10. INTRODUCTION TO DIFFERENTIAL EQUATIONS

What is a differential equation? It is an equation which involves derivatives. Differential equations are equations which relate the changes in various quantities. The natural world is filled with dynamic quantities, they depend on time. Often a differential equation will model how those quantities change with time.

The change need not be just with respect to time, we might consider something which varies as a result of time and space varying. For example, an electric or magnetic field components are functions of x,y,z and t. The differential equations which govern the electric and magnetic fields are known as Maxwell's Equations, these are *partial differential equations* (PDEs). In particular if $\vec{B} =< B_x, B_y, B_z >$ then

$$\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0.$$

Another equation you might encounter in physics is the conservative force equation. A force $\vec{F}$ is called conservative if there exists a potential function U such that $\vec{F} = -\nabla U$. In particular, if the force $\vec{F} =< F_x, F_y, F_z >$ then the potential energy function U must satisfy three PDEs:

$$\vec{F} = -\nabla U \quad \Longleftrightarrow \quad \vec{F}_x = -\frac{\partial U}{\partial x},\ \vec{F}_y = -\frac{\partial U}{\partial y},\ \vec{F}_z = -\frac{\partial U}{\partial z}.$$

We will consider how to solve this sort of equation in calculus III. More general PDEs are treated in part in the differential equations course and beyond that there is an endless supply. Some mathematicians spend a whole career just studying one special PDE, the wealth of behavior contained in a simple equation is staggering.

In this course, we study the most basic type of differential equation, the *ordinary differential equation* (ODE). An ODE has one independent and one dependent variable. Sometimes we use "x" for the independent variable, in other situations we use "t" for time. The dependent variable is usually taken to be "y" but it is also taken to be "x" (but not at the same time that "x" is the independent variable).

$$1.)\ \frac{dy}{dx} = x^2 + y + \sin(x) \qquad \text{independent variable } x, \text{ dependent variable } y$$

$$2.)\ \frac{dx}{dt} = x^2 + t + 3 \qquad \text{independent variable } t, \text{ dependent variable } x$$

$$3.)\ \frac{ds}{d\theta} = se^{\theta} \qquad \text{independent variable } \theta, \text{ dependent variable } s$$

The above are *first order differential equations* because the highest derivative that appears is the first derivative. If both the dependent variable and its derivatives appear linearly then the ODEqn is said to be linear. Equations 1.) and 3.) are linear, but the appearance of x^2 ruins it in 2.). Homogeneous linear DEqns are especially nice since the sum of solutions is again a solution..

If the highest derivative that appears in the DEqn is the second derivative then we say the DEqn is a *second order differential equation*. If the highest order that appears in the DEqn is n-th order derivative then the DEqn is said to be an *n-th order differential equation*. If the differential equation can be written as a sum of the dependent variable and its derivatives set to zero without the independent variable appearing on its own then the DEqn is said to be homogeneous, otherwise the DEqn is said to be nonhomogeneous. If the differential equation can be written as a linear combination of the dependent variable and its derivatives such that the coefficients in the sum are just numbers then it is said to be a constant coefficient DEqn.

4.) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$ second order, linear, constant coeff., homogeneous

5.) $y''' + y' + t^2 y = t$ third order, linear, nonhomogeneous

6.) $y^{(n)}(t) = y$ n-th order, linear, constant coeff., homogeneous

Ok, our vocabulary lesson is over now. Some examples in this chapter are inspired by homework problems in the excellent DEqns text by Nagle Saff and Snider.

$\dfrac{dy}{dx} = \dfrac{y(2-3x)}{x(1-3y)}$: 1st order nonlinear ODE, dependent var. = y, indep. var. = x

$x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0$: 2nd order linear ODE, dep. var. = y, indep. var. = x.

$\dfrac{\partial N}{\partial t} = \dfrac{\partial^2 N}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial N}{\partial r} + kN$: 2nd order PDE

10.1. WHAT IS A SOLUTION?

The question that titles this section begs another question; **"to what?"** We already know what the solution to an arithmetic problem is, it's a number. Or what is the solution to most algebra problems? Also a number, perhaps several. For example, $x^2 - 5x + 6 = (x-2)(x-3) = 0$ has solutions $x = 2$ and $x = 3$.

We mean to ask the question now, **"what is the solution to a differential equation?".**

- *A **explicit solution** to a differential equation is a function which satisfies the rule of the differential equation. In other words, it a function which works when substituted into the differential equation. Symbolically, if the differential equation (*) has the form $F(x, y, y', y'', \ldots, y^n(x)) = 0$ then f is a solution of (*) if and only if $F(x, f(x), f'(x), f''(x), \ldots, f^{(n)}(x)) = 0$*

- *An **implicit solution*** is some equation which satisfies the differential equation. Unless said otherwise when I say solution I mean implicit solution, but we like to find explicit solutions if possible.

- *The **general solution*** allows for all possible initial conditions. Technically speaking, it is not a function, rather it is a whole family of functions.
  .

It probably helps to see a few examples at this point.

Example 10.1.1
Let $f(x) = x^2$ then f is a solution to $\frac{dy}{dx} = 2x$ since $\frac{df}{dx} = 2x$.
Let $g(x) = x$ then g is not a solution to $\frac{dy}{dx} = 2x$ since $\frac{dg}{dx} = 1 \neq 2x$.
We call f an explicit solution since it has the graph $y = x^2$ and you can see that y is an explicit function of x, they're not mixed together.

Example 10.1.2
Let $x^2 + y^2 = 4$ this implicitly defines a solution of $\frac{dy}{dx} = -\frac{x}{y}$ since implicit differentiation of the proposed solution yields $2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$. This is called an implicit solution because we cannot just solve for y as a single function of x. It is only possible to find an explicit solution locally, this is a circle, the upper and lower pieces are separately explicit solutions $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$. The implicit solution contains both of these explicit solutions.

So how do we find solutions? I didn't mention that yet. You can see that in Example 10.1.1 the solution could be found by integration.

$$\frac{dy}{dx} = 2x \quad \Longrightarrow \quad \int \frac{dy}{dx}\,dx = \int 2x\,dx = x^2 + C$$

Select the case $C = 0$ we get $f(x) = x^2$. If you think about it, every time we integrated we solved a differential equation. Think about it:

$$\boxed{\int f(x)\,dx = y \quad \Longleftrightarrow \quad \frac{dy}{dx} = f(x)}$$

The general solution is thus analogous to the indefinite integral. In fact, when we solve an n-th order ODEn it amounts to integrating n-times. It is not surprising then that the general solution will have n-arbitrary constants of integration. When we solve a first order ODEn we get just one constant.

<u>Example 10.1.3:</u>
The general solution to $\frac{dy}{dx} = 2x$ is $y = x^2 + C$. Geometrically, we have a family of parabolas which open upward and differ just by a vertical shift.

<u>Example 10.1.4:</u>
The general solution to $\frac{dy}{dx} = -\frac{x}{y}$ is $x^2 + y^2 = R^2$. Geometrically, we have a family of circles. Example 10.1.2 was just one case. Given the differential equation we would need additional information in order to select a specific solution.

<u>Bad News?</u> Solving differential equations is not just integration in general. The differential equation in Example 10.1.1 was very special. More often than not a given DEqn will not allow us to find an equation of the form $\frac{dy}{dx}$ = stuff in x. Usually the dependent variable and its various derivatives are all jumbled together at once. We need other tricks to unravel the equation and find solutions. I suppose there are nearly as many techniques as there are types of DEqns. For this chapter there will be three main tricks to find general solutions:

- (10.2) Separate variables: $\frac{dy}{dx} = f(x)g(y) \quad \Longrightarrow \int g(y)dy = \int f(x)dx.$

- (10.4) Use Integrating Factor Technique: put in standard form $\frac{dy}{dx} + Py = Q$ Then calculate $\mu = exp\big(\int p\,dx\big)$, multiply by μ, use reverse product rule, separate and integrate.

- (10.5) 2nd order constant coefficient ODE, find quadratic auxillary equation, solve the quadratic, write down general solution.

10.2. SEPARATION OF VARIABLES

This section begins with a short proof of the method. In summary, separating variables is u-substitution. We almost did these last semester, what differs is notation and presentation of the problem. At the conclusion of this section I have a number of examples which show the wide and diverse application of first order ODEs. Let's begin.

Certain 1st order ode's can be seen to have the form

$$\frac{dy}{dx} = g(x)\, f(y)$$

In which case we can seperate the variables

$$\frac{dy}{f(y)} = g(x)\, dx$$

And then integrate

$$\int \frac{dy}{f(y)} = \int g(x)\, dx$$

When you can actually do these integrals this will implicitly (and sometimes explicitly) define y in terms of x; that means you found a sol^{n}!

Pf/ $\displaystyle\int \frac{dy}{f(y)} = \int \frac{g(x) f(y)\, dx}{f(y)}$: Changing variables to x $dy = g(x) f(y)\, dx$ by assumption

$$= \int g(x)\, dx \quad //$$

Example 10.2.1

$\boxed{E1}$ $\dfrac{dy}{dt} = ky \Rightarrow \dfrac{dy}{y} = k\,dt$ then integrate

$\displaystyle\int \frac{dy}{y} = \int k\,dt \Rightarrow \underline{\ln y = kt + c}$ (implicit sol^{n})

Thus $y(t) = e^{kt+c} = e^{c} e^{kt} = \boxed{y_0 e^{kt} = y(t)}$ (explicit sol^{n})

<u>If $y(0) = 3$ find the sol^{n}</u>

$y(0) = y_0 e^{k(0)} = \boxed{y_0 = 3} \Rightarrow \underline{y(t) = 3e^{kt}}$

(This is why y_0 is good notation here)

<u>Remark</u>: $k > 0$ exponential growth eq^{n}. $k < 0$ exponential decay eq^{n} (More on this later)

Example 10.2.2

$\boxed{E2}$ $\dfrac{dY}{dx} = a^{x+Y}$: find sol^{n} thru sep. of variables

$\int a^{-Y} dY = \int a^{x} dx$: seperate then integrate.

$\dfrac{-1}{\ln(a)} a^{-Y} = \dfrac{1}{\ln(a)} a^{x} + \tilde{C}$

$a^{-Y} = -a^{x} - \ln(a)\tilde{C} = C - a^{x}$: want to solve for Y.

$\ln(a^{-Y}) = \ln(C - a^{x})$: it is crucial to insure we are taking the ln of positive quantites! that is why we moved the minus sign to the other side.

$-Y \ln(a) = \ln(C - a^{x})$

$$\boxed{Y = \frac{-\ln(C - a^{x})}{\ln(a)}}$$

Notice that C is arbitrairy and can only be specified if we supply further demands (an intial or boundary condition)

Example 10.2.3

$\boxed{E3}$ $\dfrac{du}{d\theta} = \dfrac{2\theta + \sec^2\theta}{2u}$ find sol^{n} with $u(0) = -5$

$2u\,du = (2\theta + \sec^2\theta)d\theta$: seperated variables, now integrate,

$u^2 = \theta^2 + \tan\theta + C$: an implicit sol^{n}.

$\therefore\ u = \pm\sqrt{\theta^2 + \tan\theta + C}$: an explicit sol^{n}.

$u(0) = \pm\sqrt{C} = -5$

$\Rightarrow -\sqrt{C} = -5$ (Must choose negative sqrt. sol^{n})

$\Rightarrow C = 25$

$$\boxed{u(\theta) = -\sqrt{\theta^2 + \tan\theta + 25}}$$

Example 10.2.4

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$\boxed{E4}$ Find implicit sol$^{\underline{n}}$ to $\frac{dy}{dx} = \frac{\cos(x)}{y^4+y^2+23}$. Also find Equilbrium sol$^{\underline{n}}$'s.

$$\int (y^4 + y^2 + 23)\,dy = \int \cos(x)\,dx$$

$$\boxed{\frac{1}{5}y^5 + \frac{1}{3}y^3 + 23y = \sin(x) + C} \leftarrow$$ this implicitly solves the DEq$^{\underline{n}}$.

Here we cannot simply solve for y as a fnct. of x. The sol$^{\underline{n}}$ is still useful because it describes how x & y are related along the sol$^{\underline{n}}$ curves. Next find Equilbrium Sol$^{\underline{n}}$'s

$$\frac{dy}{dx} = 0 \Rightarrow \frac{\cos(x)}{y^4+y^2+23} = 0$$

$$\Rightarrow \cos(x) = 0$$

$$\Rightarrow \boxed{x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}.}$$

Example 10.2.4b (each time, separate then integrate)

§2.2#7 $\frac{dy}{dx} = \frac{1-x^2}{y^2} \Rightarrow y^2\,dy = (1-x^2)\,dx$

integrating both sides. $\Rightarrow \boxed{y^3/3 = x - x^3/3 + C}$ (implicit sol$^{\underline{n}}$ for DEq$^{\underline{n}}$)

§2.2#8 $\frac{dy}{dx} = \frac{1}{xy^3} \Rightarrow \int y^3\,dy = \int \frac{dx}{x} \Rightarrow \boxed{\frac{1}{4}y^4 = \ln|x| + C}$

§2.2#9 $\frac{dy}{dx} = y(2+\sin(x)) \Rightarrow \int \frac{dy}{y} = \int (2+\sin(x))\,dx \Rightarrow \boxed{\ln|y| = 2x - \cos(x) + C}$

§2.2#10 $\frac{dx}{dt} = 3xt^2 \Rightarrow \int \frac{dx}{x} = \int 3t^2\,dt \Rightarrow \boxed{\ln|x| = t^3 + C}$

Example 10.2.4c (each time, separate then integrate)

§2.2 #11 $\frac{dy}{dx} = \frac{\sec^2(y)}{1+x^2} \Rightarrow \int \frac{dy}{\sec^2(y)} = \int \frac{dx}{1+x^2}$ note $\frac{1}{\sec^2 y} = \cos^2(y)$

$$\int \frac{dy}{\sec^2 y} = \int \cos^2 y \, dy \overset{(*)}{=} \int \tfrac{1}{2}(1+\cos(2y))dy = \tfrac{1}{2}\left(y + \tfrac{1}{2}\sin(2y)\right) + C$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$$

(*) trig identity

Thus we find $\boxed{\frac{1}{2}y + \frac{1}{4}\sin(2y) = \tan^{-1}(x) + C}$

Example 10.2.4d

§2.2 #12 $x\frac{dv}{dx} = \frac{1-4v^2}{3v} \Rightarrow \frac{3v\,dv}{1-4v^2} = \frac{dx}{x}$ then consider,

$$\int \frac{3v\,dv}{1-4v^2} = \int \frac{3\,\frac{-1}{8}du}{u} = -\frac{3}{8}\ln|u| + C = -\frac{3}{8}\ln|1-4v^2| + C$$

$u = 1-4v^2$
$du = -8v\,dv$

thus $\boxed{-\frac{3}{8}\ln|1-4v^2| = \ln|x| + C}$

Example 10.2.4e

§2.2 #20 $\frac{dy}{dx} = \frac{3x^2+4x+2}{2y+1} \Rightarrow (2y+1)dy = (3x^2+4x+2)dx$

$\Rightarrow y^2 + y = x^3 + 2x^2 + 2x + C$

$y(0) = -1 \Rightarrow (-1)^2 - 1 = 0 + 2(0)^2 + 2(0) + C \Rightarrow \boxed{C = 0}$

therefore the sol^n with $y(0) = -1$ is $\boxed{y^2 + y = x^3 + 2x^2 + x}$

Example 10.2.4f

§2.2 #21 $\frac{dy}{dx} = 2\sqrt{y+1}\cos(x) \Rightarrow \int \frac{dy}{\sqrt{y+1}} = \int 2\cos(x)\,dx$

$$\int \frac{dy}{\sqrt{y+1}} = \int \frac{du}{\sqrt{u}} = \frac{u^{\frac{1}{2}}}{1/2} + C = 2\sqrt{y+1} + C$$

letting $u = y+1$
so $du = dy$

$\therefore 2\sqrt{y+1} + C = 2\sin(x)$

We know $y(\pi) = 0 \Rightarrow 2\sqrt{0+1} + C = 2\sin(\pi)$

$\Rightarrow 2 + C = 0 \Rightarrow C = -2$

$\Rightarrow \boxed{2\sqrt{y+1} - 2 = 2\sin(x)}$

Example 10.2.4g

(§2.2 #22) $x^2dx + 2y\,dy = 0 \Rightarrow \int x^2dx = \int -2y\,dy$

thus $\frac{1}{3}x^3 = -y^2 + C$. We know $y(0)=2 \Rightarrow 0 = -4 + C \therefore \underline{C=4}$

Hence $\boxed{\frac{1}{3}x^3 = y^2 + 4}$.

Example 10.2.4h

(§2.2 #23) $\frac{dy}{dx} = 2x\cos^2(y) \Rightarrow \int \frac{dy}{\cos^2 y} = \int 2x\,dx \overset{(*)}{\Rightarrow} \tan(y) = x^2 + C.$

$(*)\left(\frac{1}{\cos^2 y} = \sec^2 y \text{ and } \int \sec^2 y\,dy = \tan(y) + C\right)$ then $y(0) = \frac{\pi}{4} \Rightarrow \underline{\tan(\pi/4) = 0 + C}$

Now $\tan(\pi/4) = 1$ thus $\boxed{\tan(y) = x^2 + 1}$

Example 10.2.4i

(§2.2 #24) $\frac{dy}{dx} = 8x^3e^{-2y} \Rightarrow e^{2y}dy = 8x^3dx$ H ⑥

$\Rightarrow \frac{1}{2}e^{2y} = 2x^4 + C$

$y(1) = 0 \Rightarrow \frac{1}{2}e^{(0)} = 2(1)^4 + C \Rightarrow \frac{1}{2} = 2 + C \Rightarrow \underline{C = -3/2}$

thus $\boxed{\frac{1}{2}e^{2y} = 2x^4 - \frac{3}{2}}$ ok I'll find the explicit sol^{n}

for this one, $e^{2y} = 4x^4 - 3 \Rightarrow 2y = \ln(4x^4 - 3)$

$\Rightarrow \boxed{y = \ln\left(\sqrt{4x^4 - 3}\right)}$.

Example 10.2.4j

(§2.2 #25) $\frac{dy}{dx} = x^2(1+y) \Rightarrow \int \frac{dy}{1+y} = \int x^2dx \Rightarrow \ln|1+y| = \frac{x^3}{3} + C.$

Now $y(0) = 3 \Rightarrow \ln|4| = 0^3/3 + C \therefore C = \ln(4)$ hence

$\boxed{\ln|1+y| = \frac{1}{3}x^3 + \ln(4)}$

Example 10.2.4k

(§2.2 #26) $\sqrt{y}\,dx + (1+x)dy = 0 \Rightarrow \int \frac{dy}{\sqrt{y}} = \int \frac{-dx}{1+x}$

thus integrating yields $2\sqrt{y} = -\ln|1+x| + C$

Then $y(0) = 1 \Rightarrow 2 = -\ln(1) + C = C \therefore \boxed{2\sqrt{y} = -\ln|1+x| + 2}$

Or if you want $\boxed{y = \left[\ln\left(\frac{1}{\sqrt{|1+x|}}\right) + 2\right]^2}$

Example 10.2.5 (I often cover this in calculus I)

$\boxed{E5}$ $mg = ma = m\frac{dv}{dt} = m\frac{dx}{dt}\frac{dv}{dx} = mv\frac{dv}{dx}$

Lets solve $mg = mv\frac{dv}{dx}$ to find velocity as function of x

Cancel m to begin

$$g = v\frac{dv}{dx}$$

$$\int g\,dx = \int v\,dv$$

$$gx = \frac{1}{2}v^2 + C$$

Now if $v(x_0) = v_0$ then $gx_0 = \frac{1}{2}v_0^2 + C \quad \therefore \quad C = gx_0 - \frac{1}{2}v_0^2$

Yielding that,

$$gx = \frac{1}{2}v^2 + gx_0 - \frac{1}{2}v_0^2$$

$$g(x - x_0) = \frac{1}{2}(v^2 - v_0^2) \implies \boxed{v^2 = v_0^2 + 2g(x - x_0)}$$

relates velocity & position w/o reference to time.

Remark: Many physics problems boil down to solving some differential equation. Sometimes it is actually a *system of differential equations*. You've probably heard of Newton's Second Law; $\vec{F} = m\vec{a}$. This is actually a special case where the mass is constant. The general form of the Second Law is $\vec{F} = \frac{d\vec{P}}{dt}$ where the momentum $\vec{P} = m\vec{v}$. In the special case m is constant we find this simplifies to $\vec{F} = m\vec{a}$ since $\frac{d\vec{P}}{dt} = \frac{d}{dt}m\vec{v} = m\frac{dv}{dt} = m\vec{a}$. We'll learn the details of such calculations in calculus III, for now (other than this discussion) we have to study one-dimensional motion because I don't assume you understand the nuts and bolts of vector math. Notice that the Second Law is a system of differential equations since if $\vec{P} =< P_x, P_y, P_z >$ then

$$\begin{aligned}\vec{F} = \frac{d\vec{P}}{dt} &\iff < F_x, F_y, F_z > = < \frac{dP_x}{dt}, \frac{dP_y}{dt}, \frac{dP_z}{dt} > \\ &\iff F_x = \frac{dP_x}{dt},\ F_y = \frac{dP_y}{dt},\ F_z = \frac{dP_z}{dt}.\end{aligned}$$

For example, if $\vec{F} =< 1, t, t^2 >$ then we'd have to solve all three differential equations at once in order to find the momentum: assuming the initial momentum is zero,

$$1 = \frac{dP_x}{dt},\quad t = \frac{dP_y}{dt},\quad t^2 = \frac{dP_z}{dt} \implies P_x = t,\quad P_y = \frac{1}{2}t^2,\quad P_z = \frac{1}{3}t^3$$

This discussion is **not** part of the required material, I include it in the hopes of giving a better context to the other examples.

Example 10.2.6 (an example with variable mass)

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$\boxed{E6}$ The falling Raindrop: Imagine a drop falling then a cloud gathers water as it falls, let $m(t)$ be it's varying mass. Further assume as the drop gets bigger it gathers more & more mass proportionate to it's mass; $\frac{dm}{dt} = km$ for $k > 0$.

$F = ma$ is more generally $F = \frac{dP}{dt}$ when m varies.

$$F = \frac{dP}{dt} = \frac{d}{dt}(mv) = mg \quad \text{(fall's due to gravity)}$$

$$\frac{dm}{dt} v + m\frac{dv}{dt} = mg$$

$$\downarrow$$

$$kmv + m\frac{dv}{dt} = mg$$

$$\frac{dv}{dt} = \frac{mg - kmv}{m} = g - kv$$

$$\therefore \frac{dv}{kv - g} = -dt \Rightarrow \frac{dv}{v - g/k} = -k\,dt$$

Integrate both sides, $\ln|v - g/k| = -kt + \tilde{c}$ and exponentiate,

$$v - g/k = e^{-kt + \tilde{c}} = c e^{-kt}$$

$$\boxed{v(t) = ce^{-kt} + g/k}$$

The terminal velocity would be

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(c\underbrace{e^{-kt}}_{0} - g/k\right) = \boxed{g/k}$$

<u>Remark</u>: We took down as positive direction & Assumed no friction besides the water growth. Physically this amounts to under friction!

$$m\frac{dv}{dt} = ma = mg - kmv$$

When $v = g/k$ we have $mg - km\left(\frac{g}{k}\right) = 0$

terminal velocity happens when the <u>forces balance</u>

Orthogonal Trajectories (OT)

The word "orthogonal" means perpendicular as it applies to Euclidean geometry. To find a trajectory (just another word for a curve) which is perpendicular to a given curve $y = f(x)$ we would want a new curve $y = g(x)$ such that $g'(x) = \frac{-1}{f'(x)}$ wherever the curves interect. If the given curve $y = f(x)$ was a solution to a differential equation then the OT must be a solution to a different (but related) DEqn. Specifically, if $\frac{dy}{dx} = F(x, y)$ has solution f then $\frac{dy}{dx} = \frac{-1}{F(x,y)}$ has solutions which are orthogonal to f.

Example 10.2.7

$\boxed{E7}$ $x^2 + y^2 = R^2$ defines a circle for each $R > 0$. Let's find the orthogonal trajectories to this family of curves, diff. implicitly

$$2x + 2y\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{-x}{y} \quad \text{(for circles)}$$

Now then the O.T must have $\frac{dy}{dx} = \frac{-1}{-x/y} = \frac{y}{x}$ hence

$$\frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln|y| = \ln|x| + c$$
$$\Rightarrow y = e^{\ln(x) + c} = e^{\ln(x)}e^{c} = xe^{c}$$

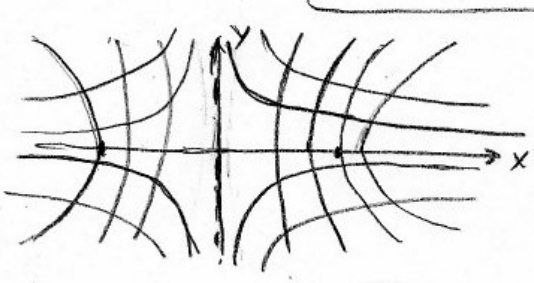
This shows that orthogonal trajectories to circles are lines through the origin.

Example 10.2.8

$\boxed{E8}$ $x^2 - y^2 = k \Rightarrow 2x - 2y\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}$

Hence the O.T. will have $\frac{dy}{dx} = \frac{-1}{x/y} = \frac{-y}{x}$

$$\therefore \frac{dy}{-y} = \frac{dx}{x} \Rightarrow -\ln|y| = \ln|x| + \tilde{c}$$
$$\Rightarrow \ln\left(\frac{1}{|y|}\right) = \ln(\bar{c}x)$$
$$\Rightarrow \frac{1}{|y|} = \bar{c}|x|$$
$$\Rightarrow \boxed{y = \frac{\pm c}{x} \text{ is the O.T.}}$$



$x^2 - y^2 = k$ is a hyperbola

$y = 0 \quad x = \pm k$

$x = 0 \quad y$ d.n.e

Example 10.2.9

E9 Mixing Problems

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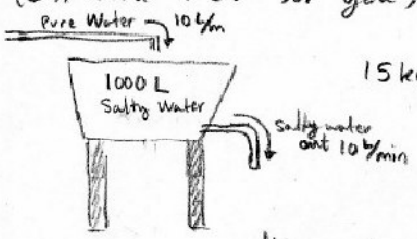
Consider some tank of fixed volume with some substance entering/exiting the tank, let $Y(t)$ be the amount of the substance at time t in the tank.

(I'll work #35 for you)



15 kg of salt at $t = 0$

Let $Y(t)$ = kg of salt in tank at time t

$$\frac{dY}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= 0 - \left(10\,\frac{L}{min}\right)\cdot\left(\frac{Y(t)}{1000L}\right)$$

$$= -\frac{1}{100}Y(t)/min \leftarrow \text{kg/min makes sense}$$

$$= -Y/100$$

Thus $\frac{dY}{Y} = -\frac{dt}{100}$ $\therefore \ln(Y) = -\frac{t}{100} + \tilde{c}$ $\therefore Y(t) = Y_0 e^{-t/100}$

$Y(0) = 15kg = Y_0$ $\therefore$ $\boxed{Y(t) = 15e^{-t/100}\,kg}$

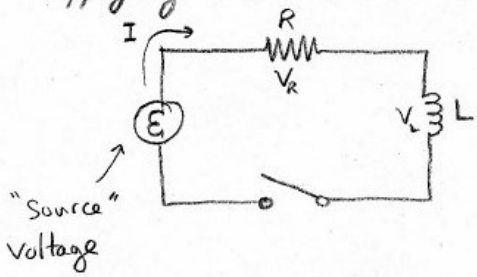
So after 20 minutes $Y(20) = 15e^{-20/100} = \boxed{12.28\,kg \text{ at } t = 20\,min}$

Example 10.2.10 (R is resistance, L is inductance)

(177)

E10 The RL - Circuit

Applying Kirchoff's Rules & a def^{n} or two we find



$$\mathcal{E} = V_R + V_L \quad \text{(Kirchoff)}$$

$$V_L = L\frac{dI}{dt} \quad \text{(Resists change in current)}$$

$$V_R = IR \quad \text{(Ohm's Law)}$$

$$\mathcal{E} = IR + L\frac{dI}{dt}$$

Now if $\mathcal{E}$ = constant and $I(0) = 0$ find $I(t)$,

$$\frac{\mathcal{E} - IR}{L} = \frac{dI}{dt} \quad \therefore \int \frac{dI}{\mathcal{E} - IR} = \int \frac{dt}{L}$$

$$\int \frac{dI}{\mathcal{E} - IR} = \frac{-1}{R}\ln(\mathcal{E} - IR)$$

$$\int \frac{dt}{L} = \frac{t}{L} + C$$

Thus $\frac{-1}{R}\ln(\mathcal{E} - IR) = \frac{t}{L} + \bar{C} \Rightarrow \mathcal{E} - IR = \bar{C}e^{-\frac{R}{L}t}$ that is

$$I = \frac{\mathcal{E}}{R}\left(1 + Ce^{-\frac{R}{L}t}\right)$$

$$I(0) = \frac{\mathcal{E}}{R}(1 + C) = 0 \quad \therefore \quad C = -1$$

$$\therefore \boxed{I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$$

Remark: the limiting current as $t \to \infty$ is $\mathcal{E}/R$. That is physically speaking the inductor is a short circuit for "long" times ($\tau = L/R$ then $5\tau \approx \infty$ pragmatically speaking.)

10.3. EXPONENTIAL GROWTH AND DECAY

We begin by studying several basic growth and decay examples. Then the logistic equation is studied in some depth.

If the growth of a population P is proportional to its size then

$$\frac{dP}{dt} = kP$$

Likewise if the rate of change of Y is proportional to Y

$$\frac{dY}{dt} = kY$$

As discussed in $\boxed{E1}$ of §7.3

$$\boxed{Y(t) = Y_0 e^{kt} \quad \text{where} \quad Y_0 = Y(0).}$$

Likewise $P(t) = P_0 e^{kt}$ where $P_0 = P(0)$. Both follow simply from sep. of variables. By the way:

$$\frac{dP}{dt} = kP \iff \frac{1}{P}\frac{dP}{dt} = k \iff \text{relative growth rate constant}.$$

So $k \equiv$ the relative growth rate.

Example 10.3.1

$\boxed{E1}$ If the population doubles every 10 yrs what is k?

$$P(0) = P_0 \quad \text{and} \quad P(10) = 2P_0 = P_0 e^{10k}$$

$$\Rightarrow \ln(2) = 10k \Rightarrow k = \frac{\ln(2)}{10\,\text{yr.}} = 0.0693\,\frac{1}{\text{yr.}}$$

The relative growth rate is 6.93%.

Exponential growth is hard to maintain for large periods of time. If the growth is exponential then the population will double for a fixed period of time. This means if we let the population grow through 10 doubling periods then the population is increased 1,024 times over. After 20 doubling periods the population will be increased 1,048,576 times over. After 40 doubling periods the population is increased 1,099,511,627,776 times. What does this mean? Exponential population growth is not a perfect model. In practice we can only assume it works over a finite period of time. The logistic equation is a more sophisticated population growth model, it assumes exponential growth for a while but then as the population approaches the so-called carrying capacity the growth slows to zero.

Comment: Over the past several centuries various carrying capacities have been proposed for the human population of earth. Again and again these have been proved incorrect. God has always allowed us to find new technologies which circumvent the doomsday scenario which was supposed to be inevitable. A model is only as good as its assumptions. The trouble with all the models of human population is they fail to acknowledge the fact that the unexpected is to be expected.

Radioactive Decay Models

Radioactive decay is a probabilistic process. The weak force allows certain particles to morph into other particles by the release or absorption of a W or Z boson. Details aside, these radioactive particles are unstable and the number of particles that decay at any time is proportional to the number of unstable particles at that time. This leads to the same mathematics as population growth, however, the "growth rate" is negative.

$\boxed{E2}$ Let $y(t) = m(t)$ be the mass of some radioactive substance then as the mass destabalizes via radiation we have

$$\frac{dm}{dt} = km \qquad (k < 0 \text{ since } m \text{ is decreasing})$$

$$\Rightarrow m(t) = m_0 e^{kt}$$

If balonium has a half-life of 1yr. then what percentage of the balonium is still in the fridge after $\frac{1}{10}$ yr. ?

$$m(1) = \frac{m_0}{2} = m_0 e^{k} \quad \therefore \quad k = \ln(1/2) = \underline{-0.693 = k}$$

$$\therefore \quad m\left(\tfrac{1}{10}\right) = m_0 e^{-0.693\left(\frac{1}{10}\right)} = (0.933)m_0 \Rightarrow \boxed{93.3\% \text{ remains}}$$

Comment: Sometimes scientists try to work this backwards. If you know the amount of a radioactive material and you know the initial abundance of the substance then you can extrapolate backwards and see how old something is. One problem, this assumes we know the initial abundance. How do we "know" such a thing? Personally I'm skeptical of what we "know" about the unrepeatable past. I have much more trust for thoroughly testable physical laws.

Comment Continued:
Please don't misunderstand, radioactive decay is a real observed phenomenon. I think where science may get into trouble is where it tries to extend past what can be tested. The same is true for creationists. We must be careful to not overstate our case. We have no reason to fear science so long as it truly seeks reality. We know the source of reality and we have a meaning for our existence. God is glorified whether or not we can "prove" Him. Of course the proof of God surrounds us every day. The question is do you accept the proofs He offers? Would it be enough that he sent His Son to appear in human form and alter the course of human history? He did that, yet the world still denies the existence of God.

I do think we are called to give a defense for the things we believe. However, I'm afraid sometimes (myself included) we are tricked into being on the defense about historical science which seems to contradict the history in Scripture. Sometimes it may be a better argument to simply say that we don't find the world's creation myth convincing. I don't believe that the universe created itself replete with physical law, logic and the plethora of beautiful mathematics which just happens to mirror nature in unexpected ways.

Truth be told, most of them don't really find their story convincing either. Why would they, it keeps changing. I don't mean to say our story of creation has not changed at all. Certainly as time goes on we may gain a better understanding of the nuts and bolts of the creation process, or perhaps not, I am not convinced one way or the other.

What I do know is that God will be at the center of our story no matter how much more information we gather about the universe. There is still much flexibility in the creationist's viewpoint. I would much rather be burdened with the supposedly troubling problem of assuming the existence of God.

Here is the difference; when we reinterpret our creation story the nature of God does not change. He is always good, just, loving, merciful and He keeps his promises without exception. When the world undertakes a scientific revolution the pure naturalist learns that his god was in fact a false god, nature is something different. I suppose he can put his faith in the ideal of perfectly modeled nature, but does that exist? What are you really trusting in when you put your trust in a changing universe? I choose to trust the unchanging God.

More on Population Models:

THE LOGISTIC EQ^{n} (179)

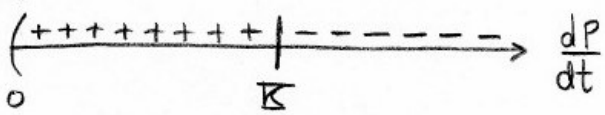
This is another model for population growth, the basic idea is that when the population P is small then $\frac{dP}{dt} = kP$ but as P gets big the resources are all used up and the population is unable to continue growing past some limiting population $K \equiv$ the carrying capacity. The simplest eq^{n} encoorporating the above features is

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \quad \text{The Logistic Eq}^{n}}$$

Notice that as $P \to K$ we have $\frac{dP}{dt} \to 0$. As we desired the growth slows to zero as we approach the carrying capacity. Additionally when $P \ll K$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \cong kP$$

So for small population this model is like exponential growth. Now lets figure out what general features the sol^{n}'s to the Logistic Eq^{n} must have, (time for some calc. I)



P increases when $P < K$

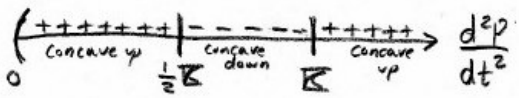
P decreases when $P > K$

What about concavity? Lets differentiate,

$$\frac{d^2P}{dt^2} = k\frac{dP}{dt}\left(1 - \frac{P}{K}\right) - \frac{k}{K}P\frac{dP}{dt}$$

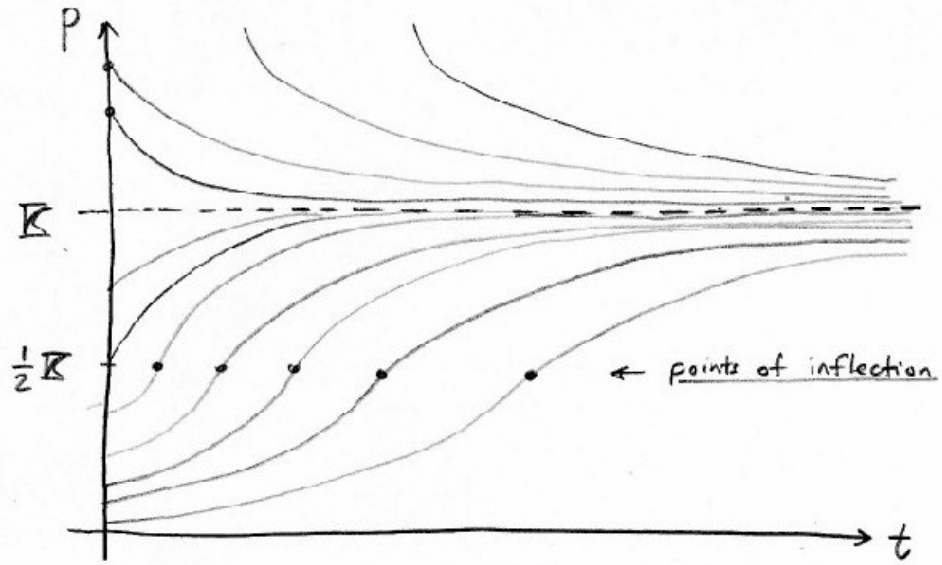
$$= k\left(1 - \frac{2P}{K}\right)\frac{dP}{dt}$$

$$= k^2\left(1 - \frac{2P}{K}\right)\left(1 - \frac{P}{K}\right)$$



Notice $\frac{dP}{dt}$ is maximized at $P = \frac{1}{2}K$.

Graph of Sol^{n}'s to Logistic Eq^{n}



Inevitably as $t \to \infty$ the sol^{n} goes to K no matter what the intial condition was.

<u>Remark</u>: We have yet to find a sol^{n}. Next we'll explicitly solve the Log. Eq^{n}. I think its interesting we can see so much just from studying the DEq^{n} directly.

Logistic Eq^{n} Analytic Sol^{n}

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \Rightarrow \frac{dP}{P(1-P/K)} = k\,dt$$

Now to integrate in P we'll use partial fractions,

$$\frac{1}{P(1-P/K)} = \frac{A}{P} + \frac{B}{1-P/K}$$

$$1 = A(1-P/K) + BP \xrightarrow{P=0} A = 1$$

$$\xrightarrow{P=K} 1 = BK \therefore B = 1/K$$

Thus $\dfrac{1}{P(1-P/K)} = \dfrac{1}{P} + \dfrac{1}{K-P}$

$$\int \frac{dP}{P(1-P/K)} = \int\left(\frac{1}{P} + \frac{1}{K-P}\right)dP = \ln|P| - \ln|K-P| = \ln\left|\frac{P}{K-P}\right|$$

$$\int k\,dt = kt + C$$

Hence $\ln\left|\dfrac{P}{K-P}\right| = kt + C \Rightarrow \left|\dfrac{P}{K-P}\right| = e^{C}e^{kt} \Rightarrow \dfrac{P}{K-P} = Ae^{kt}$

$A = \pm e^{C}$

Now solve for P

$$P = (K-P)Ae^{kt}$$

$$P(1+Ae^{kt}) = AKe^{kt} \Rightarrow P = \frac{AKe^{kt}}{1+Ae^{kt}} = \boxed{\frac{K}{1+Ae^{-kt}} = P(t)}$$

Exercise: Verify for yourself that the conclusions we reached for inc/dec concave up/down exc... are duplicated by this sol^{n}.

Remark: What ever the intial population is the final population is K

$$\lim_{t\to\infty}\left(\frac{K}{1+A\underbrace{e^{-kt}}_{\to 0}}\right) = K$$

E1 Suppose that $\frac{dP}{dt} = 0.05P - 0.0005P^2$

Then what is the carrying capacity K ? and k ?

$$\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{100}\right) = kP\left(1 - \frac{P}{K}\right)$$

Comparing we identify $\boxed{K = 100 \text{ and } k = 0.05}$

E2 Suppose the carrying capacity of the US is 1000 (million). Additionally in 1990 $P = 250$ and in 2000 $P = 275$ million. Find $P(t)$ then predict the pop. in 2010 and 2100.

$$P(t) = \frac{1000}{1 + Ae^{-kt}}$$

Let 1990 be $t = 0$. Then $P(0) = \frac{1000}{1+A} = 250 \Rightarrow \boxed{A = 3}$

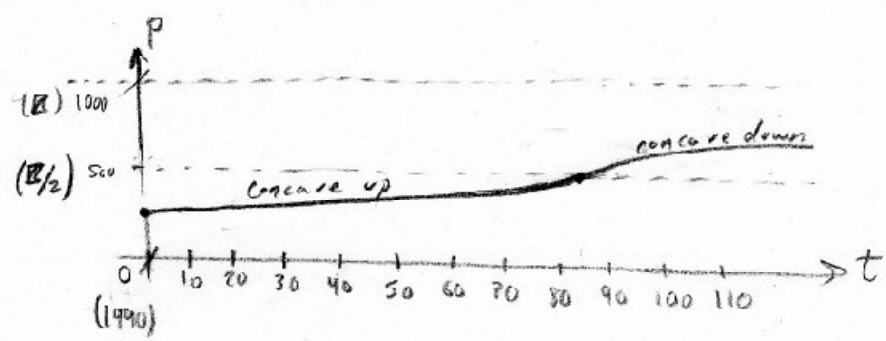
Additionally: $P(10) = \frac{1000}{1+3e^{-10k}} = 275 \Rightarrow 725 = 275(3e^{-10k})$

$$\Rightarrow \frac{725}{3 \cdot 275} = e^{-10k} = \frac{29}{33}$$

$$\Rightarrow \boxed{k = \frac{\ln(29/33)}{-10} = 0.01292}$$

$$P(20) = \frac{1000}{1+e^{-0.13(20)}} = 301 \text{ million in } 2010$$

$$P(110) = \frac{1000}{1+e^{-0.13(110)}} = 580 \text{ million in } 2100$$



$$P(t) = \frac{K}{2} = \frac{K}{1+3e^{-kt}} \Rightarrow 2 = 1 + 3e^{-kt}$$

$$\frac{1}{3} = e^{-kt} \Rightarrow \boxed{t = \frac{\ln(3)}{k} = \frac{\ln(3)}{0.01292} = 85 \Rightarrow P(85) = K/2}$$

SORRY THIS IS A LITTLE MESSY, I'VE INCLUDED THIS SECTION FOR BREADTH, NOT TESTED MOST LIKELY.

10.4. INTEGRATING FACTOR METHOD

The integrating factor method assumes that our starting point is a linear first order ODE which has been written in the so-called *standard form*

$$\boxed{\frac{dy}{dx} + Py = Q \quad \text{standard form}}$$

We assume that P, Q are continuous functions in the equation above. Notice that we cannot just separate variables and integrate. Let's see how the "integrating factor method" gets around the trouble. To start we need to define the integrating factor,

$$\boxed{\mu = exp\left(\int P(x)\,dx\right)}$$

Now observe that the integrating factor has an interesting derivative,

$$\begin{aligned}
\frac{d\mu}{dx} &= \frac{d}{dx} exp\left(\int P(x)\,dx\right) \\
&= exp\left(\int P(x)\,dx\right)\frac{d}{dx}\int P(x)\,dx \\
&= exp\left(\int P(x)\,dx\right)P(x) \\
&= \mu P
\end{aligned}$$

Now multiply the DEqn in standard form by the integrating factor and keep the identity above in mind,

$$\mu\frac{dy}{dx} + \mu Py = \mu Q \quad \Longrightarrow \quad \mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu Q$$

Notice that we can apply the product rule in reverse at this point,

$$\mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu Q \quad \Longrightarrow \quad \frac{d}{dx}\big(\,\mu y\,\big) = \mu Q \quad \Longrightarrow \quad d(\mu y) = \mu Q dx.$$

What this calculation shows is that the DEqn in standard form becomes separable if we change from y to μy, the term "separable" simply means we can separate and integrate. Integrating both sides we find,

$$\mu y = \int \mu Q\,dx \quad \Longrightarrow \quad \boxed{y = \frac{1}{\mu}\int \mu Q\,dx}$$

This formula gives a general solution for any first order ODE put in standard form. I don't want you to just use this formula. If you choose to use it then I require you to first prove it. The proof we just gave we will repeat again and again, each example follows the same pattern. The great advantage of mimicking the proof for each example is that

there is a built-in redundancy to the calculation. If you just use the formula then there is no double check on your work. It's time for examples!

Example 10.4.1

$\boxed{E1}$ Find general sol^{n} to $\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos(x)$. We know we can solve it once it's put into standard form, so make it so,

$$\frac{dy}{dx} \underbrace{- \left(\frac{2}{x}\right)}_{P(x)} y = \underbrace{x^2\cos(x)}_{Q(x)}$$

$$\mu = \exp\left(\int -\frac{2}{x}\,dx\right) = \exp\left(-2\ln|x|\right) = \exp\left(\ln\left(\frac{1}{|x|^2}\right)\right) = \frac{1}{x^2}$$

Multiplying by μ yields,

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = \cos(x)$$

Nice, $\frac{d}{dx}\left(\frac{1}{x^2}y\right) = \cos(x) \Rightarrow \frac{y}{x^2} = \sin(x) + C$

$$\Rightarrow \boxed{y = x^2(C + \sin(x))}$$

Notice that if we calculated the integrating factor incorrectly then we would not have been able to make the reverse product rule work. This is the check and balance of the method, if you are paying attention you'd have to make a pair of errors simultaneously to get it wrong. Of course it's always possible to get stuck on an integration, but I hope that will not be your stumbling stone here.

Example 10.4.2

$\boxed{E2}$ $y\frac{dx}{dy} + 2x = 5y^3$ (this one is a bit weird, we'll need to think of x as the dependent variable and y as the independent variable)

$\boxed{\text{Standard Form}}$ $\rightarrow \frac{dx}{dy} + \left(\frac{2}{y}\right)x = 5y^2$

$$\mu(y) = \exp\left(\int\left(\frac{2}{y}\right)dy\right) = \exp\left(2\ln|y|\right) = \exp\left(\ln|y|^2\right) = y^2$$

$$\underbrace{y^2\frac{dx}{dy} + 2yx}_{} = 5y^4$$ Multiplied by $\mu(y) = y^2$.

$$\frac{d}{dy}\left(y^2x\right) = 5y^4 \Rightarrow y^2x = y^5 + C$$

$$\Rightarrow \boxed{x = y^3 + C/y^2}$$

Example 10.4.3

H

§2.3 #7 (long version) $\frac{dy}{dx} - y = e^{3x}$ is in standard form with $P(x) = -1$ and $Q(x) = e^{3x}$

Calculate $\mu(x) = \exp\left(\int -1\,dx\right) = \exp(-x)$. Then multiply the DEq^{n} by $\mu(x)$,

$$\underbrace{e^{-x}\frac{dy}{dx} - e^{-x}y}_{} = e^{-x}e^{3x} = e^{2x}$$

$$\frac{d}{dx}\left(e^{-x}y\right) = e^{2x}$$

product rule in reverse

Integrate → use FTC $e^{-x}y = \frac{1}{2}e^{2x} + C$ ⇒ $\boxed{y = \frac{1}{2}e^{3x} + Ce^{x}}$

(★) (dividing by e^{-x})

Remark: Notice that we need not add a constant upon integrating $P(x)$ in $\mu(x) = exp(\int P(x)\,dx)$. If we did, it would cancel when we divide by $\mu(x)$ to solve for y. On the other hand in the final integration (marked by ★) we have no such expectation for that constant C to be cancelled. It is thus our custom to omit the integration constant when calculating the integrating factor.

Example 10.4.4

H 9

§2.3 #8

$$\frac{dy}{dx} = \frac{y}{x} + 2x + 1 \Rightarrow \overbrace{\frac{dy}{dx} - \frac{1}{x}y = 2x+1}^{\text{standard form.}}$$

$$\mu(x) = \exp\left(\int \frac{-1}{x}dx\right) = \exp\left(-\ln|x|\right) = \exp\left(\ln\left|\frac{1}{x}\right|\right) = \frac{1}{|x|}$$

Multiply by $\mu(x)$, assume $x > 0 \Rightarrow |x| = x$.

$$\underbrace{\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y}_{} = \frac{1}{x}(2x+1) = 2 + \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = 2 + \frac{1}{x}$$

integrating both sides, use for LHS that $\int \frac{d}{dx}(f(x))dx = f(x)$.

$$\frac{1}{x}y = \int\left(2 + \frac{1}{x}\right)dx = 2x + \ln|x| + C$$

$$\boxed{y = 2x^2 + x\ln(x) + Cx} \quad (x > 0)$$

Sol^{n} for $x < 0$ similar except $|x| = -x$.

Examples 10.4.5 through 10.4.8

H (10)

§2.3 #10 $\quad y + \frac{x}{2}\frac{dy}{dx} = \frac{1}{2x^3}$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^4} \quad \Rightarrow \mu(x) = \exp\left(\int \frac{2}{x}dx\right) = e^{2\ln|x|} = |x|^2 = x^2$$

$$\underbrace{x^2\frac{dy}{dx} + 2xy}_{} = \frac{x^2}{x^4}$$

$$\frac{d}{dx}\left(x^2 y\right) = \frac{1}{x^2} \Rightarrow x^2 y = \int \frac{dx}{x^2} = \frac{-1}{x} + C \quad \therefore \boxed{y = \frac{-1}{x^3} + \frac{C}{x^2}}$$

§2.3 #11 $\quad (t+y+1)dt - dy = 0 \Rightarrow \frac{dy}{dt} = t+y+1 \Rightarrow \underbrace{\frac{dy}{dt} - y = t+1}_{\text{(standard form)}}$

$$\mu = \exp\left(\int -dt\right) = \exp(-t)$$

$$\underbrace{e^{-t}\frac{dy}{dt} - e^{-t}y}_{} = e^{-t}(t+1)$$

$$\int \frac{d}{dt}\left(e^{-t}y\right) = \int\left(te^{-t} + e^{-t}\right)dt$$

$$= \int \underset{u}{t}\,\underbrace{e^{-t}dt}_{dv} + \int e^{-t}dt$$

$$= -te^{-t} + \int e^{-t}dt - e^{-t}$$

$$= -te^{-t} - e^{-t} - e^{-t} + C \Rightarrow e^{-t}y = -e^{-t}(2+t) + C$$

$$\Rightarrow \boxed{y = -2 - t + Ce^{t}}$$

§2.3 #12 $\quad \frac{dy}{dx} + 4y = x^2e^{-4x} \Rightarrow \mu(x) = \exp\left(\int 4dx\right) = e^{4x}.$

$$e^{4x}\frac{dy}{dx} + 4e^{4x}y = x^2 \Rightarrow \frac{d}{dx}\left(e^{4x}y\right) = x^2$$

$$\Rightarrow e^{4x}y = \int x^2dx = \frac{x^3}{3} + C$$

$$\Rightarrow \boxed{y = e^{-4x}\left(\tfrac{1}{3}x^3 + C\right)}$$

§2.3 #18 $\quad \frac{dy}{dx} + 4y = e^{-x} \Rightarrow \mu = \exp\left(\int 4dx\right) = e^{4x}$

$$e^{4x}\frac{dy}{dx} + 4e^{4x}y = e^{4x}e^{-x} \Rightarrow \frac{d}{dx}\left(e^{4x}y\right) = e^{3x} \Rightarrow e^{4x}y = \frac{1}{3}e^{3x} + C$$

$$y(0) = 4/3 \Rightarrow e^0\frac{4}{3} = \frac{1}{3}e^0 + C \Rightarrow \underline{C = 1}$$

$$\boxed{y = \frac{1}{3}e^{-x} + e^{-4x}}$$

Example 10.4.9 and 10.4.10

H (11)

§2.3 #22 $\sin(x)\frac{dy}{dx} + y\cos(x) = x\sin(x)$

$$\frac{dy}{dx} + \frac{\cos(x)}{\sin(x)}y = x$$

$$|\sin(x)|\frac{dy}{dx} + \frac{|\sin(x)|}{\sin(x)}\cos(x)\,y = |\sin(x)|\,x$$

$$\sin(x)\frac{dy}{dx} + \cos(x)\,y = x$$

$$\mu = \exp\left(\int \frac{\cos(x)dx}{\sin(x)}\right) = \exp\left(\int \frac{du}{u}\right) = \exp(\ln|\sin(x)|) = |\sin(x)|$$

$u = \sin(x)$, $du = \cos(x)dx$

$$\frac{d}{dx}\left(\sin(x)\,y\right) = x \Rightarrow y\sin(x) = \frac{x^2}{2} + C$$

$$y(\pi/2) = 2 \Rightarrow 2\sin(\pi/2) = \frac{\pi^2}{8} + C \quad \therefore \quad \underline{C = 2 - \pi^2/8}$$

Thus $\boxed{y = \frac{1}{\sin(x)}\left(\frac{x^2}{2} + 2 - \frac{\pi^2}{8}\right)}$

§2.3 #29 $\frac{dy}{dx} = \frac{1}{e^{4y} + 2x} \longrightarrow \frac{dx}{dy} = e^{4y} + 2x$

Then $\frac{dx}{dy} - 2x = e^{4y}$ thus $\mu(y) = \exp\left(\int -2dy\right) = e^{-2y}$.

$$\underbrace{e^{-2y}\frac{dx}{dy} - 2e^{-2y}x}_{} = e^{2y}$$

$$\frac{d}{dy}\left(e^{-2y}x\right) = e^{2y} \rightarrow e^{-2y}x = \frac{1}{2}e^{2y} + C$$

$$\rightarrow \boxed{x = \frac{1}{2}e^{4y} + Ce^{2y}}$$

Summary

1. Put linear first order ODEn into standard form, identify the "P" and "Q".
2. Calculate the integrating factor $\mu = \int P\,dx$.
3. Multiply the standard form DEqn by the integrating factor.
4. Group terms and apply the product rule in reverse.
5. Integrate both sides, don't forget to add the constant.
6. Solve for y.
7. Apply initial condition if you have one.

You could do 7.) after 5.) instead, for certain problems that is very labor-saving.

Well, it was fun, but there may be a question gnawing away as you follow the recipe. Why? Why even think to invent this integrating factor? And who thought to multiply by it and so forth? I'm not certain the history of the method, it would be interesting to research for fun. There are ways of deriving the integrating factor, but the idea stems from some fairly sophisticated geometry. We'd need to understand what a "symmetry"

of a differential equation is. Then, as I understand it, we could go looking for coordinates which make the standard form DEqn separate. There is a rather general method which will help you choose such separating coordinates. If we did that general procedure (which has nice geometric motivations) then I believe we would actually arrive at a derivation of the integrating factor. All I showed you here in this section was that it worked. That is enough of course, but sometimes we want more. I will award 10 bonus points if you can work through the argument I sketched above. (I would help get you started, just ask in office hours sometime)

<u>In-Class Exercise 10.4.11:</u> Suppose $x > 0$ and solve $x\dfrac{dy}{dx} - xy = x^2 + x$.

10.5. CONSTANT COEFFICIENT 2ND ORDER ODES

And now for the easy part. No, I'm not kidding. Once we get past my proof of why it works you'll see the examples in this section are way easier than the last section. Let me state the problem we wish to solve in general:

$$\boxed{ay'' + by' + cy = 0, \quad \text{with } a, b, c \in \mathbb{R} \text{ is a } 2^{nd}\text{-order constant coefficient ODE.}} \text{💣}$$

We call a,b,c the coefficients. Let me begin by considering a very special example, it will help me explain what we should expect generally,

Example 10.5.1 Find the general solution of $y'' = 0$, let x be the independent variable so that the notation $y'' = \frac{d^2y}{dx^2}$. Consider then that we can integrate twice to solve this one. We just need to apply the FTC twice:

$$\begin{aligned}
\frac{d^2y}{dx^2} = 0 \quad &\Longrightarrow \quad \int \frac{d}{dx}\left(\frac{dy}{dx}\right)dx = \int 0dx \\
&\Longrightarrow \quad \frac{dy}{dx} = c_1 \\
&\Longrightarrow \quad \int \frac{dy}{dx}dx = \int c_1 dx \\
&\Longrightarrow \quad \boxed{y = c_1x + c_2}
\end{aligned}$$

We can see that our general solution is actually built from two ***fundamental solutions***. In particular if we define $y_1 = x \text{ and } y_2 = 1$ the general solution is a ***linear combination*** of these two basic building blocks: $y = c_1y_1 + c_2y_2$. An example is not proof, but it turns out this is always the case, we will find every constant coefficient 2nd order ODE has a general solution which is the linear combination of two fundamental solutions.

Claim: 💣 has a general solution of the form $y = c_1y_1 + c_2y_2$ where y_1, y_2 are themselves solutions to 💣 called the fundamental solutions.

Our goal now is to find a procedure to locate these two functions. What guess can we make that includes many possibilities? One thing that comes to mind is $y = e^{\lambda x}$. This is a fairly general guess if we allow the constant λ to be a complex number. For example,

$$\begin{aligned}
\lambda &= 1 + i & e^{x+ix} &= e^x(\cos(x) + i\sin(x)) \\
\lambda &= 2i & e^{2ix} &= \cos(2x) \\
\lambda &= 0 & e^{0x} &= 1
\end{aligned}$$

Thus if the solutions we are looking for are sines, cosines or exponentials then we ought to find them with the general guess $y = e^{\lambda x}$.
Lemma: If $\lambda \in \mathbb{C}$ and then $\frac{d}{dx}\left(e^{\lambda x}\right) = \lambda e^{\lambda x}$.

Proof: In the case that $\lambda \in \mathbb{R}$ our life is easy, we just use the ordinary chain rule and that's it. The reason I have made this lemma is to discuss what happens when $\lambda = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. Let us see how the differentiation works in the complex case,

$$\begin{aligned}
\frac{d}{dx}\left(e^{\lambda x}\right) &= \frac{d}{dx}\left(e^{\alpha x + i\beta x}\right) \\
&= \frac{d}{dx}\left(e^{\alpha x}\left[\cos(\beta x) + i\sin(\beta x)\right]\right) \\
&= \frac{d}{dx}\left(e^{\alpha x}\cos(\beta x)\right) + i\frac{d}{dx}\left(e^{\alpha x}\sin(\beta x)\right) \\
&= \alpha e^{\alpha x}\cos(\beta x) - \beta e^{\alpha x}\sin(\beta x) + i\left[\alpha e^{\alpha x}\sin(\beta x) + \beta e^{\alpha x}\cos(\beta x)\right] \\
&= e^{\alpha x}\left[(\alpha + i\beta)\cos(\beta x) + (i\alpha - \beta)\sin(\beta x)\right] \\
&= e^{\alpha x}\left[(\alpha + i\beta)\cos(\beta x) + i(\alpha + i\beta)\sin(\beta x)\right] \\
&= (\alpha + i\beta)e^{\alpha x}\left[\cos(\beta x) + i\sin(\beta x)\right] \\
&= \lambda e^{\lambda x}
\end{aligned}$$

As you can see in the calculation above the reason the Lemma holds true is a fortunate interplay between the ordinary chain rule for real-valued functions and the product rule.

Remark: Consider the vector-valued function of a real variable $\vec{v}(t) =< t, t^2 >$. We can differentiate such a vector with respect to its real variable. The differentiation is done component-wise: $\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(< t, t^2 >\right) =< \frac{d}{dt}(t), \frac{d}{dt}(t^2) >=< 1, 2t >$. If we wrote that two-dimensional vector in complex notation $\vec{v} = t + it^2$ we would differentiate as follows:

$$\frac{d}{dt}\vec{v} = \frac{d}{dt}(t + it) = \frac{d}{dt}(t) + i\frac{d}{dt}(t^2) = 1 + i(2t)$$

This should be strictly understood as component-wise differentiation of a two-vector. However, it works like you would expect it to if you were not trying to be careful. That's the beauty here, you could write $\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$ and not even notice you were exiting the realm of real-valued functions of a real variable. Calculus for complex-valued functions of a real variable is deceptively simple. Now, when you take complex variables you will also discuss complex-valued functions of a complex variable. Those have all sorts of properties quite foreign to their real counterparts. You'll see. **(obviously this remark is not part of the required content of calculus II, I include it for breadth and context)**

Claim: There is at least one possibly complex-valued solution to 💣 ($ay'' + by' + cy = 0$) having the form $y = e^{\lambda x}$.

Proof: use the Lemma to see $y' = \lambda e^{\lambda x} = \lambda y$ *and* $y'' = \lambda^2 e^{\lambda x} = \lambda^2 y$. *Substitute into* 💣:

$$a\lambda^2 y + b\lambda y + cy = 0 \quad \Longrightarrow \quad (a\lambda^2 + b\lambda + c)y = 0 \quad \Longrightarrow \quad \boxed{a\lambda^2 + b\lambda + c = 0} ☺$$

You can verify that $e^{\lambda x} \neq 0$ *thus in order for our solution to work we must have that the constant* λ *is a solution to the quadratic equation ☺. We always get two solutions to the quadratic equation, although in one special case those solutions might be the same. So we always get at least one solution and the Claim is true. The ☺equation is called the auxillary or characteristic equation. I like to call it the characteristic equation because it characterizes the type of solution we get for the problem.*

Complex-Valued Solution contains two real solutions:
Our goal was to solve 💣 and what we implicitly mean by that in this course is to find real-valued solutions. Sometimes our guess yields a complex-valued solution, but this is no big deal since the following calculation shows that if $y = Re(y) + iIm(y)$ is a complex-valued solution then $y_1 \equiv Re(y)$ and $y_2 \equiv Im(y)$ are real-valued solutions.

$$\begin{aligned}
&ay'' + by' + cy = 0 \\
&\Longrightarrow \quad a\big[Re(y) + iIm(y)\big]'' + b\big[Re(y) + iIm(y)\big]' + c\big[Re(y) + iIm(y)\big] = 0 \\
&\Longrightarrow \quad aRe(y)'' + bRe(y)' + cRe(y) + i\big[aIm(y)'' + bIm(y)' + cIm(y)\big] = 0 \\
&\Longrightarrow \quad ay_1'' + by_1' + cy_1 = 0 \text{ and } ay_2'' + by_2' + cy_2 = 0
\end{aligned}$$

This is good news, now all we need to do in the case that λ is complex is to break our solution into it's real and imaginary parts. Let $\lambda = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$ then $e^{\lambda x} = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x)$ thus

$$Re(e^{i\lambda x}) = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad Im(e^{i\lambda x}) = e^{\alpha x}\sin(\beta x)$$

We find two real solutions in the complex case; $y_1 = e^{\alpha x}\cos(\beta x)$ and $y_2 = e^{\alpha x}\sin(\beta x)$.

Missing Case: there is just one case our discussion has failed to cover in general, it is the case that the solution to ☺ is repeated. From our experience with Example 10.4.1 we could have anticipated this, there is no way to get $y_1 = x$ from our guess $y = e^{\lambda x}$. The remedy is simple, just multiply our guess by x and see if it works. For example, if $\lambda = 3$ twice then the fundamental solutions will be $y_1 = e^{3x}$ and $y_2 = xe^{3x}$, these are solutions to the differential equation $y'' - 6y' + 9y = 0$. I've assigned you a homework problem that proves $xe^{\lambda x}$ is a solution in the case that λ is a repeated root. *(it's not hard, I just think you should do it because it will help you understand a little more about what is actually being said in this section)*

The Recipe: if we wish to solve $ay'' + by' + cy = 0$ then we solve the characteristic equation $a\lambda^2 + b\lambda + c = 0$. This is just a quadratic equation, we can always solve it either by factoring or via the quadratic equation. There are three things that can happen, here is what to do for each case:

1. If $a\lambda^2 + b\lambda + c = 0$ has two distinct real solutions $\alpha_1 \neq \alpha_2$ then the general solution is $\boxed{y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}}$

2. If $a\lambda^2 + b\lambda + c = 0$ has repeated real solutions $\alpha_1 = \alpha_2 = \alpha$ then the general solution is $\boxed{y = c_1 e^{\alpha x} + c_2 x e^{\alpha x}}$

3. If $a\lambda^2 + b\lambda + c = 0$ has complex solutions $\alpha \pm i\beta$ such that $\alpha, \beta \in \mathbb{R}$ then the general solution is $\boxed{y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)}$

In all three cases this is as far as we can go without more data. If we are given initial conditions then we can find values for c_1, c_2 that will fit those conditions. **THE END**

Example 10.5.2 (find the general solution of y'' + 13y' - 14y=0, y' = dy/dx)

$$\begin{aligned}
&\text{given differential equation:} && y'' + 13y' - 14y = 0 \\
&\text{corresponding characteristic eqn:} && \lambda^2 + 13\lambda + 14 = 0 \\
&\text{factor:} && (\lambda^2 + 14)(\lambda - 1) = 0 \\
&\text{find solutions:} && \lambda_1 = -14, \lambda_2 = 1 \\
&\text{Recipe case 1.) says general solution is:} && \boxed{y = c_1 e^{-14x} + c_2 e^{x}}
\end{aligned}$$

Example 10.5.2 (find the general solution of y'' + 9y = 0, y' = dy/dt)

$$\begin{aligned}
&\text{given differential equation:} && y'' + 9y = 0 \\
&\text{corresponding characteristic eqn:} && \lambda^2 + 9 = 0 \\
&\text{solve for } \lambda^2\text{:} && \lambda^2 = -9 \\
&\text{find solutions:} && \lambda = \pm\sqrt{-9} = \pm 3i \\
&\text{Recipe case 2.) says general solution is:} && \boxed{y = c_1\cos(3t) + c_2\sin(3t)}
\end{aligned}$$

In the preceding example I identified that $\alpha = 0$ and $\beta = 3$. This made the exponentials equal to one, so I didn't bother writing them. This is a special case of the complex case. It is especially interesting physically. It's the simple harmonic oscillator which is a spring w/o friction. I don't need the quadratic equation for cases like this. However, for algebra as in the next example I say the quadratic equation is the best way to go.

Examples 10.5.4 through 10.5.7

$\boxed{E4}$ $2Y'' + 6Y' - 11Y = 0$

$2\lambda^2 + 6\lambda - 11 = 0 \Rightarrow \lambda = \frac{-6 \pm \sqrt{36 + 4(2)(11)}}{4} = \frac{-6 \pm \sqrt{124}}{4}$

Thus $\lambda = \frac{-3 \pm \sqrt{31}}{2}$ which is distinct real case, (I.)

$$\boxed{Y = c_1 e^{\frac{-3+\sqrt{31}}{2}x} + c_2 e^{\frac{-3-\sqrt{31}}{2}x}}$$

$\boxed{E5}$ $\frac{d^2r}{d\theta^2} + 16r = 0$

$\lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i$ or $\alpha = 0$ and $\beta = 4$

$$\boxed{r(\theta) = c_1 \cos(4\theta) + c_2 \sin(4\theta)}$$

$\boxed{E6}$ $\varphi'' + 2\varphi' + 5\varphi = 0$ where $\varphi' \equiv \frac{d\varphi}{dt}$.

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda = \frac{-2 \pm \sqrt{4 - 4(5)}}{2} = -1 \pm \frac{\sqrt{-16}}{2} = -1 \pm 2i$, $\alpha = -1$ & $\beta = 2$

$$\boxed{\varphi(t) = e^{-t}(a\cos(2t) + b\sin(2t))}$$

$\boxed{E7}$ $\psi'' - 16\psi = 0$ where $\frac{d\psi}{dx} \equiv \psi'$,

$\lambda^2 - 16 = 0 \Rightarrow \lambda = \pm 4$

$$\boxed{\psi = c_1 e^{4x} + c_2 e^{-4x}}$$

E7 is also very special, and we should be careful not to confuse it with case 2. of the Recipe. The roots 4 and -4 are certainly distinct, they just have equal magnitudes. Also we should be careful not to confuse this with the very similar problem $\psi'' + 16\psi = 0$ which has solution $\psi = c_1 \cos(4x) + c_2 \sin(4x)$. It is worth knowing that the solution to E7 can be equivalently written as

$$\boxed{\psi = A\cosh(4x) + B\sinh(4x)} \qquad \text{or} \qquad \boxed{\psi = C\cosh(4x + D)}$$

Each of these solutions is equally general. One of your homework problems asks you to show these are in fact just the same solution. (It's not hard, you just need to remember the definition of cosh and sinh.) Similar comments apply to the case where the solution is pure imaginary. We can write $\psi = c_1 \cos(4x) + c_2 \sin(4x)$ as $\psi = A\cos(4x + \phi)$ for appropriately chosen A (amplitude) and ϕ (phase). I've asked a Collected Homework concerning that formula as well, it follows easily from a trig. Identity.

Examples 10.5.8 through 10.5.13 (we find general solution in each example)

H (23)

§4.2 #2

$$y'' - y' - 2y = 0 \Rightarrow r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0$$
$$\Rightarrow r = -1 \text{ or } r = 2 \Rightarrow \boxed{y = c_1 e^{-x} + c_2 e^{2x}}$$

§4.2 #4

$$y'' + 6y' + 9y = 0 \Rightarrow \lambda^2 + 6\lambda + 9 = 0$$
$$\Rightarrow (\lambda+3)^2 = 0 \quad \therefore \lambda = -3, -3 \Rightarrow \boxed{y = c_1 e^{-3x} + c_2 x e^{-3x}}$$

<u>Remark:</u> I like to use λ instead of r for the variable of the characteristic eq^{n}. So be warned.

§4.2 #6

$$y'' - 5y' + 6y = 0 \Rightarrow \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2) = 0$$
$$\therefore \lambda = 3 \text{ or } 2 \Rightarrow \boxed{y = c_1 e^{3x} + c_2 e^{2x}}$$

§4.2 #8

$$6y'' + y' - 2y = 0 \Rightarrow 6\lambda^2 + \lambda - 2 = 0$$
$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{1 - 4(6)(-2)}}{12} = \frac{-1 \pm \sqrt{49}}{12}$$
$$\Rightarrow \lambda_1 = \frac{-1+7}{12} \ \& \ \lambda_2 = \frac{-1-7}{12}$$
$$\Rightarrow \lambda_1 = 1/2 \text{ and } \lambda_2 = -2/3$$
$$\Rightarrow \boxed{y = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{2}{3}x}}$$

§4.2 #10

$$4y'' - 4y' + y = 0 \Rightarrow 4\lambda^2 - 4\lambda + 1 = 0$$
$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16-16}}{8} = \frac{1}{2} = \lambda_1 = \lambda_2$$
$$\Rightarrow \boxed{y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}}$$

§4.2 #12

$$3y'' + 11y' - 7y = 0$$
$$3\lambda^2 + 11\lambda - 7 = 0$$
$$\lambda = \frac{-11 \pm \sqrt{121 - 4(3)(-7)}}{6} = \frac{-11 \pm \sqrt{121 + 84}}{6} = \frac{-11 \pm \sqrt{205}}{6}$$
$$\boxed{y = c_1 \exp\left(\left(\frac{-11-\sqrt{205}}{6}\right)x\right) + c_2 \exp\left(\left(\frac{-11+\sqrt{205}}{6}\right)x\right)}$$

Examples 10.5.13 and 10.5.14

We find solution to initial value problem in each example. This means that after we deduce the form of the general solution we still need to find c_1, c_2 that fit the intial conditions.

H (24)

§4.2 #14 $\quad y'' + y' = 0$ with $y(0) = 2$ and $y'(0) = 1$.

$\lambda^2 + \lambda = \lambda(\lambda+1) = 0 \quad \therefore \lambda_1 = 0,\ \lambda_2 = -1 \quad \therefore y = c_1 + c_2 e^{-x}$

using data given $\begin{cases} y(0) = c_1 + c_2 e^0 = 2 \\ y'(0) = -c_2 e^0 = 1 \end{cases} \Rightarrow c_2 = -1 \ \& \ c_1 = 2 - c_2 = 3.$

Thus $\boxed{y = 3 - e^{-x}}$

§4.2 #18 $\quad y'' - 6y' + 9y = 0$

$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \quad \therefore \lambda_1 = \lambda_2 = 3.$

$y = c_1 e^{3x} + c_2 x e^{3x}$

$y' = 3c_1 e^{3x} + c_2(e^{3x} + 3x e^{3x})$ (product rule, remember.)

using givens. $\begin{cases} y(0) = c_1 = 2 \\ y'(0) = 3c_1 + c_2 = 6 + c_2 = 25/3 \end{cases} \Rightarrow 3c_2 = 25 - 18 = 7$

$\Rightarrow c_2 = 7/3$

$\boxed{y = 2e^{3x} + \frac{7}{3} x e^{3x}}$

Examples 10.5.15 through 10.5.17 (all complex roots)

H (26)

§4.3 #2 $\quad y'' + y = 0$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow \boxed{y = c_1 \cos(x) + c_2 \sin(x)}$

§4.3 #4 $\quad y'' - 10y' + 26y = 0$

$\lambda^2 - 10\lambda + 26 = 0 \Rightarrow \lambda = \frac{10 \pm \sqrt{100 - 4(26)}}{2} = 5 \pm \frac{\sqrt{-16}}{2} = 5 \pm 2i$

Identify $\alpha = 5$ and $\beta = 2 \Rightarrow \boxed{y = e^{5x}(c_1 \cos 2x + c_2 \sin 2x)}$

§4.3 #8 $\quad 4y'' + 4y' + 6y = 0$

$4\lambda^2 + 4\lambda + 6 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 4(4)6}}{8} = \frac{-4 \pm \sqrt{-80}}{8} = \underbrace{-\frac{1}{2}}_{\alpha} \pm i \underbrace{\sqrt{\frac{80}{64}}}_{\beta}$

$\sqrt{\frac{80}{64}} = \sqrt{\frac{10}{8}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2} = \beta \Rightarrow \boxed{y = e^{-x/2}\left(c_1 \cos\left(\frac{\sqrt{5}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{5}}{2}x\right)\right)}$

Examples 10.5.18 and 10.5.19 (both complex roots)

§4.3 #10 $y'' + 4y' + 8y = 0$

$$\lambda^2 + 4\lambda + 8 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16-32}}{2} = -2 \pm 2i = \alpha \pm i\beta.$$

$$\boxed{y = e^{-2x}(c_1 \cos(2x) + c_2 \sin(2x))}$$

§4.3 #12 $u'' + 7u = 0$

$$\lambda^2 + 7 = 0 \Rightarrow \lambda^2 = -7 \Rightarrow \lambda = \pm\sqrt{-7} = \pm 7i \quad (\alpha = 0,\ \beta = 7)$$

$$\boxed{u = c_1 \cos 7x + c_2 \sin 7x}$$

Examples 10.5.20 (complex root with initial conditions given)

§4.3 #22 $y'' + 2y' + 17y = 0$

$$\lambda^2 + 2\lambda + 17 = 0 \rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 4(17)}}{2} = -1 \pm 4i = \alpha \pm i\beta \ *$$

$$* \Rightarrow y = e^{-x}(c_1 \cos 4x + c_2 \sin 4x)$$

$$y' = -e^{-x}(c_1 \cos 4x + c_2 \sin 4x) + e^{-x}(-4c_1 \sin 4x + 4c_2 \cos 4x) \quad \text{(product rule)}$$

using the given conditions $\begin{cases} y(0) = c_1 = 1 \\ y'(0) = -c_1 + 4c_2 = -1 \Rightarrow 4c_2 = 0 \Rightarrow c_2 = 0 \end{cases}$ $\therefore \boxed{y = e^{-x} \cos 4x}$

Well I hope there enough examples in this section for you. I hope you enjoyed this section, part of the reason I included it is that it is both useful and easy. I figure you'd rather be tested on this and separation of variables. Comparatively speaking these sort of 2nd order ODEs are much easier than the 1st order ODEs which require non-trivial integration. We just do algebra here.

I should mention that 💣 covers many physically interesting examples: frictionless springs, springs with friction, circuits with inductance and capacitance (LC-circuits), circuits with inductance and capacitance and resitance (RLC-circuits) ,… If we add a forcing term to 💣 and study solutions to $ay'' + by' + cy = F(t)$ then we can cover a great multitude of physically interesting examples. I don't wish to cover them here, but I hope we've done enough here you'll be well-equipt to handle them if they come up in a science or engineering course. If you would like to see some explicit examples of these applications they're posted in my NCSU calculus II notes(I'll send you a link if you ask).

Finally, I have in mind a 10 point bonus project. I'm not certain this one has a solution. I would like to derive case 2 as a twist of Example 10.5.1. I'll explain further if you ask. I should warn you, I've tried to see it for a number of hours to no avail. Attempt only if you want a challenge.

Well, that is all for now, we'll leave the task of solving differential equations to the differential equation course. Just dipped our feet in the pond here. Time permitting we may dabble in solving DEqns via power series techniques in a few weeks. If you want to see that for certain make a point of asking me ahead of time. Thanks.

One other thing missing from these notes are pictures of direction fields for these differential equations. I plan to post the Mathematica code to plot the direction fields. If all goes well you'll actually be able to look at the graphs of the solutions we have learned how to calculate.