5. APPLICATIONS OF DERIVATIVES

Derivatives are everywhere in engineering, physics, biology, economics, and much more. In this chapter we seek to elucidate a number of general ideas which cut across many disciplines.

Linearization of a function is the process of approximating a function by a line near some point. The tangent line is the graph of the linearization.

Given some algebraic relation that connects different dynamical quantities we can differentiate implicitly. This relates the rates of change for the various quantities involved. Such problems are called "related rates problems".

The shape of a graph $y = f(x)$ can be ciphered through analyzing how the first and second derivatives of the function behave. Rolle's Theorem and the Mean Value Theorem are discussed as they provide foundational support for later technical arguments. Fermat's Theorem tells us that local extrema happen at critical points.
If a function is increasing on an interval then the derivative will be positive on that same interval. Likewise, a decreasing function will have a negative derivative. These observation lead straight to the First Derivative Test which allows us to classify critical points as being local minimas, maximas or neither. Concavity is discussed and shown to be described by the second derivative of the function. If a function is concave up on an interval then the second derivative of the function will be positive on that interval. Likewise, the second derivative is negative when the function is concave down. Concavity's connection to the second derivative gives us another test; the Second Derivative Test. Sometimes the second derivative test helps us determine what type of extrema reside at a particular critical point. However, the First Derivative Test has wider application. We also discuss the Closed Interval Method which is based on the same ideas plus the insight that when we restrict a function to a closed interval then the extreme values might occur at endpoints. In total, precalculus and college algebra skill is supplemented with new calculus-based insight. Calculus helps us graph with new found confidence.

Optimization is the application of calculus-based graphical analysis to particular physical examples. We have to find critical points then characterize them as minima or maxima depending on the problem. As always word problems pose extra troubles as the interpretation of the problem and invention of needed variables are themselves conceptually challenging. This part of calculus allows for much creativity. Often drawing a picture is an essential step to organize your ideas to forge ahead.

Finally we discuss limits at infinity. Graphically these limits tell us about horizontal asymptotes. Generally there are many different types of asymptotic behavior, we focus on the basic types. Again this helps us graph better.

5.1. LINEARIZATIONS

We have already found the linearization of a function a number of times. The idea is to replace the function by its tangent line at some point. This provides a fairly good approximation if we are near to the point. How near is near? Well, that depends on the example and what your idea of a "good approximation" should be. These are questions best left to a good numerical methods course. The linearization of a function f at a point $a \in dom(f)$ is denoted by L_f^a or simply L_f in this course,

$$\boxed{L_f^a(x) \equiv f(a) + f'(a)(x-a)}$$

The graph of L_f^a is the tangent line to $y = f(x)$ at $(a, f(a))$.

Example 5.1.1: (linearization can be used to calculate square roots)

$\boxed{E1}$ Let $f(x) = \sqrt{x}$ lets calculate $\sqrt{4.01}$ with $f(x)$'s linearization about $x = 4$. Notice $f'(x) = \frac{1}{2\sqrt{x}}$ thus

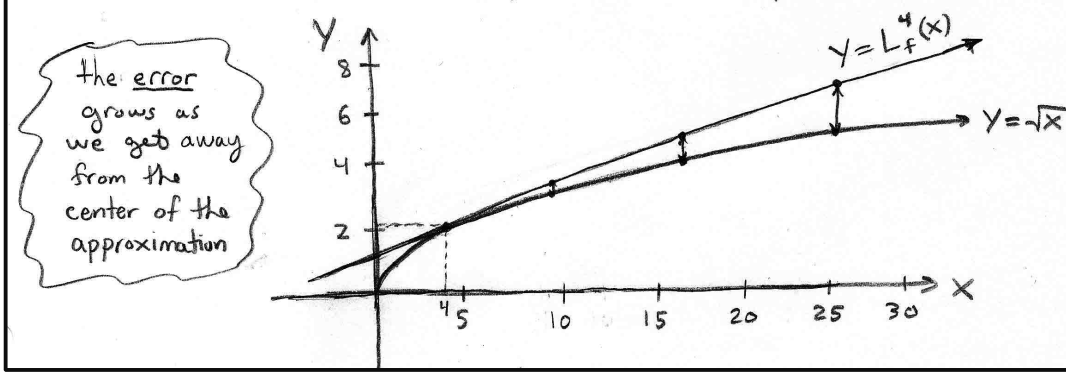
$$f(x) \cong L_f^4(x) = f(4) + f'(4)(x-4)$$
$$= \sqrt{4} + \frac{1}{2\sqrt{4}}(x-4)$$

Hence $\sqrt{4.01} \cong 2 + \frac{1}{4}(4.01-4) = 2 + \frac{0.01}{4} = \boxed{2.0025}$

This happens to exact, that is $\sqrt{4.01} = 2.0025$. What happens if we try to calculate $\sqrt{9}$ using this approximation?

$$\sqrt{9} \cong 2 + \frac{1}{4}(9-4) = 2 + \frac{5}{4} = 3.25 \neq 3$$
$$\sqrt{16} \cong 2 + \frac{1}{4}(16-4) = 2 + \frac{12}{4} = 5 \neq 4$$
$$\sqrt{25} \cong 2 + \frac{1}{4}(25-4) = 2 + \frac{21}{4} = 7.25 \neq 5$$

The further from $x = 4$ we get the worse $L_f^4(x)$ approximates,



This example shows that we can calculate good approximations to square roots, even when the computers and their robot slaves turn against us.

Example 5.1.2 and 5.1.3:

$\boxed{E2}$ Let $f(\theta) = \sin\theta$ approximate $\sin(\frac{5\pi}{16})$. We know that $\frac{5\pi}{16}$ is close to $\frac{\pi}{4}$. The values of sin and cosine at $\pi/4$ are known to us $\left(\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}\right)$ so center our approximation around $\theta = \frac{\pi}{4}$,

$$\begin{aligned} L_f^{\pi/4}(\theta) &= f(\pi/4) + f'(\pi/4)(\theta - \tfrac{\pi}{4}) \\ &= \sin(\pi/4) + \cos(\pi/4)(\theta - \tfrac{\pi}{4}) \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(\theta - \pi/4) \end{aligned}$$

Thus $\sin(\frac{5\pi}{16}) \cong \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\frac{5\pi}{16} - \frac{\pi}{4}\right) = \boxed{\frac{1}{\sqrt{2}}\left(1 + \frac{\pi}{16}\right)} \cong 0.8459$

compare that to the real value of $\sin(\frac{5\pi}{16}) = 0.8315$, not to shabby.

$\boxed{E3}$ The force of gravity on an object of mass m by the earth's mass M_E is given by NEWTON'S UNIVERSAL LAW OF GRAVITATION:

$$F = \frac{GmM_E}{r^2}$$

where r is the distance from center of earth, $r = R_E + h$ where h is the altitude and R_E is radius of earth. Now in your physics course you'll see $F = mg$ instead of the inverse square law above. How can both be true? Let F be a function of r notice $F'(r) = -\frac{2GmM_E}{r^3}$ thus

$$\begin{aligned} F(r) &\cong F(R_E) + F'(R_E)(r - R_E) \\ &= m\frac{GM_E}{R_E^2} - \frac{2GM_E}{R_E^3}m(r - R_E) \\ &\cong mg \end{aligned}$$

very small compared to other piece

So $F \cong mg$ when $r \cong R_E$, that is near the surface of the earth the force of gravity is nearly constant.

$$\frac{GM_E}{R_E^2} \cong 9.8\,\frac{m}{s^2}$$ (which should be familar to some of you.)

These examples just give you a small window into the utility of linearization. You should take our numerical methods course if you want to know more about how to perform these sorts of calculations with care. For applications, the true error in the approximation should be quantified.

5.2. RELATED RATES

Perhaps shockingly related rates problems involve rates of various quantities that are related. We consider two things which are connected through some equation. Both of those things can vary with time so we can consider the quantities as functions of time. Thus, we can differentiate the connecting equation and glean from the technique of implicit differentiation how the rates are related. Let's look at some actual examples (then come back and read this again)

Example 5.2.1 and 5.2.2:

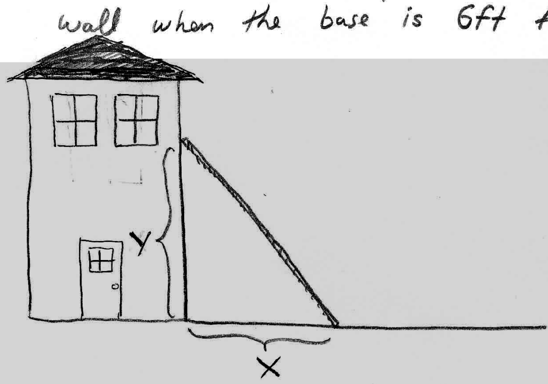
E1 Consider a spherical balloon, the volume V and radius r are related by $V = \frac{4}{3}\pi r^3$. If we know that we can blow 1cm^3 per second into the balloon then how quickly is V increasing when $r = 10\text{cm}$?

Note that V and r are functions of time. Moreover we are given that $\frac{dV}{dt} = 1\text{cm}^3$ thus

$$1\frac{\text{cm}^3}{\text{s}} = \frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1\text{cm}^3/\text{s}}{4\pi r^2} = \frac{1\text{cm}^3/\text{s}}{4\pi(10\text{cm})^2} = \boxed{0.000796\ \frac{\text{cm}}{\text{s}}}$$

E2 If a ladder 10ft long rests against a vertical wall. If the base of the ladder slides away from the wall at 1ft/s then how fast is the top of the ladder sliding down the wall when the base is 6ft from the wall? Work in ft & sec,



Given: $\frac{dx}{dt} = 1$ when $x = 6$.

Know also that: $x^2 + y^2 = 100$ since the ladder is 10ft long.

Notice $y = \sqrt{100 - x^2}$ so that when $x = 6$, $y = \sqrt{100-36} = \sqrt{64} = 8$

Differentiating w.r.t. time t gives $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

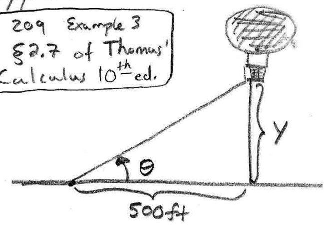
We are trying to determine $\frac{dy}{dt}$ when $x = 6$, solve for it,

$$\frac{dy}{dt} = \frac{-2x\frac{dx}{dt}}{2y} = \frac{-12\,(1\text{ft/s})}{16} = \boxed{-\frac{3}{4}\ \text{ft/s} = \frac{dy}{dt}}$$

Example 5.2.3 and 5.2.4:

E3 A balloon is rising vertically from a level field. Suppose an on-looker sees it rising at $0.14 \frac{\text{rad}}{\text{min}}$ when $\theta = \frac{\pi}{4}$ (when the ontooker is 500ft away from the launch spot).

pg. 209 Example 3 of §2.7 of Thomas' Calculus 10^{th} ed.



How fast is the balloon rising when $\theta = \pi/4$?

We know that $\frac{d\theta}{dt} = 0.14$ for $\theta = \frac{\pi}{4}$.

1. Relate y and θ : $\tan(\theta) = \frac{y}{500}$

2. Implicit Diff. to find the relation between $\frac{dy}{dt}$ and $\frac{d\theta}{dt}$:

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{500}\frac{dy}{dt}$$

3. Solve for $\frac{dy}{dt}$ and put in the givens,

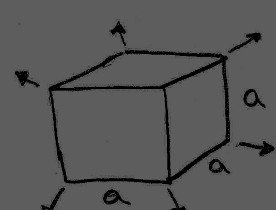
$$\frac{dy}{dt} = 500\sec^2\theta\frac{d\theta}{dt} = 500\left(\frac{1}{\cos\pi/4}\right)^2 \cdot 0.14 = 500(\sqrt{2})^2 \cdot 0.14$$

Thus putting in the units, $\boxed{\frac{dy}{dt} = 140\frac{\text{ft}}{\text{min}}}$

E4 Imagine a rubber cube is expanding so that the length of its side increases at a rate of 1 m/s when $a = 2m$ then how fast is its volume increasing when $a = 2m$?

$$V = a^3$$

$$\frac{dV}{dt} = 3a^2\frac{da}{dt} = 3(2m)^2 \cdot 1\frac{m}{s} = \boxed{12\frac{m^3}{s} = \frac{dV}{dt}}$$

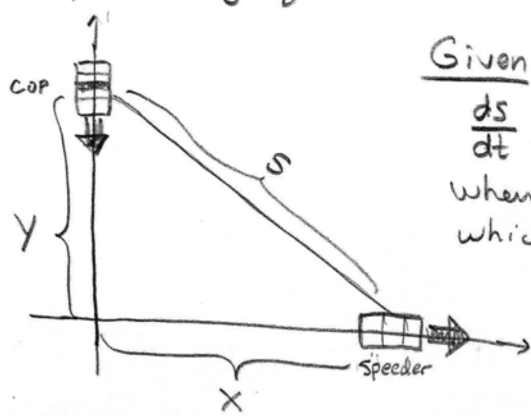


$V = a^3$

I find that in almost all word problems the first thing to do is draw a picture. Choose labels for the various quantities that are involved. Then write down any known equations. Once all of that is done then think about how to solve it. The mistake we often make is to try to see the end at the beginning. Sometimes there is something tricky in the middle that we'll not be able to circumnavigate until we have almost all the information in front of us ready to analyze.

Example 5.2.5:

$\boxed{E5}$ HIGHWAY CHASE: (pg 210 of Thomas's 10^{th} Ed.). Consider a police going 60mph sees a speeder to be going 20mph away from him when $x = 0.8$ and $y = 0.6$ in miles. How fast is the speeder going at that time?



Given:

$$\frac{ds}{dt} = 20 \quad \& \quad \frac{dy}{dt} = -60$$

when $x = 0.8$ and $y = 0.6$ which means that

$$s = \sqrt{x^2+y^2} = 1$$

Notice that $s^2 = x^2 + y^2 \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

we want to find $\frac{dx}{dt}$ so solve for it,

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{x}\left(s\frac{ds}{dt} - y\frac{dy}{dt}\right)\\ &= \frac{1}{0.8}\left(1(20) - 0.6(-60)\right)\\ &= \frac{1}{0.8}\left(20 + 36\right)\\ &= 70 \Rightarrow \boxed{\text{Speeder was going 70mph at that time}}\end{aligned}$$

I think it is fair to say that the difficult portion of these problems is how to set them up. The calculus content is not too bad. You probably have some homework which is not just a twist of one of these examples. That means you need to think about how to set it up on your own. Start with a picture.

5.3. MAXIMUM & MINIMUMS & GRAPHS

This section seeks to use calculus to better understand the shape of various graphs. We need to develop a fair amount of vocabulary. It is important that you assimilate the terms early on so we can understand each other in the examples.

Definition5.3.1:

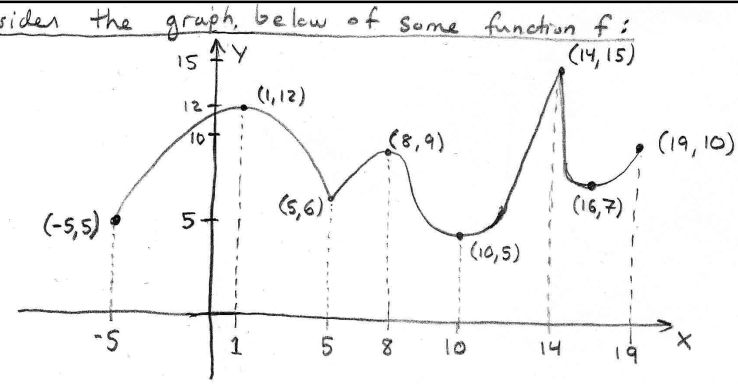
- A function f has an absolute maximum at c if $f(c) \geq f(x)$ for all $x \in dom(f)$. The absolute maximum is $f(c)$ in this case.
- A function f has an absolute minimum at d if $f(d) \leq f(x)$ for all $x \in dom(f)$. The absolute minimum is $f(d)$ in this case.
- We call the absolute maximum and minimum values the global extrema of f, a.k.a "extreme values".

Definition5.3.2:

- A function f has an local maximum at c if there exists an open subinterval J with $c \in J$ and $J \subset dom(f)$ such that $f(c) \geq f(x)$ for all $x \in J$. The local maximum is $f(c)$ in this case.
- A function f has an local minimum at d if there exists an open subinterval I with $d \in I$ and $I \subset dom(f)$ such that $f(d) \leq f(x)$ for all $x \in J$. The local minimum is $f(d)$ in this case.

Example 5.3.1:

E1 Consider the graph below of some function f:



- f has local maximums at: $x = 1, 8, 14$ and 19
  the values of the local max's are $= 12, 9, 15$ and 10 respectively.
- f obtains its absolute max at $x = 14$, its abs. max is $f(14) = \boxed{15}$
- f obtains its abs. min at $x = -5$ and 10, its abs. min is $\boxed{5}$

Theorem 5.3.1: (Extreme Value Theorem) If f is continuous on $[a, b]$ then f attains its absolute maximum of $f(c)$ on $[a, b]$ and its absolute minimum of $f(d)$ on $[a, b]$ for some $c, d \in [a, b]$. There may be multiple points where the extreme values are reached.

It is easy to see why this Theorem holds true, see Example 5.3.1. It's also easy to see why the requirement of continuity is essential. If the function had a vertical asymptote on $[a, b]$ then the function gets arbitrarily large or negative so there is no biggest or most negative value the function takes on the closed interval. Of course, if we had a vertical asymptote then the function is not continuous at the asymptote. I'll let you consult the text for a proof of the Theorem.

Look at the graph and notice we find that wherever there is a local minimum or maximum there is also a horizontal tangent line or no tangent line at all. Points where the derivative is zero or undefined are of critical importance to the analysis of graphs. Hence we define:

Definition 5.3.3: We say $x = c$ is a critical number of a function f if either $f'(c) = 0$ or $f'(c)$ does not exist. If c is a critical number then $(c, f(c))$ is a critical point.

Our observation that extrema must occur at critical points is known as Fermat's Theorem:

Theorem 5.3.2: (Fermat's Theorem) If $f(c)$ is a local extreme value for the function f then c is a critical number of f.

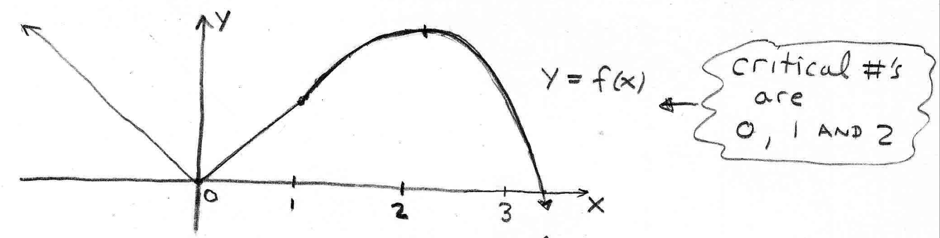
The converse of this Theorem is not true. We can have a critical number c such that $f(c)$ is **not** a local maximum or minimum. For example, $f(x) = x^3$ has critical number $x = 0$ yet $f(0) = 0$ which is neither a local max. nor min. value of $f(x) = x^3$. It turns out that (0,0) is actually an *inflection point* as we'll discuss soon. Another example of a critical point which yields something funny is a constant function; if $g(x) = c$ then $g'(x) = 0$ for each and every x. Technically, $y = c$ is both the minimum and maximum value of g. Constant functions are a sort of exceptional case in this game we are playing.

Example 5.3.2:

E2 Lets consider a piecewise defined function:

$$f(x) = \begin{cases} |x| & : x < 1 \\ -(x-2)^2+2 & : x \geq 1 \end{cases} \longrightarrow f'(x) = \begin{cases} -1 & x < 0 \\ 1 & 0 < x < 1 \\ 2(x-2) & 1 < x \end{cases}$$

Notice that $f'(0)$ and $f'(1)$ d.n.e so $x = 0, 1$ are critical #'s.
Also $2(x-2) = 0$ when $x = 2$ which is also a critical #.



Notice that this function takes its local min at $x=0$ and its local max at $x = 2$. This is no accident,

Digression: Theorems which we should mention:

I am trying to tell a story about how to take apart a function using the derivative and second derivatives. I admit these Theorems, while interesting, diverge from our main goal in this section. It is customary to cover them around this time in the calculus sequence. To be more careful I would like to spend a couple weeks really taking these apart and making delicate ϵ, δ-type proofs. However, there is not time and frankly most of you lack the mathematical maturity for such a journey at this juncture. So I say we state the Theorems, and if you want a proof there are arguments in Stewart which may suffice for now.

Again, sorry these appear in such a haphazard fashion in these notes. Rolle's Theorem goes to prove the Mean Value Theorem. Then in Stewart you'll find that the Mean Value Theorem is used to prove that a function increases where its derivative is positive. I think it is geometrically clear. Now perhaps you need to hear those arguments. If so I encourage you to read Stewart carefully, then get another text and read that as well. We will use the Mean Value Theorem in the proof for the Fundamental Theorem of Calculus.

Theorem 5.3.3: (Rolle's Theorem) Let f be a function such that

1. f is continuous on [a,b].
2. f is differentiable on (a,b).
3. $f(a) = f(b)$

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

See page 215 of Stewart for the proof. It primarily hinges on Fermat's Theorem. One interesting application worth mentioning. If the height of a particle is y(t) and it represents a particle thrown up into the air for 3 seconds meaning y(0) = y(3) = 0. Then v = dy/dt must be zero at some point during the flight of the particle. What goes up must come down, and before it comes down it has to stop going up. Thus common sense is upheld by Rolle's Theorem.

> *Theorem 5.3.4:* (Mean Value Theorem) Let f be a function such that
>
> 1. f is continuous on $[a, b]$.
> 2. f is differentiable on (a,b).
>
> Then there exists some $c \in (a, b)$ such that the derivative at that point is equal to the average rate of change over the whole interval. Meaning:
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{or if you prefer,} \quad f(b) - f(a) = f'(c)(b - a).$$

Proof: (essentially borrowed from Stewart pg. 216-217). The equation of the secant line to $y = f(x)$ on the interval [a,b] is $y = s(x)$ where $s(x)$ is defined via the point slope formula

$$s(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

The Mean Value Theorem says that there is some point on the interval [a,b] such that the slope of the tangent line is equal to the slope of the secant line $y = s(x)$. Consider a new function defined to be the difference of the secant line and the given function, call it h:

$$h(x) = f(x) - s(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Observe that $h(a) = h(b) = 0$ and h is clearly continuous on [a,b] because f is continuous and besides that the function is constructed from a sum of a polynomial with f. Additionally it is clear that h is differentiable on (a,b) since polynomials are differentiable everywhere and f was assumed to be differentiable on (a,b) to begin with. Thus Rolle's Theorem applies to h so there exists a $c \in (a, b)$ such that $h'(c) = 0$ which yields

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \quad \Longrightarrow \quad \boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}.$$

Physical Significance of the Mean Value Theorem: The term "mean" could be changed to "average". Apply the Theorem to the case that the independent variable is time t and the dependent variable is velocity v and we get the simple observation that the average velocity over some time interval is equal to the instantaneous velocity at some time during that interval of time. For example, if you go 60 miles in one hour then your average velocity is clearly 60mph. The Mean Value Theorem tells us that some time during that hour your instantaneous velocity was also 60mph.

Theorems about Constants and Derivatives:

> *Theorem 5.3.5:* If $\frac{df}{dx} = 0$ for each $x \in (a,b)$ then f is a constant function on (a,b)

Proof: apply the Mean Value Theorem. We know we can because the derivative exists at each point on the interval and this implies the function is continuous on the open interval, so it is continuous on any closed subinterval of (a,b) call it $J \subset (a,b)$. We have to apply the Mean Value Theorem to $J = [a_o, b_o]$ because we do not know for certain that the function is continuous on the endpoints. We find,

$$0 = \frac{f(b_o) - f(a_o)}{b_o - a_o} \quad \Longrightarrow \quad f(b_o) = f(a_o)$$

But this is for an arbitrary closed subinterval hence the function is constant on (a,b).

Caution: we cannot say the function is constant beyond the interval (a,b). It could do many different things beyond the interval in consideration. Piecewise continuous functions are such examples, they can be constant on the pieces yet at the points of discontinuity the function can jump from one constant to another.

> *Theorem 5.3.6:* If $\frac{df}{dx} = \frac{dg}{dx}$ for each $x \in (a,b)$ then $f(x) = g(x) + c$ for some constant c.

Proof: Apply Theorem 5.3.5 to $\frac{df}{dx} - \frac{dg}{dx} = 0$ to obtain $f(x) - g(x) = c$ hence $f(x) = g(x) + c$.

Notice that the assumption is that they are equal on an open interval. If we had that the derivatives of two functions were equal over some set which consisted of disconnected pieces then we could apply Theorem 5.3.6 to each piece separately then we would need to check that those constants from different components matched up. (for example if $\frac{df}{dx} = \frac{dg}{dx}$ on $(0,1) \cup (2,3)$ then we could have that $f(x) = g(x) + 1$ on (0,1) whereas $f(x) = g(x) + 2$ on (2,3)).

Physical Significance: If we have equal velocities over some time interval then the displacement between our positions at any time will be constant.

Theorem 5.3.2: (Closed Interval Method) If we are given function f which is continuous on a closed interval [a,b] the we can find the absolute minimum and maximum values of the function over the interval [a,b] as follows:

1. Locate all critical numbers x = c in (a,b) and calculate f(c).
2. Calculate f(a) and f(b).
3. Compare values from steps 1. and 2. the largest is the absolute maximum, the smallest (or largest negative) value is the absolute minimum of f on [a,b].

Examples 5.3.4, 5.3.4, 5.3.5 and 5.3.6 illustrate the Closed Interval Method.

Example 5.3.3 and 5.3.4:

$\boxed{E3}$ Let $f(x) = \sin(x)$ for $0 \leq x \leq 2\pi$. Note f is continous with,

$$f'(x) = \cos(x)$$

Critical numbers are sol^{n}'s to $\cos(x) = 0$ for $0 \leq x \leq 2\pi$, namely,

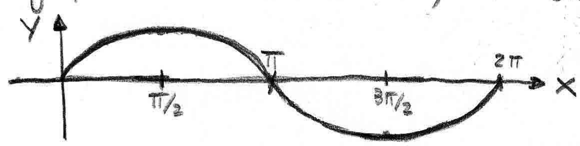
$$x = \frac{\pi}{2} \text{ and } \frac{3\pi}{2}$$

Now calculate:

$$\left.\begin{aligned} f(0) &= \sin(0) = 0 \\ f(\pi/2) &= \sin(\pi/2) = 1 \\ f(3\pi/2) &= \sin(3\pi/2) = -1 \\ f(2\pi) &= \sin(2\pi) = 0 \end{aligned}\right\}$$

Comparing we find that the max is 1 at $x = \pi/2$ and the min is -1 at $x = 3\pi/2$

Hence the graph we know and love, $y = \sin(x)$,

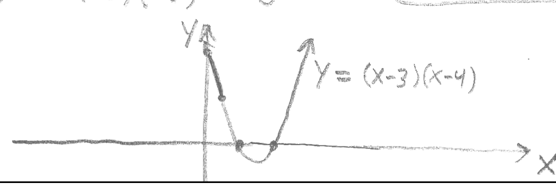


$\boxed{E4}$ Consider $f(x) = (x-3)(x-4)$ find max on $[0,1]$. Notice

$$f'(x) = (x-4) + (x-3) = 2x - 7 \Rightarrow \boxed{f'(x) = 0 \text{ for } x = 7/2}$$

Thus there are no critical #'s for f on $[0,1]$ we need only check the endpts.

$$\begin{aligned} f(0) &= (-3)(-4) = 12 \\ f(1) &= (-2)(-3) = 6 \end{aligned} \Rightarrow \boxed{f(0) = 12 \text{ is max of } f \text{ on } [0,1]}$$



Example 5.3.5:

E5

$f(x) = 3x^2 - 12x + 5$ on $[0,3]$

① $f'(x) = 6x - 12$ well-defined on $[0,3] \Rightarrow$ critical #'s only happen with $f'(c) = 0$.

$f'(c) = 6c - 12 = 0 \Rightarrow c = 2$ which is on $[0,3]$ (Critical Number)

$f(2) = 3(2)^2 - 12(2) + 5 = 12 - 24 + 5 = -7$

② $f(0) = 5$

$f(3) = 3(9) - 12(3) + 5 = 27 - 36 + 5 = -4$ } values of f at the endpoint.

③ Thus

$f(2) = -7$ is the absolute minimum of f on $[0,3]$

$f(0) = 5$ is the absolute maximum of f on $[0,3]$

Example 5.3.6:

E6

$f(x) = x^4 - 2x^2 + 3$ on $[-2, 3]$

① $f'(x) = 4x^3 - 4x$ so $f'(c) = 0$ are the only type of critical #'s the derivative exists everywhere.

$f'(c) = 4c^3 - 4c = 0$

$4c(c^2 - 1) = 0 \therefore c = 0$ & $c = \pm 1$ are critical #'s

$f(0) = 3$

$f(1) = 1 - 2 + 3 = 2$

$f(-1) = 1 - 2 + 3 = 2$

② Endpoints $f(-2) = (-2)^4 - 2(-2)^2 + 3 = 16 - 8 + 3 = 11$

$f(3) = 3^4 - 2(3)^2 + 3 = 81 - 18 + 3 = 66$

③ Comparing ① & ② we find

$f(\pm 1) = 2$ is the absolute minimum on $[-2, 3]$

$f(3) = 66$ is the abs. max. on $[-2, 3]$

Sign of derivative test: when does a function decreases or increases:

- If $\frac{df}{dx} > 0$ for all $x \in (a, b)$ then f is increasing on (a, b).
- If $\frac{df}{dx} < 0$ for all $x \in (a, b)$ then f is decreasing on (a, b).
- If $f(x) = 0$ then the function is not increasing or decreasing at x.

We defined increasing and decreasing in Definition 2.3.2. In words, a function is increasing if as x increases the values of the function likewise increase. Let's examine a few examples of this test in action:

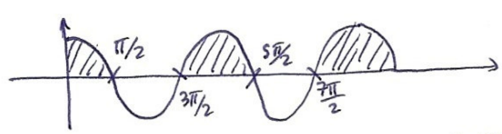
Example 5.3.7 and 5.3.8:

$\boxed{E7}$ Let $f(x) = e^x$ then $f'(x) = e^x > 0$ for $x \in \mathbb{R}$ so $f(x) = e^x$ is increasing on all of $\mathbb{R}$.

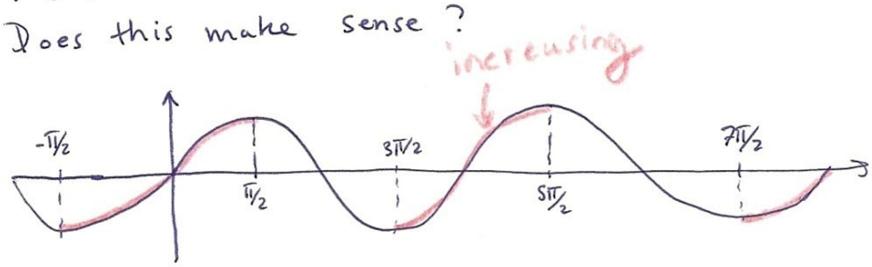
$\boxed{E8}$ Let $f(x) = x^2 - 2x + 1$ then $f'(x) = 2x - 2 = 2(x-1)$
So $f'(x) > 0$ when $x > 1$ and $f'(x) < 0$ when $x < 1$
Thus f is increasing on $(1, \infty)$ and f is decreasing on $(-\infty, 1)$.

Example 5.3.9:

$\boxed{E9}$ Let $f(x) = \sin(x)$ then $f'(x) = \cos(x)$ so we draw a graph



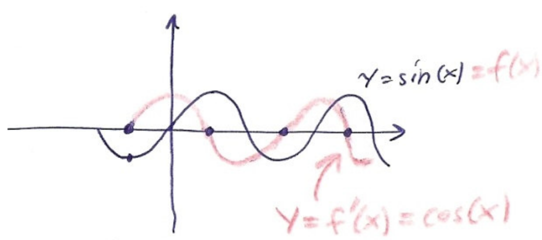
$f'(x) > 0$ when x is in $(-\pi/2, \pi/2), (3\pi/2, 5\pi/2), (7\pi/2, 9\pi/2), \ldots$
$f'(x) < 0$ when x is in $(\pi/2, 3\pi/2), (5\pi/2, 7\pi/2), (4\pi/2, 11\pi/2), \ldots$
Does this make sense?

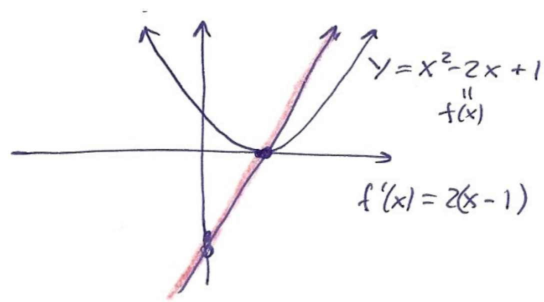


the graph above is of $y = \sin(x)$ and the red illustrates $\cos(x) > 0$ when the sine function is increasing.

Discussion: What does the sign of the derivative changing suggest?

If the derivative df/dx is a continuous function then we can conclude that the derivative must be zero on some interval if it changes sign on that interval. *(Recall that the Intermediate Value Theorem helped us find zeros of functions by examining if the function changed signs on some interval.)* Notice that when the derivative is zero at some point that means the function has a horizontal tangent line at that point. Such points are local extrema, it will be the point on the graph which is the highest or lowest point in an open neighborhood around the point. For examples:





The derivatives are graphed in red and you can see that they are zero where the function is minimized or maximized. This motivates (but does not prove!) the First Derivative Test:

> *Theorem 5.3.3:* (First Derivative Test) If we are given function f which is continuous on an open interval containing a critical number c then:
> 1. If df/dx changes signs from positive to negative at c then $f(c)$ is a local max.
> 2. If df/dx changes signs from negative to positive at c then $f(c)$ is a local min.
> 3. If df/dx does not change signs at c then $f(c)$ is not a local extrema.

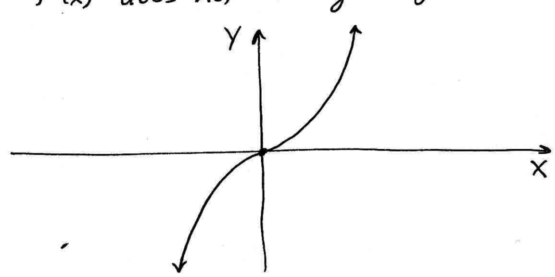
Example 5.3.10a: (this one illustrates case 3.)

$\boxed{E10}$ Example of C.) Let $f(x) = x^3$ then $f'(x) = 3x^2$

thus $c = 0$ is a critical # for f but

$f'(x) = 3x^2 > 0$

So $f'(x)$ does not change signs at $c = 0$. Why ?



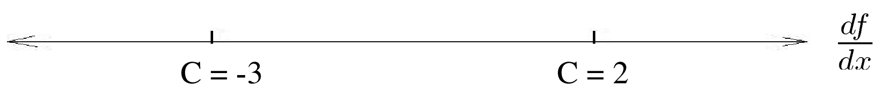
The example above is missing something I'd like to put on each example as a rule. I think we should use a ***sign chart*** to organize information about signs. Unfortunately, the homework solutions do not have this organizational aid for the most part. Look ahead to Example 5.3.12 for examples of "sign charts". Logically we don't really need them, but I think I'm going to make their use a requirement for our class. This is just so we can be on the same page when organizing our ideas. I will add many of these sign charts in lecture, so it is important to take notes on this point.

Notice that Examples 5.3.7, 5.3.8 and 5.3.9 all illustrate cases 1 and 2 of the First Derivative Test. Let me give one more examples before we go on to the Second Derivative Test.

Example 5.3.10b:
Let $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x$. Find all critical numbers and classify the critical points as local maximums, minimums, or neither. Observe,

$$f'(x) = x^2 + x - 6 = (x-2)(x+3).$$

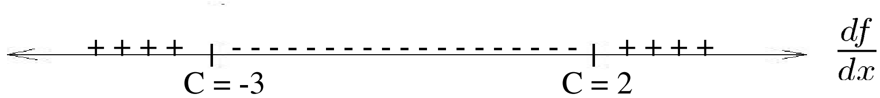
We have two critical numbers; c = 2 and c = -3. Lets draw the sign chart,



Then we test a point somewhere in the interior of each region,

$$f'(-4) = (-4-2)(-2+3) = 8 > 0$$
$$f'(0) = (-2)(3) = -6 < 0$$
$$f'(3) = (3-2)(3+3) = 6 > 0$$

Which suggests we fill in the sign chart as follows:



By the First Derivative Test we conclude,
$f(-3) = -27/3 + 9/2 - 6(-3) = 27/2$ is a local maximum (case 1),
$f(2) = 8/3 + 4/2 - 6(2) = -22/3$ is a local minimum (case 2).

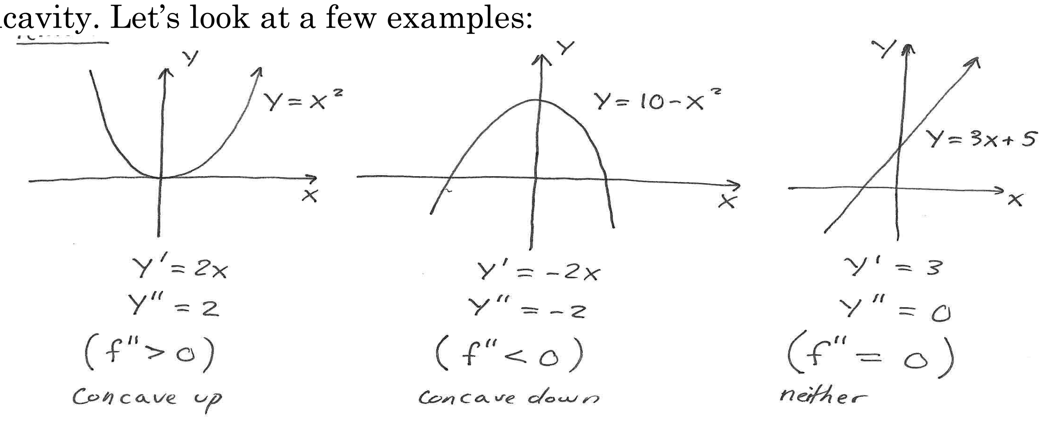
<u>*Question:*</u> *Observe that* $f(x) = x\frac{\sqrt{(x-1)^2}}{x-1}$ *for* $x \neq 1$ *and* $f(1) = 1$ *has critical number c = 1. Moreover, the derivative changes sign from* $df/dx = -1$ *for* $x < 1$ *to* $df/dx = 1$ *for* $x > 1$. *Is it in fact the case that* f *obtains the local maximum* $f(1) = 1$ *at (1,1) ? Does this contradict the First Derivative Test ? Explain.*

Concave-up and Concave-down:

The change in a function is captured by the first derivative; increasing functions have positive derivative functions while decreasing functions have negative derivative functions. It turns out that the *concavity* of a function is captured by the second derivative. A function is *concave up* on an interval J if the function has the shape of a bowl which is right-side up over that interval J. A function is *concave down* on an interval J if the function has the shape of a bowl which is up-side down over that interval J. I personally find these sort-of geometric definitions rather unsatisfying. I challenge you to find a clear definition stated in Stewart's text. That said, I say we take the following as the real definition for concave up or down.

> *Definition 5.3.4:* Let f be a function with $f'(x)$ and $f''(x)$ well-defined for each $x \in J$ then we say that
>
> - $f(x)$ is concave-up on J if $f''(x) > 0$ for each $x \in J$
> - $f(x)$ is concave-down on J if $f''(x) < 0$ for each $x \in J$
>
> A function is concave up(or down) at a point if there exists a neighborhood about the point for which the function is concave up(or down). We say the function changes concavity at a point c if the function is concave-up(down) to the left of the point and concave-down(up) to the right of the point (c, f(c)). If the function changes concavity at a point (c, f(c)) then we say that (c , f(c)) is a point of *inflection.*

Consider this, if a function has a the shape of a bowl right side up then the slopes of the tangent lines will increase as we increase x. On the other hand, if a function has the shape of a bowl upside down then the slopes of the tangent lines will decrease as x increases. In other words, the derivative is an increasing function where the function is concave-up and the derivative is a decreasing function where the function is concave-down. This proves the definition given above is equivalent to the geometric bowl-based definition for concavity. Let's look at a few examples:



- *The line is an exceptional case, then thing between concave up and down.*

(by the way, the term "convex" used to be used for concave down, this term is still used in physics particularly in the study of optics)

Concavity test: when is a function concave up or down:

- If $f''(x) > 0$ for all $x \in (a, b)$ then f is concave-up on (a, b).
- If $f''(x) < 0$ for all $x \in (a, b)$ then f is concave-down on (a, b).
- If $f''(c) = 0$ then the function ***might*** have an inflection point at c.

I emphasize that when the second derivative is zero we might find an inflection point, but it doesn't have to be the case(see picture on last page for example). Also, it could be that we find an inflection point where the second derivative does not exist. There are many possibilities. The same is true for critical points. When a critical point is not at a local max or min it could be an inflection point, but it might be something else, there are countless other options. ***Bonus Point:*** *find me an example of a continuous function which has a nonzero derivative and a critical point which is neither at a local maximum, minimum or inflection point.*

Theorem 5.3.3: (Second Derivative Test) If a function f has $\frac{df}{dx}$ continuous on an open neighborhood containing c such that $f'(c) = 0$ then:

1. If $f''(c) < 0$ then $f(c)$ is a local maximum.
2. If $f''(c) > 0$ then $f(c)$ is a local minimum.
3. If $f''(c) = 0$ then the Second Derivative Test is inconclusive.

Proof: Suppose that $f''(c) < 0$ and $f'(c) = 0$ then $(c, f(c))$ has a horizontal tangent line and the function is upside-down bowl-shaped near the point, hence it is geometrically clear that the point is a local maximum. Likewise if $f''(c) > 0$ and $f'(c) = 0$ then $(c, f(c))$ has a horizontal tangent line and the function is rightside-up bowl-shaped near the point, hence it is geometrically clear that the point is a local minimum. Technically, this proof leaves something to be desired, but this is the heart of it.

$\boxed{E11}$ Consider $f(x) = x^3 - 12x - 5$

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$

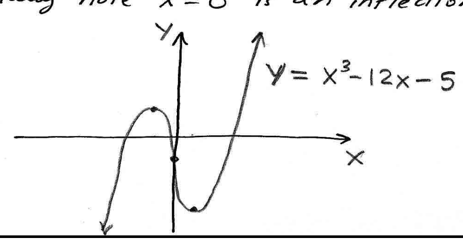
$$f''(x) = 6x$$

Critical #'s are $x = \pm 2$ where $f'(\pm 2) = 0$. Then

$f''(2) = 6(2) = 12 \quad \therefore \quad f(2) = 8 - 24 - 5 = -21$ local min.

$f''(-2) = 6(-2) = -12 \quad \therefore \quad f(-2) = -8 + 24 - 5 = 11$ local max

We can additionally note $x = 0$ is an inflection point. Sketching f,



Example 5.3.12:

E12 We will study a # of questions about $f(x) = \dfrac{x}{(1+x)^2}$. I will introduce some notational helps to keep track of signs of f, f' and f''. If you have your own system for finding zeroes exc... from precalculus then feel free to use it. Find the following,

a.) CRITICAL POINTS
b.) intervals on which f is decreasing / increasing
c.) local maximums / minimums
d.) intervals on which f is concave up / down
e.) inflection points
f.) zeroes of function (precalculus here)
g.) graph f carefully using a → f.

Lets begin by calculating f' and f'',

$$f'(x) = \frac{\frac{dx}{dx}(1+x)^2 - x\frac{d}{dx}(1+x)^2}{(1+x)^4}$$

$$= \frac{1+\cancel{2x}+x^2-\cancel{2x}-2x^2}{(1+x)^4}$$

$$= \frac{1-x^2}{(1+x)^4} \quad : \quad 1-x^2 = (1+x)(1-x) \text{ thus,}$$

$$= \boxed{\frac{1-x}{(1+x)^3} = f'(x)}$$

a.) Notice $f'(1) = 0$ and $f'(-1)$ d.n.e thus the critical #'s of f are $\boxed{c = \pm 1}$. Now we know that f' can only change sign at critical #'s thus we can just check a pt. from each region to completely describe f' positive/neg.

$$\underset{}{- - - - - - -} \underset{-1}{|} + + + + + + + + + \underset{1}{|} - - - - - - - \to f'$$

b.) from the sign-chart of f' we can read off that

f is decreasing on $(-\infty, -1)$ and $(1, \infty)$
f is increasing on $(-1, 1)$

Example 5.3.12 continued:

c.) We use the 1^{st} derivative test to conclude that since f' changes from $(+)$ to $(-)$ at $x = 1$ the value $f(1) = \frac{1}{2^2} = \frac{1}{4}$ is a local max of f at $x = 1$. ($x = -1$ is a vertical asymptote so although its a critical point $f(-1)$ can't be a local max, it d.n.e.)

d.) We need to find f'' and how it behaves,

$$f''(x) = \frac{d}{dx}\left(\frac{1-x}{(1+x)^3}\right) = \frac{2(x-2)}{(x+1)^4}$$

Notice that f'' can only change sign at its zeroes and vertical asymptotes which occur at $x = 2$ and $x = -1$. We need to check 3 points to generate the following,

$$\underset{-1}{- - - - - - |} - - - - - - \underset{2}{|} + + + + + + \longrightarrow f''$$

We can read off the sign chart for f'' that

f is concave down on $(-\infty, 2)$
f is concave up on $(2, \infty)$

e.) f changes concavity at $x = 2$ which is the only inflection point.

f.) arguably I should've put this 1^{st}, oh well anyway $f(0) = 0$ and $f(-1)$ d.n.e, f must change signs either 0 or -1 so we again just need to check 3 testpts.

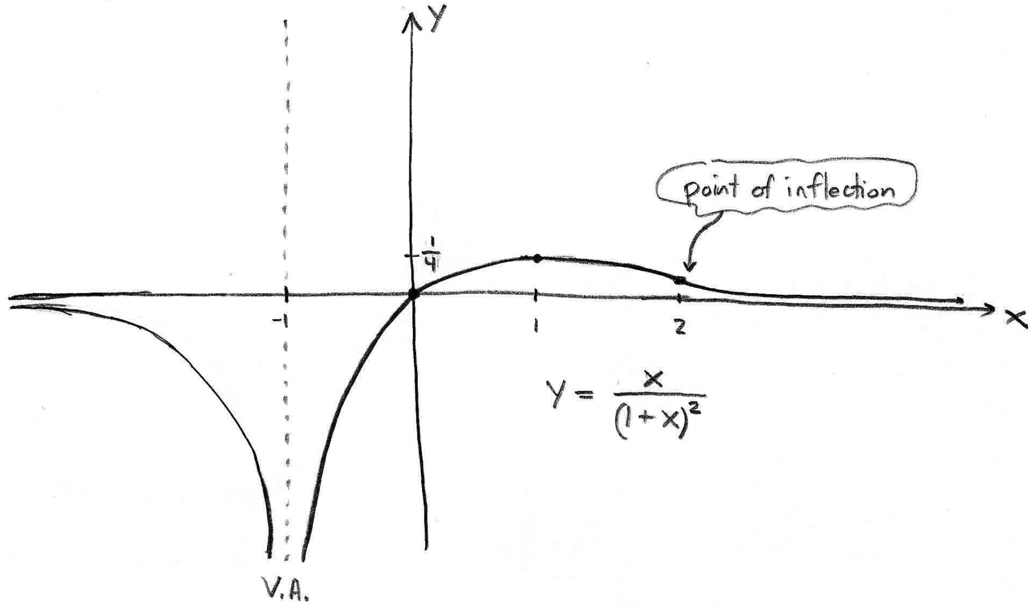
$$\underset{-1}{- - - - - - |} - - - - - - - - \underset{0}{|} + + + + + + + + \longrightarrow f$$

ok, I only asked for the zeroes of f but this will help us graph f next

Remark: a function which is continuous everywhere except at vertical asymptotes can only change sign at its zeroes and vertical asymptotes. This idea applies to the function, its derivative and its second derivative. This is a consequence of the Intermediate Value Theorem and plain-old common sense.

Example 5.3.12 conclusion:

$\boxed{\text{E12 conclusion}}$ Now lets assemble the graph of f using all the data,



Remark on case 3 of 2nd Derivative Test: Maybe you are wondering, what is an example of a function which falls into case 3 of the derivative test? One simple example is a line $f(x) = mx + b$ which has $f'(x) = m$. Clearly df/dx is continuous everywhere. Notice $f''(x) = 0$ for each $x \in \mathbb{R}$. There are two cases:

1. $m = 0$ thus $f(x) = b$ and $y = b$ is the maximum and minimum value of the function at all points.

2. $m \neq 0$ then $f(x) = mx + b$ and the function has no local or global extreme values. If we restrict attention to a closed interval then the line takes its extreme values with respect to that interval on the endpoints.

Example 5.3.13:

$\boxed{E13}$ Do the same as $\boxed{E12}$ for xe^{-x},

$f(x) = xe^{-x} \Rightarrow$ $- - - - - \underset{0}{|} + + + + + + + + \rightarrow f$

$f'(x) = e^{-x}(1-x) \Rightarrow$ $+ + + + + + + \underset{1}{|} - - - - - - - \rightarrow f'$

$f''(x) = e^{-x}(x-2) \Rightarrow$ $- - - - - - - \underset{2}{|} + + + + + + + + \rightarrow f''$

Therefore,

a.) critical point is $x = 1$

b.) f increases on $(-\infty, 1)$ and decreases on $(1, \infty)$

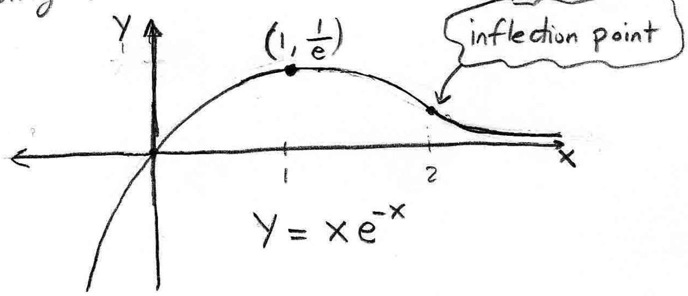
c.) f has a local max of $f(1) = \frac{1}{e}$ at $x = 1$ by 1^{st} Der. Test.

d.) f concave up on $(2, \infty)$ and concave down on $(-\infty, 2)$.

e.) $x = 2$ is an inflection point of f.

f.) $x = 0$ is the only zero of f

g.) graph →



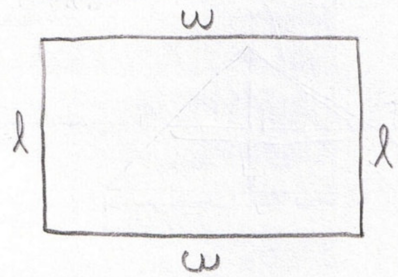
We have at our disposal all the tools we need to figure out what a function looks like locally. Given the formula we simply take a derivative or two and <u>*think*</u>. The global picture of the function requires one more piece of information: the asymptotic behavior of the function. Such behavior is captured by the limits at $\pm\infty$. We will discuss this last topic on graphing in the section 5.5.

5.4. OPTIMIZATION

This section seeks to use calculus to answer questions like: what is the biggest, smallest, cheapest longest, shortest, hottest, coldest, best, worst, etc … Given some word problem or equation which models a particular physical problem what are the possibilities, what are the extremes? The analysis developed and discussed in the preceding section allows us to tackle such questions in a way that was unavailable before the advent of calculus.

Example 5.4.1:

E1: Given 400ft of fencing what dimensions should you give a rectangular pen as to maximize the area?



$$2\ell + 2w = 400$$

$$A = \ell w$$

Notice that $\ell = 200 - w$

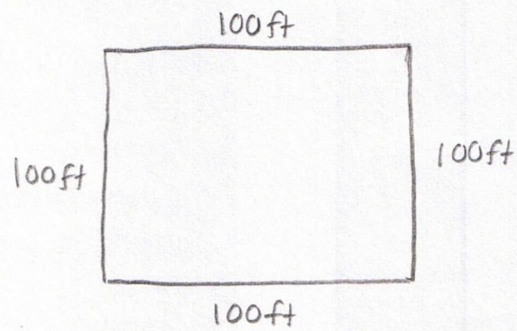
$$\Rightarrow \boxed{A = (200-w)w = 200w - w^2}$$

Maximize A as a function of w then,

$$\frac{dA}{dw} = 200 - 2w \Rightarrow \frac{dA}{dw} = 0 \text{ when } w = 100$$

Then note $\frac{d^2A}{dw^2} = -2 < 0 \Rightarrow A(100)$ is a maximum.

Thus by 2nd derivative test $w = 100$ gives max. area. Thus

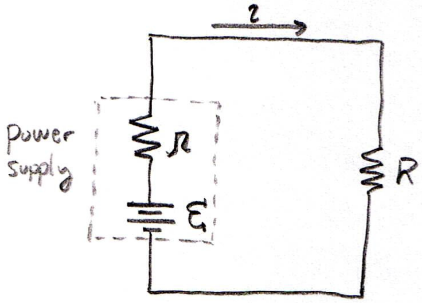


$$\ell = 200 - 100 = 100$$

These dimensions maximize the area of the pen.

Example 5.4.2:

E2 Given a power supply with fixed voltage $\mathcal{E}$ and internal resistance r what load R will maximize the power delivered to the load?



$\mathcal{E} = ir + iR$: Kirchoff's Voltage Law

$P = i^2 R$: Power delivered to R by current i

Solve to find $i = \dfrac{\mathcal{E}}{r+R}$ then substitute into $P = i^2 R = \dfrac{\mathcal{E}^2 R}{(r+R)^2}$

Now do calculus on P as a function of R,

$$\frac{dP}{dR} = \mathcal{E}^2 \frac{d}{dR}\left(\frac{R}{(r+R)^2}\right)$$

$$= \mathcal{E}^2\left[\frac{(r+R)^2 - 2(r+R)R}{(r+R)^4}\right]$$

$$= \frac{\mathcal{E}^2}{(r+R)^4}\left[r^2 + \cancel{2rR} + R^2 - \cancel{2rR} - 2R^2\right]$$

$$= \frac{\mathcal{E}^2}{(r+R)^4}\left[r^2 - R^2\right]$$

$$= \frac{\mathcal{E}^2}{(r+R)^2}\left[(r-R)(r+R)\right]$$

: $r, R > 0$ for physical reasons so $R = r$ is only interesting critical point.

$$\underset{0}{|}\;+++++++++++\;\underset{r}{|}\;-----------\rightarrow \frac{dP}{dR}$$

Thus by 1st derivative test the max power is delivered to R when $R = r$. This is a simple case of the

"Max Power Transfer Th$^{\underline{m}}$"

Notice it implies that the Max Efficiency of a simple power supply is 50%. Electrical engineers can build more efficient power supplies called "switching" power supplies, in those supplies the output is automatically taylored as to be best for the load. It's a tricky buisness.

Example 5.4.3:

E3 The range of a projectile fired at angle θ with velocity V can be shown to be $R = \frac{V\sin(2\theta)}{g}$ assuming no friction and $g = 9.8\ m/s^2$. What angle θ maximizes the range. What is R_{max} ?

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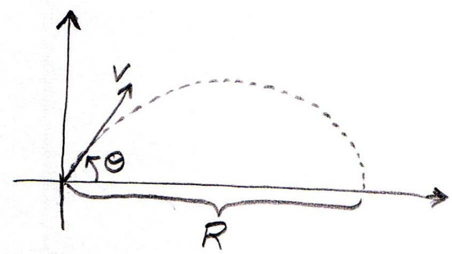
We notice that V and g are constants, R is the dependent (like Y) variable and θ is the independent variable (like x).

$$\frac{dR}{d\theta} = \frac{d}{d\theta}\left(\frac{V}{g}\sin(2\theta)\right) = \frac{V}{g}\frac{d}{d\theta}\left(\sin(2\theta)\right) = \frac{2V}{g}\cos(2\theta)$$



For physical reasons we restrict θ to $0 \leq \theta \leq \pi/2$.

$\cos(2\theta) = 0$ for $2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$

only $\theta = \pi/4$ is a physically interesting critical value.

Calculate $\frac{d^2R}{d\theta^2} = \frac{-4V}{g}\sin(2\theta) \Rightarrow \frac{d^2R}{d\theta^2}(\theta = \frac{\pi}{4}) = \frac{-4V}{g}\sin(\pi/4)$

Thus we note $\frac{d^2R}{d\theta^2}(\theta = \frac{\pi}{4}) = \frac{-4V}{\sqrt{2}g} < 0$ hence by 2^{nd} derivative test we have that $R(\pi/4)$ is the max. range. $\boxed{\theta = \pi/4}$

$$R_{max} = \frac{V\sin(2\frac{\pi}{4})}{g} = \boxed{\frac{V}{g} = R_{max}}$$

Example 5.4.4:

E4 The height of a projectile launched straight up with velocity V_0 is given by:

$$Y = V_0 t - \frac{1}{2} g t^2$$

What is the maximum height and at what time is it reached?

Notice that V_0 and g are just numbers so,

$$\frac{dY}{dt} = \frac{d}{dt}\left(V_0 t - \frac{1}{2} g t^2\right) = V_0 - gt \quad \text{(the velocity)}$$

$$\frac{d^2Y}{dt^2} = \frac{d}{dt}\left(V_0 - gt\right) = -g \quad \text{(the acceleration)}$$

Now $\frac{dY}{dt} = 0 = V_0 - gt \implies \boxed{t_{max} = V_0/g \text{ is critical time.}}$

We know that $Y(t_{max}) = Y_{max} = V_0\left(\frac{V_0}{g}\right) - \frac{1}{2} g \left(\frac{V_0}{g}\right)^2 = \boxed{\frac{V_0^2}{2g} = Y_{max}}$ is indeed the max height because $Y'' = -g < 0$. (2^{nd} der. test.)

Example 5.4.5:

E5 Find the point on the line $Y = 4x + 7$ that is closest to the origin $(0,0)$.

We want to minimize the distance $S = \sqrt{x^2 + y^2}$ where $Y = 4x + 7$ so consider

$$\frac{d}{dx}\left(\sqrt{x^2 + (4x+7)^2}\right) = \frac{2x + 8(4x+7)}{2\sqrt{x^2 + (4x+7)^2}} = \frac{ds}{dx}$$

So then $\frac{ds}{dx} = 0 \iff 2x + 8(4x+7) = 0 \iff 34x = -56$

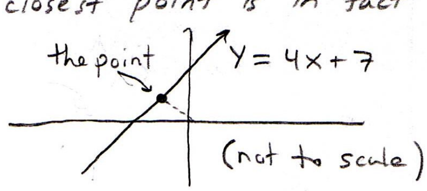
Thus $x = -28/17$ is the only critical point.

$$\overset{- - - - - - \;|\; + + + + + + + + +}{\longleftarrow\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\underset{-\frac{28}{17}}{+}\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\longrightarrow} \quad \frac{ds}{dx}$$

Thus $s(-28/17 = x)$ is the min. distance to $Y = 4x + 7$ from the origin by 1^{st} derivative test. The Y-coordinate is

$$Y\left(\frac{-28}{17}\right) = 4\left(\frac{-28}{17}\right) + 7 = \frac{7}{17}$$

So the closest point is in fact $\boxed{\left(\frac{-28}{17}, \frac{7}{17}\right)}$



Example 5.4.6:

What lengths x_{min} and x_{max} give a mass-spring system obeying Hooke's Law $F = -kx$ the minimum and maximum Kinetic energy K? Recall that energy is conserved such a system and in particular we have that $K = \frac{1}{2}mv^2$ and $U = \frac{1}{2}kx^2$ with $K + U = E_o$ (a positive constant).

Solution: Solve for K,

$$K = E_o - \frac{1}{2}kx^2$$

Look for critical numbers:

$$\frac{dK}{dx} = 0 - \frac{1}{2}k(2x) = -kx \quad \Longrightarrow \quad x = 0 \text{ only critical number}$$

Furthermore, notice

$$\frac{d^2K}{dx^2} = -k < 0 \text{ the spring constant } k > 0 \text{ by convention}$$

Which shows that $x_{max} = 0$ yields the maximum Kinetic energy of $K = E_o$. How did I miss x_{min}? What am I not paying attention to about the Kinetic energy function? Let me give a colossal hint: what is the domain for K? We know that mass is positive and so is v^2 so it stands to reason that $K \geq 0$ hence $E_o - \frac{1}{2}kx^2 \geq 0$ or simply $E_o \geq \frac{1}{2}kx^2$ thus $x^2 \leq 2E_o/k$ which indicates that $-\sqrt{2E_o/k} \leq x \leq \sqrt{2E_o/k}$. We need to use the closed interval method for this problem. Perhaps that was not obvious from the start. Lets check K on the endpoints,

$$K\big(x = -\sqrt{2E_o/k}\big) = E_o - \frac{1}{2}k\big(-\sqrt{2E_o/k}\big)^2 = 0$$
$$K\big(x = \sqrt{2E_o/k}\big) = E_o - \frac{1}{2}k\big(\sqrt{2E_o/k}\big)^2 = 0$$

Thus $x_{min} = \pm\sqrt{2E_o/k}$ where the Kinetic energy is zero.

Moral of story: *domains matter, its not "just" a math thing. It can happen that the interesting case is at an endpoint and not at a critical point.*

Example 5.4.7:

E7 Find two positive numbers whose product is 100 and whose sum is a minimum

Let m and n be the numbers. We have that

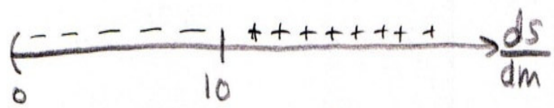
$$mn = 100$$

$$S = m + n = m + \frac{100}{m}$$

Now minimize S as a function of m,

$$\frac{dS}{dm} = 1 - \frac{100}{m^2} = 0 \Rightarrow m = \pm 10$$ are critical values of m.

Notice that $m=0$ is also a critical value. However only $m=10$ matters because we're looking for positive #'s.



Thus S is minimized for $m=10$. Thus the numbers are $\boxed{m=10 \text{ and } n=10}$.

Example 5.4.8: when you take differential equations you will learn how to solve Newton's equations in the case of a velocity dependent friction force. In the following example we analyze the solution to see how the spring vibrates.

E8

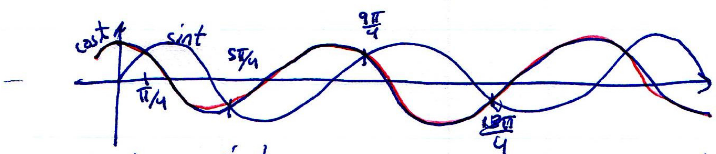
Newton's 2nd Law. $ma = F_{spring} + F_{friction}(velocity)$ has sol^{n} below,

$$Y = A\sin(t)e^{-t}$$

$$Y' = (A\cos t)e^{-t} + (A\sin t)(-e^{-t})$$
$$= Ae^{-t}(\cos t - \sin t)$$

$Y'(t) = 0 \Rightarrow \cos t - \sin t = 0 \Rightarrow \tan(t) = 1$

$t = \pi/4,\ 5\pi/4,\ 9\pi/4,\ 13\pi/4, \dots$



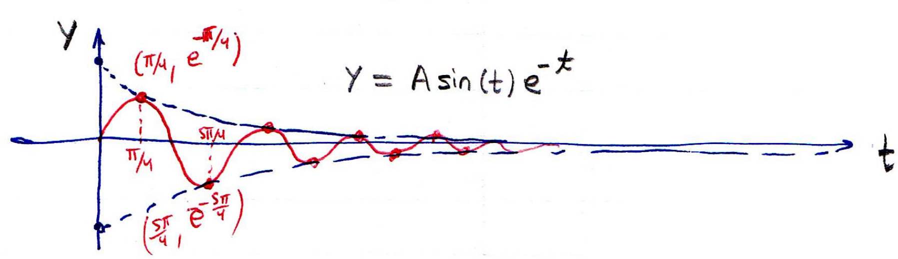
$\tan(t) = \dfrac{\sin t}{\cos t} = 1$ when graphs match.

$$Y'' = -Ae^{-t}(\cos t - \sin t) + Ae^{-t}(-\sin t - \cos t)$$
$$= Ae^{-t}(\cancel{\sin t} - \cos t - \cancel{\sin t} - \cos t)$$
$$= Ae^{-t}(-2\cos t)$$
$$= -Ae^{-t}\cos t$$

$Y''(t) < 0$ for $t = \pi/4,\ 9\pi/4, \dots$ (Maximums)

$Y''(t) > 0$ for $t = 5\pi/4,\ 13\pi/4, \dots$ (Minimums)

So $Y(\pi/4)$ is the absolute maximum and $Y(5\pi/4)$ is abs. minimum all subsequent points get smaller & smaller thanks to e^{-t}.



5.5. LIMITS AT INFINITY

The behavior a function at very very very… very big values of x is captured by the limit of the function at ∞. Now infinity is not a number so such a limit has not yet been discussed. In Chapter 3 of these notes we learned how to calculate the limit of a function at some finite point. Sometimes the output of the limit turned out to be $\pm\infty$, now we turn our attention to the case that the argument of the function tends toward a very big input or a very large negative input.

Definition5.5.1:
The limit at ∞ for a function f is $L \in \mathbb{R}$ if the values $f(x)$ get arbitrarily close to L as the input x gets arbitrarily large. We write

$$\lim_{x\to\infty} f(x) = L$$

in this case. To be more precise we should say that $\lim_{x\to\infty} f(x) = L$ if for each $\epsilon > 0$ there exists $N > 0$ such that if $x > N$ then $|f(x) - L| < \epsilon$.

The precise definition essentially says that if we pick a band of width 2ϵ about the line $y = L$ then for points to the right of $x = N$ the graph fits inside the band. Stewart has nice pictures of this, go look at them in section 4.4 pg. 238.

Example 5.5.1:
Let $f(x) = \frac{1}{x}$. Calculate the limit of $f(x)$ at ∞. Observe that,

$$f(10) = 0.1, \qquad f(100) = 0.01, \qquad f(1000) = 0.001.$$

We see that the values of the function are getting closer and closer to zero as x gets larger and larger. This leads us to conclude,

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

In other words, if we divide something nonzero by a very big number then we get something very small. This sort of limit is not ambiguous, to determine the answer we either need to think about a table of values or perhaps a graph.

Or if you want to be picky you can argue as follows: Let $\epsilon > 0$ choose $N = 1/\epsilon$ and observe that for $x > N = 1/\epsilon$ we have that $1/x < \epsilon$. Consequently, $|f(x) - 0| = |1/x| = 1/x < \epsilon$. Hence by the precise definition

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

We should also define limits at minus infinity.

Definition5.5.2:
The limit at $-\infty$ for a function f is $L \in \mathbb{R}$ if the values $f(x)$ get arbitrarily close to L as the input x gets arbitrarily negative and large. We write

$$\lim_{x \to -\infty} f(x) = L$$

in this case. To be more precise we should say that $\lim_{x \to -\infty} f(x) = L$ if for each $\epsilon > 0$ there exists $M < 0$ such that if $x < M$ then $|f(x) - L| < \epsilon$.

Let's look at a simple example:

Example 5.5.2:
Let $f(x) = \frac{1}{x}$. Calculate the limit of $f(x)$ at $-\infty$. Observe that,

$$f(-10) = -0.1, \qquad f(-100) = -0.01, \qquad f(-1000) = -0.001.$$

We see that the values of the function are getting closer and closer to zero as x gets larger and negative. This leads us to conclude,

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

In other words, if we divide something nonzero by a very big negative number then we get something very small and negative. This sort of limit is not ambiguous, to determine the answer we either need to think about a table of values or perhaps a graph.

Or if you want to be picky you can argue as follows: Let $\epsilon > 0$ choose $M = -1/\epsilon$ and observe that for $x < M = -1/\epsilon$ we have that $-1/x < \epsilon$ and clearly $x < 0$ since $x < -1/\epsilon < 0$ so $|x| = -x$ thus:

$$|f(x) - 0| = |1/x| = -1/x < \epsilon$$

Hence by the precise definition

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

I do not require you to be proficient in picky ϵ type arguments, I include them here just in case the common sense arguments fail to convince you. Let us consider an example which we will rely on in later examples quite frequently.

Example 5.5.3:
Let $f(x) = 1/x^n$ where $n > 0$. Calculate the limit of $f(x)$ at ∞. Observe that,

$$f(10) = 1/10^n, \qquad f(100) = 1/100^n, \qquad f(1000) = 1/1000^n.$$

We see that the values of the function are getting closer and closer to zero as x gets larger and larger. This leads us to conclude,

$$\lim_{x\to\infty} \frac{1}{x^n} = 0$$

In other words, if we divide something nonzero by a very big number then we get something very small. This sort of limit is not ambiguous, to determine the answer we either need to think about a table of values or perhaps a graph.

Let $\epsilon > 0$ choose $N = 1/\epsilon^{\frac{1}{n}}$. Suppose $x > N = 1/\epsilon^{\frac{1}{n}}$ thus $1/x < \epsilon^{\frac{1}{n}}$ which implies $1/x^n < (\epsilon^{\frac{1}{n}})^n = \epsilon$. Consider then, for $x > N$

$$|f(x) - 0| = |1/x^n| = 1/x^n < \epsilon$$

Therefore by the precise definition for limits at infinity,

$$\lim_{x\to\infty} \frac{1}{x^n} = 0.$$

The graphical significance of all three examples thus far considered is that the function has a *horizontal asymptote* of $y = 0$ at infinity or minus infinity.

Definition5.5.3:
If $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$ if then the function is said to have a horizontal asymptote of $y = L$. Likewise, if $\lim_{x\to-\infty} f(x) = L \in \mathbb{R}$ if then the function is said to have a horizontal asymptote of $y = L$.

Example 5.5.4:
Let $f(x) = \tan^{-1}(x)$. We saw back in section 2.4.5 page 17 of these notes that the inverse tangent has horizontal asymptotes of $y = \pi/2$ and $y = -\pi/2$. (*Those facts follow from the graph which can be obtained from flipping* $y = \tan(x)$ *about the line* $y = x$*)*. We find that

$$\lim_{x\to\infty} \left(\tan^{-1}(x) \right) = \frac{\pi}{2} \qquad \lim_{x\to-\infty} \left(\tan^{-1}(x) \right) = \frac{-\pi}{2}.$$

Vertical asymptotes of the function correspond to horizontal asymptotes for the inverse function. Challenge: what are the horizontal asymptotes of $y = \tan^{-1}(3x)$? We can also discuss limits which go to infinity at infinity.

<u>*Definition5.5.4:*</u>
The limit at ∞ for a function f is ∞ if the values $f(x)$ get arbitrarily large as the input x gets arbitrarily large. We write

$$\lim_{x\to\infty} f(x) = \infty$$

in this case. Likewise, the limit at $-\infty$ for a function f is ∞ if the values $f(x)$ get arbitrarily large as the input x gets arbitrarily large and negative. We write

$$\lim_{x\to-\infty} f(x) = \infty$$

Finally, if the function takes on increasingly large negative values as the input becomes increasing large or negative then we write

$$\lim_{x\to-\infty} f(x) = \infty \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = -\infty$$

These definitions are very natural, lets examine a number of easy cases. I will just state the answers in this example since they are derived from plain-old common sense for the most part. As before we will have to show more work for the limits which are indeterminant, we discuss those trickier examples later in this section.

<u>Example 5.5.5</u>: the last three columns we dealt with earlier but I thought it might be instructive to include those here as well just to make sure you understand the difference between limits which go to infinity and those which are taken at infinity.

$$\begin{array}{ccccc}
\displaystyle\lim_{x\to\infty}\frac{1}{x} = 0 & \displaystyle\lim_{x\to-\infty}\frac{1}{x} = 0 & \displaystyle\lim_{x\to0^+}\frac{1}{x} = \infty & \displaystyle\lim_{x\to0^-}\frac{1}{x} = -\infty & \displaystyle\lim_{x\to0}\frac{1}{x} = dne \\
\displaystyle\lim_{x\to\infty}\frac{1}{x^2} = 0 & \displaystyle\lim_{x\to-\infty}\frac{1}{x^2} = 0 & \displaystyle\lim_{x\to0^+}\frac{1}{x^2} = \infty & \displaystyle\lim_{x\to0^-}\frac{1}{x^2} = \infty & \displaystyle\lim_{x\to0}\frac{1}{x} = \infty \\
\displaystyle\lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0 & \displaystyle\lim_{x\to-\infty}\frac{1}{\sqrt{x}} =? & \displaystyle\lim_{x\to0^+}\frac{1}{\sqrt{x}} = \infty & \displaystyle\lim_{x\to0^-}\frac{1}{\sqrt{x}} =? & \displaystyle\lim_{x\to0}\frac{1}{\sqrt{x}} = dne \\
\displaystyle\lim_{x\to\infty}\sqrt{x} = \infty & \displaystyle\lim_{x\to-\infty}\sqrt{x} =? & \displaystyle\lim_{x\to0^+}\sqrt{x} = 0 & \displaystyle\lim_{x\to0^-}\sqrt{x} =? & \displaystyle\lim_{x\to0}\sqrt{x} = dne \\
\displaystyle\lim_{x\to\infty}x^2 = \infty & \displaystyle\lim_{x\to-\infty}x^2 = \infty & \displaystyle\lim_{x\to0^+}x^2 = 0 & \displaystyle\lim_{x\to0^-}x^2 = 0 & \displaystyle\lim_{x\to0}x^2 = 0 \\
\displaystyle\lim_{x\to\infty}x^3 = \infty & \displaystyle\lim_{x\to-\infty}x^3 = -\infty & \displaystyle\lim_{x\to0^+}x^3 = 0 & \displaystyle\lim_{x\to0^-}x^3 = 0 & \displaystyle\lim_{x\to0}x^3 = 0
\end{array}$$

I have used "?" instead of d.n.e. in a few places just to make it fit. Those limits are taken at a limit point which is not in the domain of the function, in some cases not even on the boundary of the function. If we can't take values close to the limit point then by default the limit is said to not exist, in which case we use d.n.e. as a shorthand (or just for this example "?")

We can also have limits which fail to exist at plus or minus infinity due to oscillation. All of the functions in the next example fall into that category.

Example 5.5.6: the following limits all involve cyclic functions. They never settle down to one value for large positive or negative input values so the limits d.n.e.

$$\lim_{x\to\infty}\sin(x) = d.n.e. \quad \lim_{x\to\infty}\cos(x) = d.n.e. \quad \lim_{x\to\infty}\tan(x) = d.n.e.$$
$$\lim_{x\to-\infty}\sin(x) = d.n.e. \quad \lim_{x\to-\infty}\cos(x) = d.n.e. \quad \lim_{x\to-\infty}\tan(x) = d.n.e.$$
$$\lim_{x\to\infty}\csc(x) = d.n.e. \quad \lim_{x\to\infty}\sec(x) = d.n.e. \quad \lim_{x\to\infty}\sec(x) = d.n.e.$$
$$\lim_{x\to-\infty}\csc(x) = d.n.e. \quad \lim_{x\to-\infty}\sec(x) = d.n.e. \quad \lim_{x\to-\infty}\sec(x) = d.n.e.$$

I should probably not neglect the other common elementary functions which we encounter often in examples. These limits I would consider basic, I recall these on the basis of the graph of the function and a little common sense.

Example 5.5.7: return to Chapter 2 if you are rusty on the graphs and basic properties of these functions. The interplay between a function and its inverse is especially enlightening for $\ln(x), \sin^{-1}(x), \cos^{-1}(x)$.

$$\lim_{x\to\infty}\sin^{-1}(x) = d.n.e. \quad \lim_{x\to\infty}\cos^{-1}(x) = d.n.e. \quad \lim_{x\to\infty}\tan^{-1}(x) = \pi/2$$
$$\lim_{x\to-\infty}\sin^{-1}(x) = d.n.e. \quad \lim_{x\to-\infty}\cos^{-1}(x) = d.n.e. \quad \lim_{x\to-\infty}\tan^{-1}(x) = -\pi/2$$
$$\lim_{x\to\infty} e^{x} = \infty \quad \lim_{x\to\infty} e^{-x} = 0 \quad \lim_{x\to\infty}(1/2)^{x} = 0$$
$$\lim_{x\to-\infty} e^{x} = 0 \quad \lim_{x\to-\infty} e^{-x} = \infty \quad \lim_{x\to-\infty}(1/2)^{x} = \infty$$

The domain of $\sin^{-1}(x)$ and $\cos^{-1}(x)$ will be the range of sine and cosine respectively; that is $dom(\sin^{-1}(x)) = [-1,1]$ and $dom(\cos^{-1}(x)) = [-1,1]$ so clearly the limits at plus and minus infinity are not sensible as $\sin^{-1}$ and $\cos^{-1}$ are not even defined at ± 2. In contrast the range of the exponential function is all positive real numbers and $\ln(x)$ is the inverse function of e^x thus

$$\lim_{x\to-\infty}\ln(x) = d.n.e. \qquad \lim_{x\to 0^+}\ln(x) = -\infty \qquad \lim_{x\to\infty}\ln(x) = \infty$$

For $x < 0$ the $ln(x)$ is not real, the middle limit you should have thought about in the earlier discussion of limits. The last one is true although an uncritical appraisal of the graph $y = \ln(x)$ gives the appearance of a horizontal asymptote, but appearances can be deceiving.

I don't believe we defined limits going to infinity precisely before, but it is a simple matter. We say that $\lim_{x\to a} f(x) = \infty$ if for each $M > 0$ there exists $\delta > 0$ such that if $0 < |x-a| < \delta$ then $f(x) \geq M$. What then does $\lim_{x\to\infty} f(x) = \infty$ mean precisely? It means that for each $M > 0$ there exists a $N > 0$ such that for $x > N$ we find $f(x) > M$. Take $f(x) = \ln(x)$ for example. Let $M > 0$ and choose $N = e^M > 0$. Suppose $x > N = e^M$ then $\ln(x) > \ln(e^M) = M$. Hence the natural log goes to infinity at infinity. The fact that natural log didn't mess up the inequality follows from the observation that $\frac{d}{dx}\ln(x) = \frac{1}{x} > 0$ so $\ln(x)$ is an increasing function on $(0, \infty)$. **The analysis in this paragraph is not a required topic. I have included it just in case you were curious how we can make these arguments rigorous.**

Indeterminant Limits at Infinity

Up to this point I have attempted to catalogue the basic results. I'm sure I forgot something important, but I hope these examples give you enough of a basis to do those limits which are unambiguous at plus or minus infinity. There is another category of problems where the limits which are given are not obvious, there is some form of interminancy. All the same indeterminant forms (see section 3.3 page 33) arise again and most of the algebraic techniques we used back in section 3.5 will arise again here although perhaps in a slightly altered form.

The good news is that limits at infinity enjoy all the same properties as limits which are taken at a finite limit point, at least in as much as the properties make sense. Of course we can only apply the limit properties when the values of the limit are finite. For example,

$$\lim_{x\to\infty}(x - 2x) = \lim_{x\to\infty}(x) + \lim_{x\to\infty}(-2x) = \infty - \infty$$

is not valid because you might be tempted to cancel and find $\lim_{x\to\infty}(x - 2x) = 0$ yet $\lim_{x\to\infty}(x - 2x) = \lim_{x\to\infty}(-x) = -\infty$ is the correct result. So we should only split limits by the limit laws when the subsequent limits are finite. That said, I do admit there are certain cases it doesn't hurt to apply the limit laws even though the limits are infinite. In particular if $\lim f = \infty$ then $\lim cf = c\lim f = c\infty$ and if we agree to understand that $c\infty = \infty$ for $c > 0$ whereas $c\infty = -\infty$ if $c < 0$. Such statements are dangerous because the reader may be tempted to apply laws of arithmetic to expressions involving ∞ and its just not that simple.

Example 5.5.8: this one is of type ∞/∞ to begin.

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x+3}{x-2}\right) &= \lim_{x\to\infty}\left(\frac{\frac{x}{x}+\frac{3}{x}}{\frac{x}{x}-\frac{2}{x}}\right) && \text{divided top and bottom by } x \\
&= \lim_{x\to\infty}\left(\frac{1+0}{1-0}\right) && c/x \to 0 \text{ as } x \to \infty \\
&= 1.
\end{aligned}$$

Here is another example of the same type.

Example 5.5.9: this one is of type ∞/∞ to begin.

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x^3+3x-2}{x^4-2x+1}\right) &= \lim_{x\to\infty}\left(\frac{\frac{1}{x}+\frac{3}{x^3}-\frac{2}{x^4}}{1-\frac{2}{x^3}+\frac{1}{x^4}}\right) && \text{divided top and bottom by } x^4 \\
&= \lim_{x\to\infty}\left(\frac{0+0-0}{1-0+0}\right) && \text{for } n=1,2,4,\ c/x^n\to 0 \text{ as } x\to\infty \\
&= 0.
\end{aligned}$$

Please sir may I have another? Yes.

Example 5.5.10: this one is of type ∞/∞ to begin.

$$\begin{aligned}
\lim_{x\to-\infty}\left(\frac{x^3+3x-2}{x^2-x+7}\right) &= \lim_{x\to-\infty}\left(\frac{x+\frac{3}{x}-\frac{2}{x^2}}{1-\frac{2}{x}+\frac{7}{x^2}}\right) && \text{divided top and bottom by } x^4 \\
&= \lim_{x\to-\infty}\left(\frac{x}{1}\right) && \text{for } n=1,2,\ c/x^n\to 0 \text{ as } x\to-\infty \\
&= -\infty.
\end{aligned}$$

Another way of thinking about this one is to put in very big negative values of x. For example, when $x=-1000$ we find

$$\frac{x^3+3x-2}{x^2-x+7} = \frac{1000^3+3000-2}{1000^2-1000-2} \approx \frac{1000^3}{1000^2} = 1000 = x$$

This sort of reasoning is a good method to try if you are lost as to what algebraic step to apply. There are problems which no amount of algebra will fix, sometimes considering numerical evidence is the best way to figure out a limit.

Example 5.5.11: this one is of type ∞/∞ to begin.

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{\sqrt{2x^4+3x-2}}{x^2-x+7}\right) &= \lim_{x\to\infty}\left(\frac{\frac{1}{x^2}\sqrt{2x^4+3x-2}}{\frac{1}{x^2}\left(x^2-x+7\right)}\right) \\
&= \lim_{x\to\infty}\left(\frac{\sqrt{\frac{2x^4+3x-2}{x^4}}}{1-\frac{1}{x}+\frac{7}{x^2}}\right) \\
&= \lim_{x\to\infty}\left(\frac{\sqrt{2+\frac{3}{x^3}-\frac{2}{x^4}}}{1-\frac{1}{x}+\frac{7}{x^2}}\right) \\
&= \sqrt{2}
\end{aligned}$$

Example 5.5.12: this one is of type $0 \cdot \infty$ to begin.

$$\begin{aligned}
\lim_{x\to\infty} \left(\, e^{-x} 2^x \, \right) &= \lim_{x\to\infty} \left(e^{\ln(2^x)} e^{-x} \right) \qquad \text{sneaky step} \\
&= \lim_{x\to\infty} \left(e^{x\ln(2)} e^{-x} \right) \\
&= \lim_{x\to\infty} \left(e^{x(\ln(2)-1)} \right) \\
&= 0.
\end{aligned}$$

In the last step I noticed $\ln(2) - 1 \approx 0.692 - 1 < 0$ thus the limit amounts to the exponential function evaluated at ever increasing large negative values, this gives zero. This example really belongs in the section with L'Hopital's Rule, I

We see that limits of type ∞/∞ can result in many different final answers depending on how the indeterminancy is resolved. The next example is more general, I think it is healthy to think about something a little more abstract from time to time. The strategy used is essentially identical other examples'.

Example 5.5.13: let P be a polynomial of degree p and let Q be a polynomial of degree q. This means there exist real coefficients $a_p, a_{p-1}, \dots, a_1, a_0$ and $b_q, b_{q-1}, \dots, b_1, b_0$ such that $a_p \neq 0$ and $b_q \neq 0$ where

$$P(x) = a_p x^p + \cdots + a_1 x + a_0 \qquad Q(x) = b_q x^q + \cdots + b_1 x + b_0$$

Consider $f(x) = P(x)/Q(x)$. There are three cases. Let's begin with $p > q$ so $p - q > 0$

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{P(x)}{Q(x)}\right) &= \lim_{x\to\infty}\left(\frac{a_p x^p + \cdots + a_1 x + a_0}{b_q x^q + \cdots + b_1 x + b_0}\right) \\
&= \lim_{x\to\infty}\left(\frac{a_p x^{p-q} + \cdots + \frac{a_1}{x^{q-1}} + \frac{a_0}{x^q}}{b_q + \cdots + \frac{b_1}{x^{q-1}} + \frac{b_0}{x^q}}\right) \\
&= \lim_{x\to\infty}\left(\frac{a_p}{b_q}x^{p-q} + \cdots + \frac{a_1}{b_q x^{q-1}} + \frac{a_0}{b_q x^q}\right) \\
&= \pm\infty.
\end{aligned}$$

If $a_p/b_q > 0$ then we get $+\infty$ whereas if $a_p/b_q < 0$ we obtain $-\infty$. The next case is that the denominator has a larger degree, that is to say $p < q$ thus $q - p > 0$

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{P(x)}{Q(x)}\right) &= \lim_{x\to\infty}\left(\frac{a_p x^p + \cdots + a_1 x + a_0}{b_q x^q + \cdots + b_1 x + b_0}\right) \\
&= \lim_{x\to\infty}\left(\frac{a_p x^{p-q} + \cdots + \frac{a_1}{x^{q-1}} + \frac{a_0}{x^q}}{b_q + \cdots + \frac{b_1}{x^{q-1}} + \frac{b_0}{x^q}}\right) \\
&= \lim_{x\to\infty}\left(\frac{a_p}{b_q}\frac{1}{x^{q-p}} + \cdots + \frac{a_1}{b_q x^{q-1}} + \frac{a_0}{b_q x^q}\right) \\
&= 0.
\end{aligned}$$

Finally it could be that the numerator and denominator have equal degree; $p = q$

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{P(x)}{Q(x)}\right) &= \lim_{x\to\infty}\left(\frac{a_p x^p + \cdots + a_1 x + a_0}{b_q x^q + \cdots + b_1 x + b_0}\right) \\
&= \lim_{x\to\infty}\left(\frac{a_p x^{p-q} + \cdots + \frac{a_1}{x^{q-1}} + \frac{a_0}{x^q}}{b_q + \cdots + \frac{b_1}{x^{q-1}} + \frac{b_0}{x^q}}\right) \\
&= \lim_{x\to\infty}\left(\frac{a_p}{b_q} + \cdots + \frac{a_1}{b_q x^{q-1}} + \frac{a_0}{b_q x^q}\right) \\
&= \frac{a_p}{b_q}.
\end{aligned}$$

In each case my goal was to simplify the denominator so I could focus on the behavior of the numerator. Very similar arguments will work for $x \to -\infty$.

Example 5.5.14: we can throw away a bounded function in a sum when the other function in the sum is unbounded, here are two examples of this idea in action:

$$\lim_{x\to\infty}(\sin(x)+e^x) = \lim_{x\to\infty}(e^x) = \infty$$

$$\lim_{x\to-\infty}(x+2) = \lim_{x\to-\infty}(x) = -\infty$$

Example 5.5.15: if we take a function $f(x)$with a known limit of $L \in \mathbb{R}$ or $\pm\infty$ as $x \to \pm\infty$ then the limit of $f(x+a)$ is the same for $x \to \pm\infty$. For example,

$$\lim_{x\to\infty}(e^x) = \infty \implies \lim_{x\to\infty}(e^{x+3}) = \infty$$

$$\lim_{x\to-\infty}\left(\frac{1}{x^2}\right) = 0 \implies \lim_{x\to-\infty}\left(\frac{1}{(x-7)^2}\right) = 0$$

$$\lim_{x\to\infty}(\tan^{-1}(x)) = \frac{\pi}{2} \implies \lim_{x\to\infty}(\tan^{-1}(x+2)) = \frac{\pi}{2}$$

Example 5.5.16: in a contest between power functions the largest degree wins.

$$\lim_{x\to\infty}(x^3-x^2) = \lim_{x\to\infty}(x^3) = \infty.$$

$$\lim_{x\to\infty}(x^3-x^4) = \lim_{x\to\infty}(-x^4) = -\infty.$$

On the other hand the exponential function will win against a polynomial because eventually the exponential function's values will totally dwarf the power function's values.

$$\lim_{x\to\infty}(x^3-e^x) = \lim_{x\to\infty}(e^x) = \infty.$$

$$\lim_{x\to-\infty}(2^{-x}+x) = \lim_{x\to-\infty}(2^{-x}) = \infty.$$

You might ask why? *Well, I challenge you to prove these claims for a bonus point.*

Example 5.5.17: that "rationalization" idea comes up again here:

$$\begin{aligned}
\lim_{x\to\infty}(x-\sqrt{x}) &= \lim_{x\to\infty}\left(\frac{x+\sqrt{x}}{x+\sqrt{x}}(x-\sqrt{x})\right) \\
&= \lim_{x\to\infty}\left(\frac{x^2-x}{x+\sqrt{x}}\right) \\
&= \lim_{x\to\infty}\left(\frac{x-1}{1+\frac{\sqrt{x}}{x}}\right) \\
&= \lim_{x\to\infty}\left(\frac{x-1}{1+\sqrt{1/x}}\right) \\
&= \lim_{x\to\infty}(x) \\
&= \infty.
\end{aligned}$$

Example 5.5.18: that "rationalization" idea comes up again here: to begin the limit is of type $-\infty + \infty$ but as you can see below the $-\infty$ wins in the end.

$$\begin{aligned}
\lim_{x\to-\infty}(x+\sqrt{x^2+4x}) &= \lim_{x\to-\infty}\left(\frac{x-\sqrt{x^2+4x}}{x-\sqrt{x^2+4x}}(x+\sqrt{x^2+4x})\right) \\
&= \lim_{x\to-\infty}\left(\frac{x^2-x^2-4x}{x-\sqrt{x^2+4x}}\right) \\
&= \lim_{x\to-\infty}\left(\frac{-4x}{2x-\sqrt{x^2+4x}}\right) \\
&= \lim_{x\to-\infty}\left(\frac{-4}{1-\sqrt{1+4/x}}\right) \\
&= -\infty.
\end{aligned}$$

That need not be the case, consider a similar problem that looks about the same on the surface but works out quite differently:

$$\begin{aligned}
\lim_{x\to-\infty}\left(x-\sqrt{2x^2+4x}\right) &= \lim_{x\to-\infty}\left(\frac{x+\sqrt{2x^2+4x}}{x+\sqrt{2x^2+4x}}\left(x-\sqrt{2x^2+4x}\right)\right) \\
&= \lim_{x\to-\infty}\left(\frac{x^2-2x^2-4x}{x+\sqrt{2x^2+4x}}\right) \\
&= \lim_{x\to-\infty}\left(\frac{-x^2-4x}{x+\sqrt{2x^2+4x}}\right) \\
&= \lim_{x\to-\infty}\left(\frac{-x-4/x}{1+\sqrt{2-4/x}}\right) \\
&= \infty.
\end{aligned}$$

Example 5.5.19: when dealing with square roots it is important that you remember that the laws of exponents indicate $\frac{1}{x}\sqrt{a+b} = \sqrt{\frac{1}{x^2}(a+b)}$. We assume that $a, c > 0$ in this problem. Consider,

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{\sqrt{ax^2+bx+c}}{\sqrt{cx^2+dx+e}}\right) &= \lim_{x\to\infty}\left(\frac{\frac{1}{x}\sqrt{ax^2+bx+c}}{\frac{1}{x}\sqrt{cx^2+dx+e}}\right) \\
&= \lim_{x\to\infty}\left(\frac{\sqrt{\frac{ax^2+bx+c}{x^2}}}{\sqrt{\frac{cx^2+dx+e}{x^2}}}\right) \\
&= \lim_{x\to\infty}\left(\frac{\sqrt{a+b/x+c/x^2}}{\sqrt{c+d/x+e/x^2}}\right) \\
&= \sqrt{\frac{a}{c}}.
\end{aligned}$$

Example 5.5.20: The Squeeze Theorem applies to limits at $\pm\infty$. Suppose we are given a function f such that

$$\frac{2}{\pi}\tan^{-1}(x) \leq f(x) \leq \frac{\sqrt{4x^2+1}}{x-3}$$

for all $x \geq 1,000,000$. We can calculate the limit at ∞ via the Squeeze Theorem. Observe that

$$\lim_{x\to\infty}\left(\frac{2}{\pi}\tan^{-1}(x)\right) = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1$$

$$\lim_{x\to\infty}\left(\frac{\sqrt{4x^2+1}}{2x-3}\right) = \lim_{x\to\infty}\left(\frac{\sqrt{4+1/x^2}}{2-3/x}\right) = \sqrt{4}/2 = 1.$$

Therefore, by the Squeeze Theorem,

$$\lim_{x\to\infty} f(x) = 1.$$

Trading limits at infinity for limits at zero:

Notice that if $x = 1/t$ then $t = 1/x$ it stands to reason that if $x \to \infty$ then $t \to 0^+$. We can make the following substitution with that exchange in mind :

$$\boxed{\lim_{x\to\infty} f(x) = \lim_{t\to 0^+} f(1/t)}$$

Homework problems 29 and 30 of section 4.4 of Stewart's 6th-Ed. are based on this exchange of limits. It is rather neat that the infinitesimal and the infinite are linked together in this way. I spent about an hour trying to get the following example to be pretty to no avail. I can do it with the technique of L'Hopital's Rule.

Example 5.5.21: the infinite limit view of e. Consider the following limit:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

If you can show that this definition is compatible with our previous implicit definition

$$\lim_{h\to 0}\left(\frac{e^h-1}{h}\right) = 1$$

for e I will award 3 bonus points if you can do it without using L'Hopital's