, a_1, a_0$  and  $b_q, b_{q-1}, \dots, b_1, b_0$  such that  $a_p \neq 0$  and  $b_q \neq 0$  where

$$P(x) = a_p x^p + \dots + a_1 x + a_0 \quad Q(x) = b_q x^q + \dots + b_1 x + b_0$$

Consider  $f(x) = P(x)/Q(x)$ . There are three cases. Let's begin with  $p > q$  so  $p - q > 0$

$$\begin{aligned} \lim_{x \rightarrow \infty} \left( \frac{P(x)}{Q(x)} \right) &= \lim_{x \rightarrow \infty} \left( \frac{a_p x^p + \dots + a_1 x + a_0}{b_q x^q + \dots + b_1 x + b_0} \right) \\ &= \lim_{x \rightarrow \infty} \left( \frac{a_p x^{p-q} + \dots + \frac{a_1}{x^{q-1}} + \frac{a_0}{x^q}}{b_q + \dots + \frac{b_1}{x^{q-1}} + \frac{b_0}{x^q}} \right) \\ &= \lim_{x \rightarrow \infty} \left( \frac{a_p}{b_q} x^{p-q} + \dots + \frac{a_1}{b_q x^{q-1}} + \frac{a_0}{b_q x^q} \right) \\ &= \pm \infty. \end{aligned}$$

If  $a_p/b_q > 0$  then we get  $+\infty$  whereas if  $a_p/b_q < 0$  we obtain  $-\infty$ . The next case is that the denominator has a larger degree, that is to say  $p < q$  thus  $q - p > 0$

$$\begin{aligned} \lim_{x \rightarrow \infty} \left( \frac{P(x)}{Q(x)} \right) &= \lim_{x \rightarrow \infty} \left( \frac{a_p x^p + \dots + a_1 x + a_0}{b_q x^q + \dots + b_1 x + b_0} \right) \\ &= \lim_{x \rightarrow \infty} \left( \frac{a_p x^{p-q} + \dots + \frac{a_1}{x^{q-1}} + \frac{a_0}{x^q}}{b_q + \dots + \frac{b_1}{x^{q-1}} + \frac{b_0}{x^q}} \right) \\ &= \lim_{x \rightarrow \infty} \left( \frac{a_p}{b_q} \frac{1}{x^{q-p}} + \dots + \frac{a_1}{b_q x^{q-1}} + \frac{a_0}{b_q x^q} \right) \\ &= 0. \end{aligned}$$

Finally it could be that the numerator and denominator have equal degree;  $p = q$

$$\begin{aligned} \lim_{x \rightarrow \infty} \left( \frac{P(x)}{Q(x)} \right) &= \lim_{x \rightarrow \infty} \left( \frac{a_p x^p + \dots + a_1 x + a_0}{b_q x^q + \dots + b_1 x + b_0} \right) \\ &= \lim_{x \rightarrow \infty} \left( \frac{a_p x^{p-q} + \dots + \frac{a_1}{x^{q-1}} + \frac{a_0}{x^q}}{b_q + \dots + \frac{b_1}{x^{q-1}} + \frac{b_0}{x^q}} \right) \\ &= \lim_{x \rightarrow \infty} \left( \frac{a_p}{b_q} + \dots + \frac{a_1}{b_q x^{q-1}} + \frac{a_0}{b_q x^q} \right) \\ &= \frac{a_p}{b_q}. \end{aligned}$$

In each case my goal was to simplify the denominator so I could focus on the behavior of the numerator. Very similar arguments will work for  $x \rightarrow -\infty$ .

**Example 5.5.14:** we can throw away a bounded function in a sum when the other function in the sum is unbounded, here are two examples of this idea in action:

$$\begin{aligned}\lim_{x \rightarrow \infty} (\sin(x) + e^x) &= \lim_{x \rightarrow \infty} (e^x) = \infty \\ \lim_{x \rightarrow -\infty} (x + 2) &= \lim_{x \rightarrow -\infty} (x) = -\infty\end{aligned}$$

**Example 5.5.15:** if we take a function  $f(x)$  with a known limit of  $L \in \mathbb{R}$  or  $\pm\infty$  as  $x \rightarrow \pm\infty$  then the limit of  $f(x + a)$  is the same for  $x \rightarrow \pm\infty$ . For example,

$$\begin{aligned}\lim_{x \rightarrow \infty} (e^x) = \infty &\implies \lim_{x \rightarrow \infty} (e^{x+3}) = \infty \\ \lim_{x \rightarrow -\infty} \left(\frac{1}{x^2}\right) = 0 &\implies \lim_{x \rightarrow -\infty} \left(\frac{1}{(x-7)^2}\right) = 0 \\ \lim_{x \rightarrow \infty} (\tan^{-1}(x)) = \frac{\pi}{2} &\implies \lim_{x \rightarrow \infty} (\tan^{-1}(x+2)) = \frac{\pi}{2}\end{aligned}$$

**Example 5.5.16:** in a contest between power functions the largest degree wins.

$$\begin{aligned}\lim_{x \rightarrow \infty} (x^3 - x^2) &= \lim_{x \rightarrow \infty} (x^3) = \infty. \\ \lim_{x \rightarrow \infty} (x^3 - x^4) &= \lim_{x \rightarrow \infty} (-x^4) = -\infty.\end{aligned}$$

On the other hand the exponential function will win against a polynomial because eventually the exponential function's values will totally dwarf the power function's values.

$$\begin{aligned}\lim_{x \rightarrow \infty} (x^3 - e^x) &= \lim_{x \rightarrow \infty} (e^x) = \infty. \\ \lim_{x \rightarrow -\infty} (2^{-x} + x) &= \lim_{x \rightarrow -\infty} (2^{-x}) = \infty.\end{aligned}$$

You might ask why? *Well, I challenge you to prove these claims for a bonus point.*

**Example 5.5.17:** that “rationalization” idea comes up again here:

$$\begin{aligned}\lim_{x \rightarrow \infty} (x - \sqrt{x}) &= \lim_{x \rightarrow \infty} \left( \frac{x + \sqrt{x}}{x + \sqrt{x}} (x - \sqrt{x}) \right) \\ &= \lim_{x \rightarrow \infty} \left( \frac{x^2 - x}{x + \sqrt{x}} \right) \\ &= \lim_{x \rightarrow \infty} \left( \frac{x - 1}{1 + \frac{\sqrt{x}}{x}} \right) \\ &= \lim_{x \rightarrow \infty} \left( \frac{x - 1}{1 + \sqrt{1/x}} \right) \\ &= \lim_{x \rightarrow \infty} (x) \\ &= \infty.\end{aligned}$$

**Example 5.5.18:** that “rationalization” idea comes up again here: to begin the limit is of type  $-\infty + \infty$  but as you can see below the  $-\infty$  wins in the end.

$$\begin{aligned}
 \lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 4x}) &= \lim_{x \rightarrow -\infty} \left( \frac{x - \sqrt{x^2 + 4x}}{x - \sqrt{x^2 + 4x}} (x + \sqrt{x^2 + 4x}) \right) \\
 &= \lim_{x \rightarrow -\infty} \left( \frac{x^2 - x^2 - 4x}{x - \sqrt{x^2 + 4x}} \right) \\
 &= \lim_{x \rightarrow -\infty} \left( \frac{-4x}{2x - \sqrt{x^2 + 4x}} \right) \\
 &= \lim_{x \rightarrow -\infty} \left( \frac{-4}{1 - \sqrt{1 + 4/x}} \right) \\
 &= -\infty.
 \end{aligned}$$

That need not be the case, consider a similar problem that looks about the same on the surface but works out quite differently:

$$\begin{aligned}
 \lim_{x \rightarrow -\infty} (x - \sqrt{2x^2 + 4x}) &= \lim_{x \rightarrow -\infty} \left( \frac{x + \sqrt{2x^2 + 4x}}{x + \sqrt{2x^2 + 4x}} (x - \sqrt{2x^2 + 4x}) \right) \\
 &= \lim_{x \rightarrow -\infty} \left( \frac{x^2 - 2x^2 - 4x}{x + \sqrt{2x^2 + 4x}} \right) \\
 &= \lim_{x \rightarrow -\infty} \left( \frac{-x^2 - 4x}{x + \sqrt{2x^2 + 4x}} \right) \\
 &= \lim_{x \rightarrow -\infty} \left( \frac{-x - 4/x}{1 + \sqrt{2 + 4/x}} \right) \\
 &= \infty.
 \end{aligned}$$

**Example 5.5.19:** when dealing with square roots it is important that you remember that the laws of exponents indicate  $\frac{1}{x} \sqrt{a + b} = \sqrt{\frac{1}{x^2}(a + b)}$ . We assume that  $a, c > 0$  in this problem. Consider,

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \left( \frac{\sqrt{ax^2 + bx + c}}{\sqrt{cx^2 + dx + e}} \right) &= \lim_{x \rightarrow \infty} \left( \frac{\frac{1}{x} \sqrt{ax^2 + bx + c}}{\frac{1}{x} \sqrt{cx^2 + dx + e}} \right) \\
 &= \lim_{x \rightarrow \infty} \left( \frac{\sqrt{\frac{ax^2 + bx + c}{x^2}}}{\sqrt{\frac{cx^2 + dx + e}{x^2}}} \right) \\
 &= \lim_{x \rightarrow \infty} \left( \frac{\sqrt{a + b/x + c/x^2}}{\sqrt{c + d/x + e/x^2}} \right) \\
 &= \sqrt{\frac{a}{c}}.
 \end{aligned}$$

**Example 5.5.20:** The Squeeze Theorem applies to limits at  $\pm\infty$ . Suppose we are given a function  $f$  such that

$$\frac{2}{\pi} \tan^{-1}(x) \leq f(x) \leq \frac{\sqrt{4x^2 + 1}}{x - 3}$$

for all  $x \geq 1,000,000$ . We can calculate the limit at  $\infty$  via the Squeeze Theorem. Observe that

$$\begin{aligned} \lim_{x \rightarrow \infty} \left( \frac{2}{\pi} \tan^{-1}(x) \right) &= \frac{2}{\pi} \cdot \frac{\pi}{2} = 1 \\ \lim_{x \rightarrow \infty} \left( \frac{\sqrt{4x^2 + 1}}{2x - 3} \right) &= \lim_{x \rightarrow \infty} \left( \frac{\sqrt{4 + 1/x^2}}{2 - 3/x} \right) = \sqrt{4}/2 = 1. \end{aligned}$$

Therefore, by the Squeeze Theorem,

$$\lim_{x \rightarrow \infty} f(x) = 1.$$

### **Trading limits at infinity for limits at zero:**

Notice that if  $x = 1/t$  then  $t = 1/x$  it stands to reason that if  $x \rightarrow \infty$  then  $t \rightarrow 0^+$ . We can make the following substitution with that exchange in mind :

$$\boxed{\lim_{x \rightarrow \infty} f(x) = \lim_{t \rightarrow 0^+} f(1/t)}$$

Homework problems 29 and 30 of section 4.4 of Stewart's 6<sup>th</sup>-Ed. are based on this exchange of limits. It is rather neat that the infinitesimal and the infinite are linked together in this way. I spent about an hour trying to get the following example to be pretty to no avail. I can do it with the technique of L'Hopital's Rule.

**Example 5.5.21:** the infinite limit view of  $e$ . Consider the following limit:

$$\lim_{x \rightarrow \infty} \left( 1 + \frac{1}{x} \right)^x = e$$

If you can show that this definition is compatible with our previous implicit definition

$$\lim_{h \rightarrow 0} \left( \frac{e^h - 1}{h} \right) = 1$$

for  $e$  I will award 3 bonus points if you can do it without using L'Hopital's