16. FURTHER APPLICATIONS OF CALCULUS

This chapter focuses on a few applications we missed along the way. I have taken the liberty of just cutting and pasting my old NCSU notes. I think they're pretty decent as they stand. Don't be frightened by the physics, I have included those examples to aid those of you who choose to do the optional physics homework problems.

16.1. AVERAGE VALUE OF A FUNCTION

If we take the area under $y=f(x)$ on $a \le x \le b$ and divide by the width of the interval which is $b-a$ this gives the average,

$$f_{ave} \equiv \frac{1}{b-a}\int_a^b f(x)\,dx$$

this is the natural generalization to discrete averages (see pg. 473)

E1 Let $f(x) = \sin^3(x)$ find f_{ave} on $[0, \pi]$

$$\begin{aligned} f_{ave} &= \frac{1}{\pi}\int_0^\pi \sin^3(x)\,dx \\ &= \frac{1}{\pi}\int_0^\pi (1-\cos^2(x))\sin(x)\,dx \\ &= \frac{1}{\pi}\int_1^{-1}(1-u^2)(-du) \end{aligned}$$

$u = \cos(x)$
$du = -\sin(x)\,dx$
$u(0) = \cos(0) = 1$
$u(\pi) = \cos(\pi) = -1$

$$\begin{aligned} &= \frac{1}{\pi}\left(\frac{1}{3}u^3 - u\right)\Big|_1^{-1} \\ &= \frac{1}{\pi}\left[\left(\frac{-1}{3}+1\right) - \left(\frac{1}{3}-1\right)\right] \\ &= \boxed{\frac{4}{3\pi} = f_{ave}} \end{aligned}$$

E2 Is the average velocity the average of the instantaneous velocity?

$$\begin{aligned} V_{ave.} &= \frac{1}{t_2-t_1}\int_{t_1}^{t_2} V(t)\,dt \\ &= \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\frac{dx}{dt}\,dt \\ &= \frac{1}{t_2-t_1}\int_{x(t_1)}^{x(t_2)} dx \\ &= \boxed{\frac{x(t_2)-x(t_1)}{t_2-t_1} = \frac{\Delta x}{\Delta t}} \end{aligned}$$

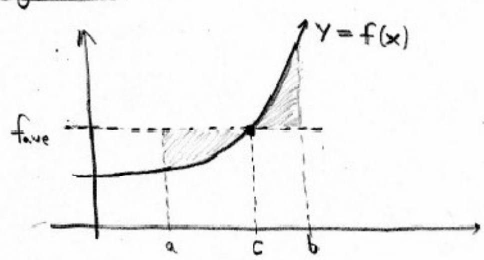
Our new concept of averaging matches the old as it should.

152

Thm (Mean Value Thm for integrals) Let f be continuous on $[a, b]$ then $\exists\, c \in [a, b]$ so that $f(c) = f_{ave}$. That is,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \xrightarrow[\text{aka}]{} (b-a)f(c) = \int_a^b f(x)\,dx$$

"Proof" by Picture:



graphically $f_{ave} = Y$ should have equal over/under estimates of the area. Meaning the shaded areas should be equal.

E3 If a ball has velocity $V = 30 - 9.8t$ when its thrown straight up what is the average velocity during the first 2 seconds? At what time does the instantaneous velocity match the average?

$$V_{ave} = \frac{1}{2}\int_0^2 (30 - 9.8t)\,dt$$

$$= \frac{1}{2}\left(30t - 4.9t^2\right|_0^2$$

$$= \frac{1}{2}\left(60 - 19.6\right)$$

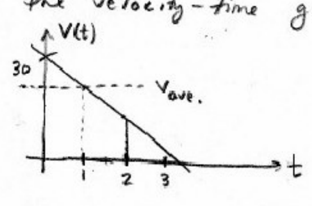
$$= \boxed{20.2}$$

$V_{ave} = V_{instantaneous}$ when $20.2 = 30 - 9.8t$

$9.8t = 9.8$

$\boxed{t = 1}$

Not to surprising if you consider the velocity-time graph



$\boxed{E4}$ (#15 pg. 470 §6.4) The voltage in U.S. houses typically varies sinusoidally from 155V to −155V with a frequency of 60 Hz. That is the voltage in your wall-socket as a function of time in seconds,

$$\mathcal{E}(t) = 155 \sin(120\pi t)$$

a.) The average voltage is zero because, (average over 1 cycle)

$$\mathcal{E}_{ave.} = \frac{1}{\frac{1}{60}} \int_0^{1/60} 155 \sin(120\pi t)\, dt$$

$$= \frac{60(155)}{120\pi}\left(-\cos(120\pi t)\right)\Big|_0^{1/60}$$

$$= \frac{60(155)}{120\pi}\Big(-\cos(2\pi) + \cos(0)\Big)$$

$$= \boxed{0 = \mathcal{E}_{ave.}}$$

b.) The voltage is not zero of course and is effectively like V_{rms} if we were to replace it with a constant voltage,

root mean square

$$\mathcal{E}_{rms} = \sqrt{\frac{1}{\frac{1}{60}} \int_0^{1/60} (\mathcal{E}(t))^2\, dt}$$

$$= \sqrt{(60)(155)^2 \int_0^{1/60} \sin^2(120\pi t)\, dt}$$

$$= \sqrt{(60)(155)^2 \int_0^{1/60} \tfrac{1}{2}\big(1 - \cos(240\pi t)\big)\, dt}$$

$$= \sqrt{(60)(155)^2 \Big[\tfrac{1}{2}\Big(t - \tfrac{1}{240\pi}\sin(240\pi t)\Big|_0^{1/60}}$$

$$= \sqrt{(60)(155)^2 \Big[\tfrac{1}{2}\Big(\tfrac{1}{60} - \tfrac{1}{240\pi}\sin(4\pi) - 0 + \tfrac{1}{240\pi}\sin(0)\Big)\Big]}$$

$$= \frac{155}{\sqrt{2}}$$

$$= \boxed{110\,V = \mathcal{E}_{rms}}$$

You could replace $\mathcal{E}(t)$ with a battery of $\mathcal{E} = 110V$ and deliver the same power to a given resistor.

16.2. DEFINITION OF A PROBABILITY DENSITY

165b

Probability

Let X be a continous random variable then we can describe the probability of a certain range of X occuring via an integral

$$P[a \le X \le b] = \int_a^b f(x)\,dx$$

The function $f(x)$ is the probability density function. In particular for this to make sense the sum of all prob. should be one

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \text{Normalization Condition}}$$

$\boxed{E}$ Let $f(x) = Ae^{-x/\tau}$ for $x \in D = [0,\infty)$ then what value must we assign to A to make $f(x)$ a prob. density and what is $P[0 \le x \le \tau]$, $P[0 \le x \le 2\tau]$, ..., $P[0 \le x \le 5\tau]$ ($f(x) = 0$ $x < 0$).

$$\int_0^\infty Ae^{-x/\tau}dx = \lim_{t\to\infty}\int_0^t Ae^{-x/\tau}dx =$$

$$= \lim_{t\to\infty} A\left(-\tau e^{-x/\tau}\Big|_0^t\right)$$

$$= \tau A = 1 \quad \therefore \quad \boxed{A = 1/\tau}$$

$$P(0 \le x \le \tau) = \int_0^\tau \frac{1}{\tau}e^{-x/\tau}dx = -e^{-x/\tau}\Big|_0^\tau = 1 - e^{-1} = 0.632$$

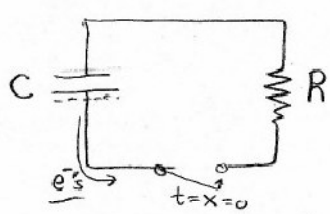
$$P(0 \le x \le 2\tau) = \int_0^{2\tau}\frac{1}{\tau}e^{-x/\tau}dx = 1 - e^{-2} = 0.865$$

$$P(0 \le x \le 3\tau) = 1 - e^{-3} = 0.950$$

$$P(0 \le x \le 4\tau) = 1 - e^{-4} = 0.982$$

$$P(0 \le x \le 5\tau) = 1 - e^{-5} = 0.993$$

This could model the chance a particular charge had flowed from a discharging capacitor ($\tau = RC$ in this case)



Typical values say
$R = 1k\Omega$
$C = 1\mu F$
$\tau = RC = 1ms$

(165c)

$\boxed{\text{Def}^{n}/ \text{ The average value of } X \text{ is } \mu = \int_{-\infty}^{\infty} x f(x)\,dx}$ also called mean value

This is just a continous weighted average, think about it.//

$\boxed{E2}$ What's average of last example

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx \quad : \quad f(x) = \begin{cases} 0 & x<0 \\ \frac{1}{\tau}e^{-x/\tau} & x \geq 0 \end{cases}$$

$$= \lim_{t\to\infty} \int_0^t \frac{x}{\tau} e^{-x/\tau}\,dx$$

Notice $\int \underbrace{x}_{u} \underbrace{e^{ax}dx}_{dV} = \frac{xe^{ax}}{a} - \int \frac{e^{ax}}{a}dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$ thus $a = -1/\tau$

$$\mu = \lim_{t\to\infty} \frac{1}{\tau}\left[-\tau x e^{-x/\tau} - \tau^2 e^{-x/\tau} \Big|_0^t \right]$$

$$= \lim_{t\to\infty} \frac{1}{\tau}\left[-\tau t e^{-t/\tau} - \tau^2 \underbrace{e^{-t/\tau}}_{0}\right] + \lim_{t\to\infty}\left[\tau^2 e^0\right]$$

$$\lim_{t\to\infty}\left(\frac{t}{e^{t/\tau}}\right) \stackrel{L}{=} \lim_{t\to\infty}\left(\frac{1}{\frac{1}{\tau}e^{t/\tau}}\right) = 0$$ (Need L-Hopital's Rule)

$\therefore \boxed{\mu = \tau^2}$

$\boxed{\text{Def}^{n}/ \text{ The median value of } X \text{ is the } \# \, m \text{ such that } \int_m^{\infty} f(x)\,dx = \frac{1}{2}}$

$\boxed{E3}$ What's the median value of our example?

$$\frac{1}{2} = \int_m^{\infty} \frac{1}{\tau} e^{-x/\tau}dx = \lim_{t\to\infty} \int_m^t \frac{e^{-x/\tau}}{\tau}dx = \lim_{t\to\infty}\left(-\underbrace{e^{-t/\tau}}_{0} + e^{-m/\tau}\right) = e^{-m/\tau}$$

$$\frac{1}{2} = e^{-m/\tau} \Rightarrow \ln(1/2) = -m/\tau \quad \therefore \boxed{m = \tau \ln(2)}$$ $(0.693\tau = m)$

16.3. CALCULUS APPLIED TO PHYSICS

(158)

E5 Find magnitude of work required to lift a 10m cable with a uniformly distributed mass of 10 kg.

Lets define ℓ to be the length of the cable being lifted, this will vary from $\ell = 10\,m \longrightarrow \ell = 0m$ as the cable is lifted. We should find the work dW done as we lift length ℓ a distance $d\ell$,

$$dW = F d\ell$$
$$= -mg\, d\ell$$
$$= -\lambda \ell g\, d\ell$$

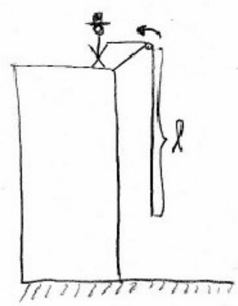
$\lambda = \frac{\text{mass}}{\text{length}} = \frac{10\,kg}{10\,m} = 1\,kg/m$

"λ is linear mass density"

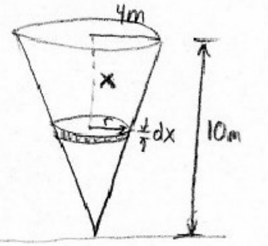
$m = \lambda \ell$.

Now add up the work

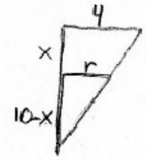
$$W = \int_{10}^{0} -\lambda g\, \ell\, d\ell$$
$$= -\lambda g \left(\tfrac{1}{2}\ell^2\right|_{10}^{0}$$
$$= \lambda g\, \tfrac{1}{2}(10)^2$$
$$= 1 \cdot 9.8 \cdot 50$$
$$= \boxed{490\,J}$$



E7 (Example 4 in text modified slightly.) Find minimum work to pump water out of a full cone.



$$dm = \rho\, dV = \rho \pi r^2 dx$$



$$\frac{4}{10} = \frac{r}{10-x}$$

$$r = \frac{2}{5}(10-x)$$

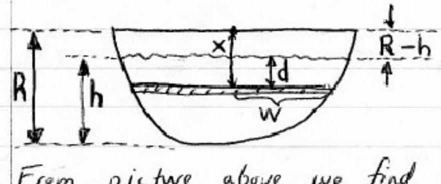
$$\begin{aligned} dW &= (dm)g \cdot x \\ &= (\rho \pi r^2 dx)\, g\, x \\ &= \rho g \pi \frac{4(10-x)^2}{25} x\, dx \\ &= \frac{4\pi\rho g}{25}(100x - 20x^2 + x^3)\, dx \end{aligned}$$

Remark: this is a magic cone, we cannot just tip it over or drill a hole in the bottom. We are forced to pump out the top ☺.

$$\begin{aligned} W &= \int_0^{10} \frac{4\pi\rho g}{25}(100x - 20x^2 + x^3)\, dx \\ &= \frac{4\pi\rho g}{25}\left(50\times10^2 - \frac{20}{3}\times10^3 + \frac{10^4}{4}\right) \\ &= \frac{4\pi(1000)(9.8)}{25}\left(50\times10^2 - \frac{20}{3}\times10^3 + \frac{10^4}{4}\right) \\ &= \boxed{4.11\times10^6\ \mathrm{J}} \end{aligned}$$

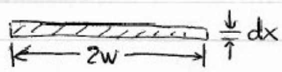
(Notice the text integrates from $2 \to 10$ because their cone isn't assumed to be full, think about it.)

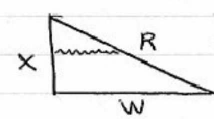
Find hydrostatic force on half-barrel pictured below, well just the end piece. The radius of barrel is R and the water of density ρ is filled to height h. Lets find force dF on a strip of area dA at position x and <u>depth</u> d below the surface.



$$\Rightarrow d = x - (R-h)$$

From picture above we find $P = \rho g d = \rho g (x - R + h)$.
(I set up the pressure wrong in notes, it is the <u>depth</u> that should determine the pressure, specifically $P = \rho g d$.) Next;





$w = \sqrt{R^2 - x^2}$

$dA = 2w\,dx = 2\sqrt{R^2-x^2}\,dx$

Ok so our choice of x makes dA relatively pretty, (you can try defining x differently but it'll make the square root nasty...) Ok, we know $P = \frac{dF}{dA}$ so $dF = P\,dA$

$$dF = \rho g (x - R + h)\, 2\sqrt{R^2 - x^2}\,dx$$

Now we just need to add-up the forces, $R-h \le x \le R$

$$\begin{aligned} F &= \int_{R-h}^{R} 2\rho g (x - R + h)\sqrt{R^2 - x^2}\,dx \\ &= \int_{R-h}^{R} 2\rho g x\sqrt{R^2-x^2}\,dx + \int_{R-h}^{R} 2\rho g (h-R)\sqrt{R^2-x^2}\,dx \\ &= 2\rho g \underbrace{\int_{R-h}^{R} x\sqrt{R^2-x^2}\,dx}_{u-\text{substitute}} + 2\rho g (h-R)\underbrace{\int_{R-h}^{R}\sqrt{R^2-x^2}\,dx}_{\text{trig-substitute.}} \end{aligned}$$

HINT • The homework problem results in the same type of integration.

(ignore comments about homework and notes in this example)

E8 continued

$$\int x\sqrt{R^2-x^2}\,dx = \int \sqrt{w}\,\frac{dw}{-2} \qquad w = R^2 - x^2$$

$$= -\frac{1}{2}\int w^{\frac{1}{2}}\,dw \qquad dw = -2x\,dx$$

$$= -\frac{1}{2}\,\frac{2}{3}\,w^{\frac{3}{2}} + C \qquad x\,dx = \frac{-dw}{2}$$

$$= -\frac{1}{3}\left(R^2 - x^2\right)^{3/2} + C$$

$$\int \sqrt{R^2-x^2}\,dx = \int (R\cos\theta)(R\cos\theta\,d\theta) \qquad x = R\sin\theta$$

$$= \int R^2\cos^2\theta\,d\theta \qquad dx = R\cos\theta\,d\theta$$

$$= \int \frac{R^2}{2}\left(1+\cos(2\theta)\right)d\theta \qquad R^2 - x^2 = R^2\cos^2\theta$$

$$\sqrt{R^2-x^2} = R\cos\theta$$

$$= \frac{R^2}{2}\int d\theta + \frac{R^2}{2}\int \cos(2\theta)\,d\theta$$

$$= \frac{R^2}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$$

$$F = 2\rho g\int_{R-h}^{R} x\sqrt{R^2-x^2}\,dx + 2\rho g(h-R)\int_{R-h}^{R}\sqrt{R^2-x^2}\,dx$$

$$= \frac{-2\rho g}{3}\left(R^2-x^2\right)^{3/2}\Big|_{R-h}^{R} + \rho g(h-R)R^2\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{x=R-h}^{x=R}$$

$$= +\frac{2}{3}\rho g\left(R^2-(R-h)^2\right)^{3/2} + \rho g(h-R)R^2\left(\sin^{-1}\left(\frac{x}{R}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{x}{R}\right)\right)\right)\Big|_{R-h}^{R}$$

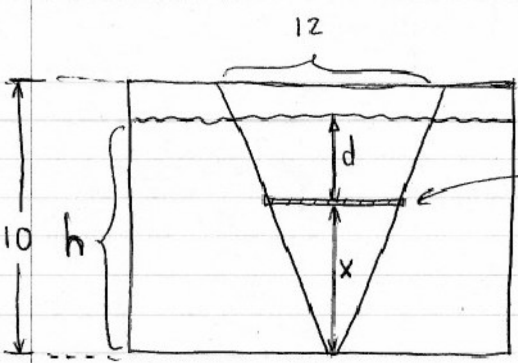
$$= \frac{2}{3}\rho g\left(R^2-(R-h)^2\right)^{3/2} + \rho g(h-R)R^2\left(\sin^{-1}\left(\frac{R}{R}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{R}{R}\right)\right)\right)$$

$$- \rho g(h-R)R^2\left(\sin^{-1}\left(\frac{R-h}{R}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{R-h}{R}\right)\right)\right)$$

$$= \boxed{\frac{2}{3}\rho g\left(R^2-(R-h)^2\right)^{3/2} + \rho g(h-R)R^2\left[\frac{\pi}{2} + \frac{1}{2}\sin\left(2\,\frac{\pi}{2}\right) - \sin^{-1}\left(\frac{R-h}{R}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{R-h}{R}\right)\right)\right]}$$

$\boxed{\text{E9}}$ Find the hydrostatic force on the triangular region pictured below. Assume the water of density ρ is upto level h

(163)



width is w which clearly depends linearly on x.

$w = mx + b$

$$w(0) = m(0) + b = 0 \quad \therefore \quad b = 0$$
$$w(10) = m(10) + 0 = 12 \quad \therefore \quad m = \frac{12}{10} = \frac{6}{5} \quad \therefore \quad \boxed{w = \frac{6}{5}x}$$

This formula for w checks because $w(10) = \frac{6}{5}(10) = 12$ as it should. Then the area of strip is $dA = w\,dx = \frac{6}{5}x\,dx$.

Now set up the pressure $P = \rho g d$,

$$x + d = h \Rightarrow d = h - x \Rightarrow P = \rho g(h-x)$$

Again the connection between force & pressure is $P = \frac{dF}{dA}$ so $dF = P\,dA$, thus

$$dF = \rho g(h-x)\frac{6}{5}x\,dx$$

Now sum the forces for the strips at x in $0 \le x \le h$,

$$\begin{aligned} F &= \int_0^h \frac{6\rho g}{5}(hx - x^2)\,dx \\ &= \frac{6\rho g}{5}\left(\frac{1}{2}hx^2 - \frac{1}{3}x^3\right|_0^h \\ &= \frac{6\rho g}{5}\left(\frac{1}{2}h^3 - \frac{1}{3}h^3\right) \\ &= \boxed{\frac{1}{5}\rho g h^3} \end{aligned}$$

$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

§6.5 Moments & Centers of Mass (com)

In the discrete case we define $\vec{r}_{cm} = \dfrac{\sum m_i \vec{r}_i}{\sum m_i}$ — the sum is over all the particles.

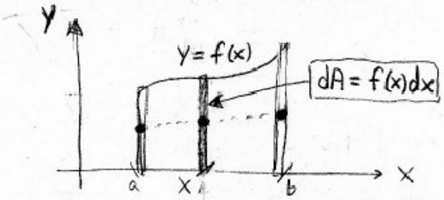
For 2 dimensional case we have

$$\vec{r}_{cm} = (X_{cm}, Y_{cm})$$

$$X_{cm} = \left(\sum m_i x_i\right) / \left(\sum m_i\right) = M_x / M \qquad M_x = \text{moment of intertia w.r.t } x\text{-axis.}$$

$$Y_{cm} = \left(\sum m_i y_i\right) / \left(\sum m_i\right) = M_y / M \qquad M = \text{total mass.}$$

What then is the generalization of this to a continous region.
Lets see how to find the c.o.m. of the region bounded by $y = 0$, $y = f(x)$ and $x = a$ and $x = b$. Assume uniform density.



Each strip has its c.o.m. at $(x, \frac{1}{2}f(x))$. We can treat it like a bunch of particles with dm each

$$\rho = \frac{dm}{dA} \rightarrow dm = \rho\, dA$$

So we find the c.o.m. of this system in the natural way,

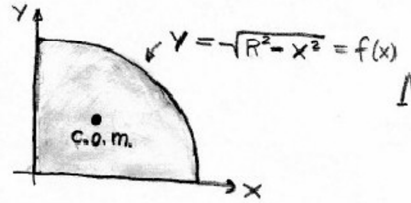
$$\overline{x} = \frac{\int_a^b \rho\, x f(x)dx}{\int_a^b \rho f(x)dx} \quad \text{AND} \quad \overline{y} = \frac{\int_a^b \rho \frac{1}{2} f(x) f(x)dx}{\int \rho f(x)\, dx}$$

Or in terms of moments & mass we have

$M_y \equiv \rho \int x\, f(x)dx$	$\overline{x} = \dfrac{M_y}{M}$
$M_x \equiv \rho \int \frac{1}{2}[f(x)]^2 dx$	$\overline{y} = \dfrac{M_x}{M}$
$M = \rho \int f(x)dx = \rho\,(\text{AREA})$	

We assume ρ to be a constant.

$\boxed{E10}$ Find the moments of intertia and c.o.m for a quarter-circle of <u>uniform</u> density ρ with radius R



$$M = \int_0^R \rho f(x)\,dx$$
$$= \rho \int_0^R f(x)\,dx$$
$$= \boxed{\frac{\pi}{4}\rho R^2 = M}$$

mass is simply area times density.

$$M_y = \rho \int_0^R x\sqrt{R^2 - x^2}\,dx$$
$$= \rho \int_{R^2}^0 \frac{-1}{2}\sqrt{u}\,du$$

$u = R^2 - x^2 \qquad u(R) = 0$
$du = -2x\,dx \qquad u(0) = R^2$

$$= \frac{\rho}{2}\,\frac{2}{3}\,u^{3/2}\Big|_0^{R^2}$$
$$= \frac{\rho}{3}\left(R^2\right)^{3/2}$$
$$= \boxed{\frac{1}{3}\rho R^3 = M_x}$$

$$M_x = \rho \int_0^R \frac{1}{2}\left(R^2 - x^2\right)dx$$
$$= \frac{\rho}{2}\left(R^2 x - \frac{1}{3}x^3\right|_0^R$$
$$= \boxed{\frac{1}{3}\rho R^3 = M_y}$$

Now we can find the center of mass,

$$X_{cm} = \frac{M_y}{M} = \frac{\frac{1}{3}\rho R^3}{\frac{\pi}{4}\rho R^2} = \frac{4}{3\pi}R \cong 0.4244R$$

$$Y_{cm} = \frac{M_x}{M} = \frac{\frac{1}{3}\rho R^3}{\frac{\pi}{4}\rho R^2} = \frac{4}{3\pi}R \cong 0.4244R$$

From the symmetry of the region we could have anticipated $X_{cm} = Y_{cm}$.

THE END