7. APPLICATIONS OF THE INTEGRAL

We begin this chapter by discussing the net change theorem. This theorem says that the integral can be used to add up all the infinitesimal changes to get the total change. Many applications of integration are based on this general principle: the integral is a sum. The change in position is the velocity. The integral of velocity gives the position. Likewise the acceleration is the change in velocity. The velocity is generated through an integral of the acceleration. I try to give a general flavor of how integration is use to extend average concepts to instantaneous concepts. I give a few examples of how average concepts still hold true at the infinitesimal level.

Once we are finished discussing generalities of the infinitesimal method, we focus our efforts in the remainder of the chapter on the tasks of calculating areas and volumes.

Recall that we began our discussion of integral calculus by asking the question: “what is the area of some region with curvy edges”. We found that it was actually fairly easy to calculate the signed area under a large number of curves. If the curve in question was the graph of a function which possessed an antiderivative then the FTC produced the area with ease. Of course, if the curve is the graph of a function with no easy antiderivative then we have no recourse except some numerical method such as left, right or midpoint rule. In later chapters we will learn a few more tools towards tackling ugly integrals.

In this chapter we return to the problem of calculating the area of some region. In contrast to signed area, this will really be the area which is positive. We will find that infinitesimal arguments provide an efficient shorthand for writing limiting processes. Almost every problem is begun by drawing a graph of the region and a typical approximating rectangle. Then we find the net area by adding up all the infinitesimal areas. This “adding up” is integration, we should think of integration as a continuous sum.

Once the area problem is settled we turn to the task of calculating the volumes of solids which possess a certain regularity. If a solid is such that the cross-sectional area is of the same type at each value of the axis perpendicular to the cross-section then we can add up the volume of each slice and get the total volume. For example, a sphere has a cross-sections which are disks if we use a diameter as an axis. A tetrahedron has triangular cross-sections relative to the axis which extends perpendicularly from one its faces to an opposing vertex. A solid of revolution has cross-sections which are disks or washers relative to the axis of revolution. Of course, you should look at the pictures in this chapter before you get too worried about the meaning of this paragraph. Integration provides the technology needed to add together all the tiny volumes.

7.1. INFINITESIMAL METHODS

The “Total Change Theorem” is that

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

In other words, the integral of a rate of change of f with respect to x over an interval [a,b] gives the net change in f over that interval. Let us suppose that $\frac{df}{dx} = g$ then we can act as though dx was a number and find $df = g\ dx$. Thus,

$$\int df = \int g\ dx$$

Now we are really interested in definite integrals. Notice this is a funny equation, we have df on the LHS and dx on the RHS, so we’ll need upper/lower bounds with respect to f on the LHS and upper/lower bounds with respect to x on the RHS. Thus,

$$\int_{f(a)}^{f(b)} df = \int_a^b g\ dx \quad \Longrightarrow \quad f(a) - f(b) = \int_a^b g\ dx$$

We are using the U-substitution theorem here. However, it is equivalent to use infinitesimal arguments.

Concept: *average concepts still hold true at the infinitesimal level. Integration then extends these microscopic arguments to macroscopic rules. Often we can find a relation that holds true over an instant dt of time or a little displacement dx.*

For example, the displacement for a particle moving with velocity v during a time dt is simply the product of velocity and time; $dx = vdt$. This makes sense because the velocities $v(t)$ and $v(t+dt)$ are equal. To be more precise I should say they are equal in the limit that $dt \to 0$. During an instant of time the velocity is constant so we can use the constant velocity formula.

$$\begin{aligned} dx = v\ dt \ \Longrightarrow \ & \int_{x_o}^{x(t)} dx = \int_0^t v(u)\ du \\ \Longrightarrow \ & x(t) = x_o + \int_0^t v(u)du. \end{aligned}$$

Another example is current $I = dQ/dt$. If we wish to calculate the net charge that has flowed from time zero to time t then we simply integrate the current,

$$\begin{aligned} I = \frac{dQ}{dt} &\implies dQ = I\ dt \\ &\implies \int_{Q_o}^{Q(t)} dQ = \int_0^t I(u)\ du \\ &\implies Q(t) = Q_o + \int_0^t I(u) du. \end{aligned}$$

This idea also applies to things which are not from some rate of change necessarily. For example, the work done by a force F over a distance x is given by the formula $W = Fx$. Now this formula is only for constant forces which act in the direction of the displacement. What would we do if the force was a function of position? Then we could not just use the formula since the force is not constant. However, if we look at $F(x)$ and $F(x + dx)$ then those forces are equal in the limit that $dx \to 0$. So we can conclude that the simple work equation holds at the infinitesimal level; the work dW done by a force $F(x)$ over a displacement dx will be $dW = F(x)dx$. If the force does work from $x = a$ to $x = b$ then the net work done will be the sum of all the infinitesimal works dW, in other words,

$$W = \int dW = \int_a^b F(x) dx$$

Another application is hydrostatic force. The force on a dam is due to water pressure. The definition of pressure is that it is force per unit area, this gives us

$$P = \frac{F}{A}$$

Now this only makes sense so long as the same force is applied over the whole area. We cannot just apply this equation to the force due to the water pressure on a dam. The pressure at the bottom of the dam is large than that at the top. In fact it is known that $P = \rho g d$, it is proportional to the depth d. Different depth gives different pressures, hopefully this is a familiar fact to everyone. So, if we wish to calculate the net-force due to pressure (this is called the "hydrostatic force") then we should consider horizontal strips of area dA. These will have the same pressure all along them so the equation makes sense to apply to the strip, we have

$$P = \rho g d = \frac{dF}{dA}$$

The little force dF is due to the pressure P acting on dA. Then

$$dF = \rho g d\ dA \quad \implies \quad F = \int_R \rho g d\ dA$$

where $\int_R$ is shorthand for a detailed integration based on the geometry of the dam. I have no intention of testing you on the physical examples here, I just wanted to give a flavor of how these argument go in real-world applications.

7.2. POSITION, VELOCITY, ACCELERATION

Differentiation and integration provide the link between position, velocity and acceleration. Let us collect a few definitions for one-dimensional motion.

Definition 7.2.1: Let the position of some object travelling in one dimension be denoted by x. If the position at time $t = t_1$ is $x = x_1$ and the position at time $t = t_2$ is $x = x_2$ the *duration* between these times is denoted $\triangle t = t_2 - t_1$ and the *displacement* during that duration is denoted $\triangle x = x_2 - x_1$. The position x is the *displacement from the origin*. The position will be a function of time t. Let's define speed and velocity:

$$\boxed{v \equiv \frac{dx}{dt} = \dot{x} \qquad \text{speed} = |v|.}$$

The speed is the magnitude of the velocity. Acceleration is a,

$$\boxed{a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x}.}$$

The "dot" and "double-dot" notation are rather old-fashioned but still very much in vogue in certain circles.

Differentiation allows us to find the velocity and acceleration if we are given the position.

Example 7.2.1:
Suppose that $x = 1 - t - t^2$ is the position of an object travelling in one dimension.

$$\text{velocity at time } t\text{:} \;\; v(t) = \frac{d}{dt}\left(1 - t - t^2\right) = -1 - 2t.$$

$$\text{speed at time } t\text{:} \;\; |v(t)| = |-1 - 2t| = 2t + 1.$$

$$\text{acceleration at time } t\text{:} \;\; a(t) = \frac{d}{dt}\left(-1 - 2t\right) = -2.$$

What does integration do for us? Let's consider a simple example.

Example 7.2.2: Let $a(t) = 2$. What can we say about the velocity at time t in this case? Let's integrate over the time interval $[0, t]$,

$$\int_0^t a(u)du = \int_0^t \frac{dv}{du}du = v(u)\Big|_0^t = v(t) - v(0).$$

But we can also just integrate directly, $\int_0^t 2du = 2t$. We find

$$v(t) - v(0) = 2t \quad \Longrightarrow \quad \boxed{v(t) = v(0) + 2t.}$$

We need to know the initial velocity $v(0)$ to give an unambiguous formula.

Let me work out a rather famous problem, the constant acceleration problem.

Example 7.2.3: Let $a = \ddot{x} = g$ where g is a constant. Suppose the initial velocity and position are also given. Denote the initial velocity $v(0) = \dot{x}(0) = v_o$ and the initial position $x(0) = x_o$. Let us find the position and velocity as a function of time. Integrate over the time interval $[0, t]$,

$$\int_0^t a(u)du = \int_0^t \frac{dv}{du}du = v(u)\bigg|_0^t = v(t) - v(0) = v(t) - v_o.$$

But we can also just integrate directly, $\int_0^t g du = gt$. We find

$$v(t) - v_o = gt \quad \Longrightarrow \quad \boxed{v(t) = v_o + gt.}$$

Next we integrate the velocity to find the position,

$$\int_0^t v(u)du = \int_0^t \frac{dx}{du}du = x(u)\bigg|_0^t = x(t) - x(0) = x(t) - x_o$$

But we can also just integrate directly, $\int_0^t (v_o + gu)du = v_o t + \frac{1}{2}gt^2$. We find

$$x(t) - x_o = v_o t + \frac{1}{2}gt^2 \quad \Longrightarrow \quad \boxed{x(t) = x_o + v_o t + \frac{1}{2}gt^2.}$$

This example can be repeated for any acceleration and set of given initial conditions. There are a few worked problems in the homework solutions if you need to see more of these. One last idea that's a little trick, for one dimensional non-stop motion we can solve the position function for time and rewrite the velocity as a function of position instead of time. Now you can do that algebraically, its not too hard but the calculus argument is slicker:

$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} \quad \Longrightarrow \quad adx = vdv$$

Now apply this to the case $a = g$. We have $gdx = vdv$ thus,

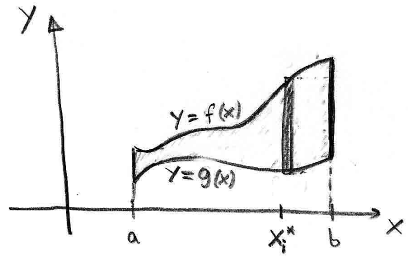
$$\int_{x_o}^{x_f} gdx = \int_{v_o}^{v_f} vdv \quad \Longrightarrow \quad g(x_f - x_o) = \frac{1}{2}v_f^2 - \frac{1}{2}v_o^2$$

Perhaps some of you recall using the formula $v_f^2 = v_0^2 + 2g(x_f - x_o)$ in the study of projectile motion in highschool. This is where is comes from. Well there is a little more physics behind our starting point here, but we'll save that for physics.

7.3. HOW TO FIND THE AREA

Example 7.3.1:

E1 Consider a generic example to begin, find the area bounded by $x = a$, $x = b$ and $f(x)$ and $g(x)$ where $f(x) > g(x)$ for all x on $[a, b]$. A picture helps



we can find the area by dividing the area up into ∞ly many infinitesimal rectangles. We've drawn a typical box you can see it's height is $f(x) - g(x)$, it has area $\Delta A_i = [f(x_i) - g(x_i)]\Delta x$.

$$A = \text{AREA} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*) \right) \Delta x$$

But this is precisely the integral of $f(x) - g(x)$,

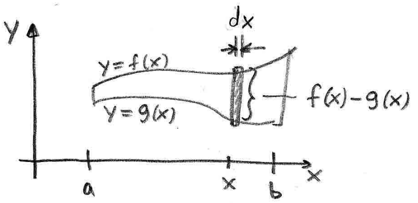
$$\boxed{A = \int_a^b (f(x) - g(x))\, dx}$$

- Technically, we should begin all problems with argument about Δx and ΔA_i and so on, these are finite. Then once the object of interest is suitably approximated by n-rectangles we pass to the limit $n \to \infty$ and find the Σ becomes an $\int$ and Δx becomes a dx. For all calulational purposes the first steps with "Δ"'s are unecessary. We can make these arguments using infinitesimals from the beginning and I will from now on. I mention this so you can understand the connection between the method in these notes and the more cumbersome arguments in your text

The general strategy is to draw a picture to get a handle on the problem, then find the formula for a typical infinitesimal rectangle and then add all the little areas together by integrating dA.

Example 7.3.1 (again, but clearer):

$\boxed{E1}$ Revisted using infinitesimal arguments.



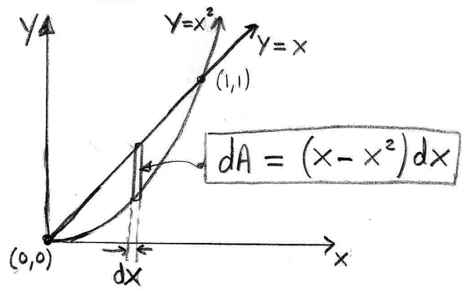
$$\underbrace{dA = \big(f(x) - g(x)\big)\,dx}_{\text{typical area of infinitesimal rectangle.}}$$

To obtain area simply add up all the infinitesimal areas,

$$A = \int dA = \int_a^b \big(f(x) - g(x)\big)\,dx$$

Example 7.3.2:

$\boxed{E2}$ Area bounded by $y = x$ and $y = x^2$ is what?



• Need to find intersection points here we can simply equate y,

$$x = x^2$$

$$x^2 - x = x(x-1) = 0$$

$$x = 0 \quad \text{or} \quad x = 1$$

The points of intersection are $(0,0)$ and $(1,1)$.

$$A = \int dA = \int_0^1 (x - x^2)\,dx$$

$$= \left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right|_0^1$$

$$= \tfrac{1}{2} - \tfrac{1}{3}$$

$$= \boxed{\tfrac{1}{6}}$$

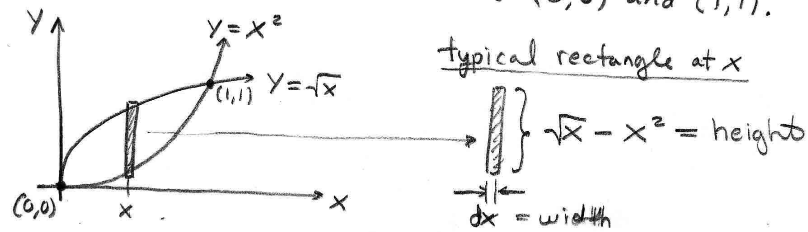
Example 7.3.3:

$\boxed{E3}$ find area bounded by $y = x^2$ and $y = \sqrt{x}$

Lets find where these curves intersect, set $y = y$ yielding,

$$x^2 = \sqrt{x}$$
$$x^4 = x$$
$$x^4 - x = 0$$
$$x(x^3 - 1) = 0 \Rightarrow \underline{x = 0 \text{ or } x = 1}$$

Thus the points of intersection are $(0,0)$ and $(1,1)$.



The area of this tiny rectangle is, $dA = (\sqrt{x} - x^2)\,dx$

Thus
$$A = \int_0^1 (\sqrt{x} - x^2)\,dx$$
$$= \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right|_0^1$$
$$= \frac{2}{3} - \frac{1}{3}$$
$$= \boxed{1/3}$$

Some Advice:

Strategy

1. Graph region carefully, use algebra to find points of intersection.
2. Draw a typical tiny rectangle and find its area (dA).
3. Add the areas of the tiny rectangles by integrating.

Example 7.3.4:

E4 Find area bounded by $y^2 = x$ and $2y = x - 3$

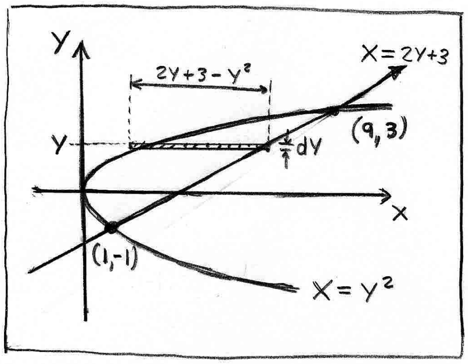
Notice that at intersection $x = x \Rightarrow y^2 = 2y + 3$

$\Rightarrow y^2 - 2y - 3 = 0$

$\Rightarrow (y-3)(y+1) = 0$

$\Rightarrow \underline{y = 3 \text{ or } y = -1}$

$\Rightarrow$ points of intersection are $(9,3)$ and $(1,-1)$



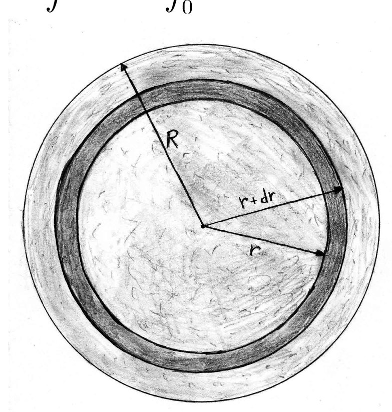
- its better to use horizontal strips because we don't need to break up into cases. If we used vertical strips it would be tricky because $0 \le x \le 1$ is different than $1 \le x \le 9$ for vertical strips

$$dA = (2y + 3 - y^2)\,dy$$

Thus
$$\begin{aligned} A &= \int_{-1}^{3} (2y + 3 - y^2)\,dy \\ &= \left(y^2 + 3y - \tfrac{1}{3}y^3\right)\Big|_{-1}^{3} \\ &= \left(9 + 9 - \tfrac{1}{3}(27)\right) - \left(1 - 3 + \tfrac{1}{3}\right) \\ &= 9 + 5/3 = \boxed{32/3 \cong 10.67} \end{aligned}$$

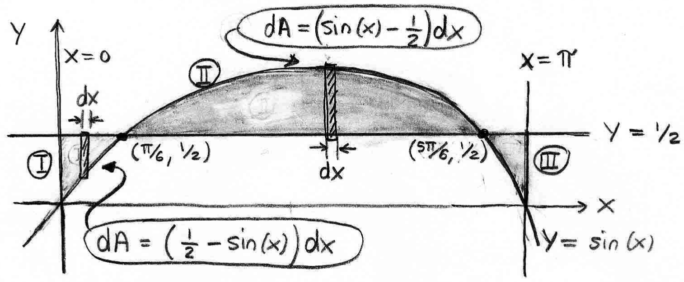
Remark: The choice of rectangle is primarily one of convenience. We could use another shape which neatly divides the shape into pieces. For example, a disk can be seen as the union of washers of thickness dr and circumference $2\pi r$ so $dA = 2\pi r dr$ (this is the area of the shaded washer region pictured below). If the disk is of radius R then we can find the area by adding up all the infinitesimal areas,

$$A = \int dA = \int_0^R 2\pi r dr = \boxed{\pi R^2.}$$



Example 7.3.5:

$\boxed{E5}$ Find area bounded by $y = \sin(x)$ and $y = \frac{1}{2}$ and $x = 0$ and $x = \pi$ (136)



Notice that $\sin(30°) = \sin(\pi/6) = 1/2$ and by the symmetry of $\sin(x)$ about $x = \pi/2$ its clear $\sin(5\pi/6) = 1/2$. This reveals the interception points are $(\pi/6, 1/2)$ and $(\frac{5\pi}{6}, 1/2)$. Clearly we need to divide up into cases I, II and III,

$$A = \int_0^{\pi/6}\left(\frac{1}{2} - \sin(x)\right)dx + \int_{\pi/6}^{5\pi/6}\left(\sin(x) - \frac{1}{2}\right)dx + \int_{5\pi/6}^{\pi}\left(\frac{1}{2} - \sin(x)\right)dx$$

$$= \frac{\pi}{12} + \cos(x)\Big|_0^{\pi/6} - \frac{4\pi}{12} - \cos(x)\Big|_{\pi/6}^{5\pi/6} + \frac{\pi}{12} + \cos(x)\Big|_{5\pi/6}^{\pi}$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 - \frac{4\pi}{12} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}$$

$$= \frac{-2\pi}{12} + 2\sqrt{3} - 2$$

$$= 2(\sqrt{3} - 1) - \pi/6$$

$$\cong \boxed{0.9405 = A}$$

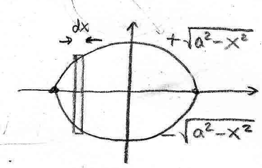
Example 7.3.6: (if on test antiderivative would be provided)

$\boxed{E6}$ Lets find the area of a circle (again)

The eq's of the circle in parametric form are,

$x = a\cos\theta$ and $y = a\sin\theta$

Where $x^2 + y^2 = a^2 \Rightarrow y = \pm\sqrt{a^2 - x^2}$



$$dA = 2\sqrt{a^2 - x^2}\,dx$$

$$A = \int_{-a}^{a} 2\sqrt{a^2 - x^2}\,dx$$

$$= \int_{\pi}^{0} 2\sqrt{a^2 - a^2\cos^2\theta}\,(-a\sin\theta\, d\theta)$$

$x = a\cos\theta$
$dx = -a\sin\theta\, d\theta$
$x = a \Rightarrow \theta = 0$
$x = -a \Rightarrow \theta = +\pi$

$$= 2a^2\int_0^{\pi}\sin^2\theta\, d\theta$$

$$= 2a^2\left(\int_0^{\pi}\frac{1}{2}(1 - \cos 2\theta)\,d\theta\right)$$

$$= 2a^2\left(\frac{\pi}{2} - \frac{\sin 2\theta}{4}\Big|_0^{\pi}\right)$$

$$= \boxed{\pi a^2}$$

Example 7.3.7:(integration too hard for test, would provide antiderivative)

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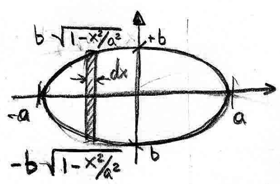
$\boxed{E7}$ Area of Ellipse



Standard Ellipse
$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

Ellipse in Polar Coord.
$$\boxed{\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}}$$

$$y = \pm b\sqrt{1-(x/a)^2} \Rightarrow \text{height of box} = 2b\sqrt{1-(x/a)^2}$$

$$dA = 2b\sqrt{1-(x/a)^2}\,dx$$

$$A = \int_{-a}^{a} 2b\sqrt{1-\left(\tfrac{x}{a}\right)^2}\,dx$$

$$\boxed{\begin{aligned} x &= a\sin\theta & x=a &\Rightarrow \theta = \pi/2 \\ dx &= a\cos\theta\,d\theta & x=-a &\Rightarrow \theta = -\pi/2 \end{aligned}}$$

$$= 2b\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2\theta}\;a\cos\theta\,d\theta$$

$$= 2ab\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

$$= 2ab\int_{-\pi/2}^{\pi/2}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= ab\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= ab\left[\left(\pi/2 + \tfrac{1}{2}\underbrace{\sin(\pi)}_{0}\right) - \left(-\pi/2 + \tfrac{1}{2}\underbrace{\sin(-\pi)}_{0}\right)\right]$$

$$= \boxed{\pi ab}$$

Notice when $a = b = r$ we get $A = \pi r^2$ which is good.

Digression: What is the eq$^{\underline{n}}$ of this ellipse in polar coordinates?

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Longrightarrow \frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

$$r^2\left(\cos^2\theta + \frac{a^2}{b^2}\sin^2\theta\right) = a^2$$

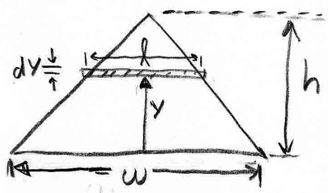
$$\boxed{r = \frac{a}{\sqrt{\cos^2\theta + \left(\frac{a}{b}\right)^2\sin^2\theta}}}$$

We can again notice when $a=b$ we get $r=a$ a very sensible eq$^{\underline{n}}$ for a circle at the origin.

Remark: this problem I could ask you to set-up, but the integral is too hard to execute this semester. We need trig. substitution which is a topic from calculus II. I might give a problem like this and provide the antiderivative. That way you could still calculate the answer so long as you set it up correctly.

Example 7.3.8: (area of a triangle from calculus argument)

$\boxed{E8}$ let's find the width of a slice in a triangular region of width w and height h as a function of y.



Again it's clear that for this shape that ℓ varies linearly with y, so $\ell = my + b$. Additionally we know

$$y = 0 \Rightarrow \ell = w = m(0) + b \;\therefore\; \boxed{b = w}$$
$$y = h \Rightarrow \ell = 0 = mh + w \;\therefore\; \boxed{m = -w/h}$$

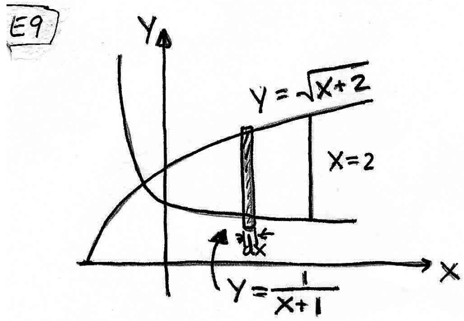
Thus we find $\boxed{\ell = \frac{-w}{h}y + w}$.

We see that $dA = \ell\, dy = (-\frac{w}{h}y + w)\,dy$ thus

$$\begin{aligned} A &= \int_0^h \left(-\frac{w}{h}y + w\right)dy \\ &= \left(\frac{-w}{2h}y^2 + wy\right|_0^h \\ &= -\frac{1}{2}wh + wh \\ &= \boxed{\frac{1}{2}wh = \frac{1}{2}(\text{base})(\text{height})} \end{aligned}$$

I don't mean to suggest this calculation is necessary to prove the area of a triangle is $\frac{1}{2}bh$. The proof of that formula falls easily from purely geometric arguments. This example is just to show that calculus is consistent with known geometrical formulas. After all, if we claim that the integral gives area then we ought to be able to find all the area formulas that precede calculus logically. In fact we can find those formulas and much much more.

Example 7.3.9:

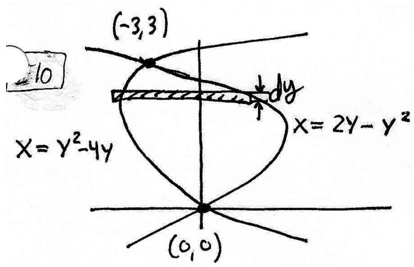


$$dA = (y_T - y_B)dx$$
$$= \left(\sqrt{x+2} - \frac{1}{x+1}\right)dx$$

There is a vertical slice at each x from 0 to 2. Thus

$$A = \int_0^2 \left(\sqrt{x+2} - \frac{1}{x+1}\right)dx = \left[\frac{2}{3}(x+2)^{3/2} - \ln|x+1|\right|_0^2$$
$$= \left(\frac{2}{3}(4)^{3/2} - \ln(3)\right) - \left(\frac{2}{3}(2)^{3/2} - \ln(1)\right)$$
$$= \boxed{2.349}$$

Example 7.3.10:



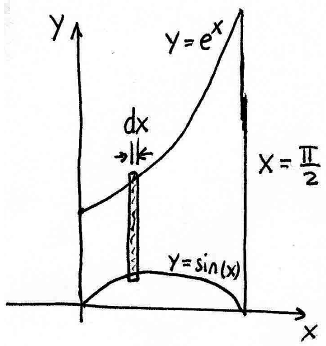
$$dA = (x_R - x_L)\,dy$$
$$= \left[(2y - y^2) - (y^2 - 4y)\right]dy$$
$$= (6y - 2y^2)dy$$

Notice it's easier to do it horizontally here, and also the integration simplifies if you simplify dA before integrating.

$$A = \int_0^3 (6y - 2y^2)dy = \left[3y^2 - \frac{2}{3}y^3\right|_0^3$$
$$= \left[3(9) - \frac{2}{3}(3)^3\right] - [0 - 0]$$
$$= \left(27 - \frac{2}{3}(27)\right)$$
$$= \frac{1}{3}(27)$$
$$= \boxed{9}$$

Example 7.3.11:

139c

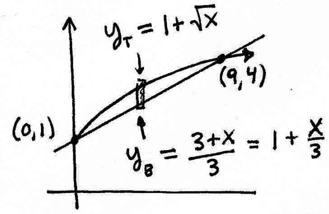


$$dA = (y_T - y_B)\,dx = (e^x - \sin(x))\,dx$$

Therefore,

$$A = \int_0^{\pi/2} (e^x - \sin(x))\,dx = \left[e^x + \cos(x)\right]_0^{\pi/2} = \boxed{e^{\pi/2} - 2 = 2.81}$$

Example 7.3.12:



$$dA = (y_T - y_B)\,dx = \left(1 + \sqrt{x} - 1 - \frac{x}{3}\right)dx = \left(\sqrt{x} - \frac{1}{3}x\right)dx$$

Therefore, using algebra below to explain bounds,

$$A = \int_0^9 \left(\sqrt{x} - \frac{1}{3}x\right)dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{6}x^2\right|_0^9 = \left[\frac{2}{3}(9)^{3/2} - \frac{9^2}{6}\right] = \boxed{4.5}$$

Points of intersection followed from $y_B = y_T$ because,

$$1 + \sqrt{x} = 1 + \frac{x}{3}$$
$$\sqrt{x} = \frac{x}{3}$$
$$3\sqrt{x} = x$$
$$9x = x^2$$

Cannot just divide by x you lose information by doing that. (x might be zero)

$$x^2 - 9x = 0$$
$$x(x-9) = 0$$
$$\therefore\ \underline{x = 0 \text{ or } x = 9}$$

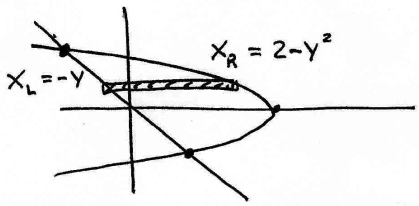
Example 7.3.13:

E13 $\quad x = 2 - y^2$

$x = -y$

Just swith the roles of x & y to see why the graph should be a line and sideways parabola.

(139d)



Points of intersection follow from $x_L = x_R$,

$$2 - y^2 = -y$$

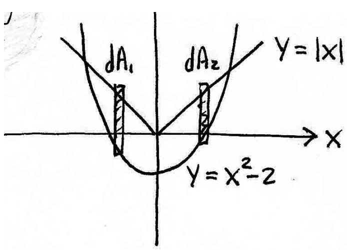
$$y^2 - y - 2 = 0$$

$$(y-2)(y+1) = 0 \quad \therefore \ \underline{y = 2 \ \& \ (\text{or}) \ y = -1}$$

$$\begin{aligned} dA &= (x_R - x_L)\,dy \\ &= (2 - y^2 - (-y))\,dy \\ &= (2 + y - y^2)\,dy \end{aligned} \quad \therefore \quad A = \int_{-1}^{2} (2+y-y^2)\,dy = \left[2y + \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right|_{-1}^{2}$$

Evaluation yields $\boxed{A = 4.5}$

Example 7.3.14:



$$\begin{aligned} dA_1 &= (-x - (x^2 - 2))\,dx \\ &= (2 - x - x^2)\,dx \end{aligned} \qquad (x < 0)$$

$$\begin{aligned} dA_2 &= (x - (x^2 - 2))\,dx \\ &= (2 + x - x^2)\,dx \end{aligned} \qquad (x > 0)$$

Points of intersection

$\underline{x > 0}$ $\quad x = x^2 - 2 \quad \Rightarrow x^2 - x - 2 = (x-2)(x+1) = 0$

$\therefore \ x = 2$ or $\cancel{x = -1}$ throw that out because $x \ngtr 0$.

$\underline{x < 0}$ $\quad -x = x^2 - 2 \quad \Rightarrow x^2 + x - 2 = (x+2)(x-1) = 0$

$\therefore \ x = -2$ or $\cancel{x = 1}$ throw out that sol$^{\underline{n}}$ because $x \nless 0$.

$$\begin{aligned} A &= \int_{-2}^{0} (2 - x - x^2)\,dx + \int_{0}^{2} (2 + x - x^2)\,dx \\ &= \left(2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right|_{-2}^{0} + \left(2x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right|_{0}^{2} = \tfrac{10}{3} + \tfrac{10}{3} = \boxed{\tfrac{20}{3}} \end{aligned}$$

Example 7.3.15:

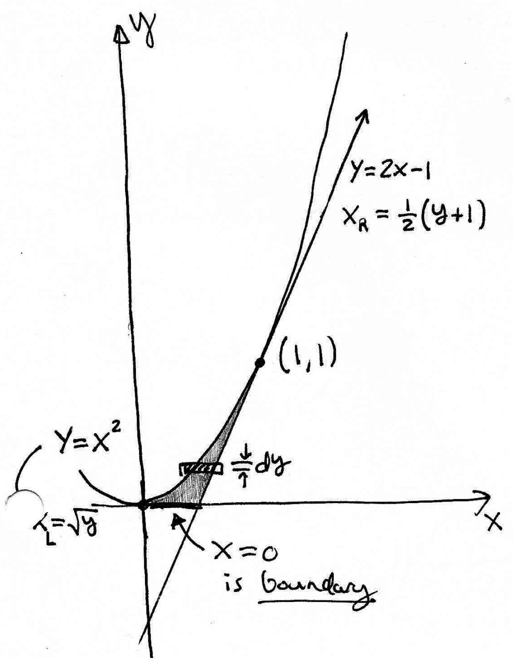
E15 Find area bounded by $y=x^2$ and the tangent to $y=f(x)=x^2$ at $(1,1)$, and the x-axis. (139e)

Recall we can write the tangent to $y=f(x)$ at $(a,f(a))$ in general (provided $f'(a)$ exists)

$$y = f(a) + f'(a)(x-a)$$

Then we have $a=1$ and $f'(x)=2x \Rightarrow f'(1)=2$ then

$$y = 1 + 2(x-1) = \underline{2x-1 = y} \quad \text{(tangent line)}$$



$$dA = (x_R - x_L)\,dy$$
$$= \left(\tfrac{1}{2}y + \tfrac{1}{2} - \sqrt{y}\right)dy$$

Points of intersection:

$$2x-1 = x^2$$
$$x^2-2x+1 = (x-1)^2 = 0$$

$\therefore$ $\underline{x=1}$ $\Rightarrow$ $y=1$ at intersection

that is the pt. of intersection is $(1,1)$

Therefore, noting $0 \le y \le 1$ for area,

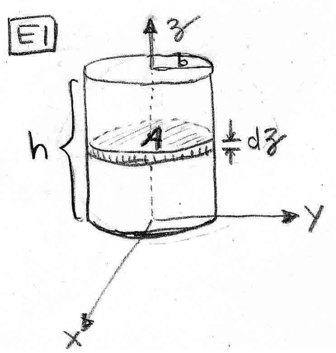
$$A = \int_0^1 \left(\tfrac{1}{2}y + \tfrac{1}{2} - \sqrt{y}\right)dy$$
$$= \left(\tfrac{1}{4}y^2 + \tfrac{1}{2}y - \tfrac{2}{3}y^{3/2}\right)\Big|_0^1 = \tfrac{1}{4} + \tfrac{1}{2} - \tfrac{2}{3} = \boxed{\tfrac{1}{12}}$$

(Unfortunately the hole punch ate half of the $x_L = \sqrt{y}$.)

7.4. CALCULATING VOLUME

Example 7.4.1:

E1



$$A = \pi b^2$$

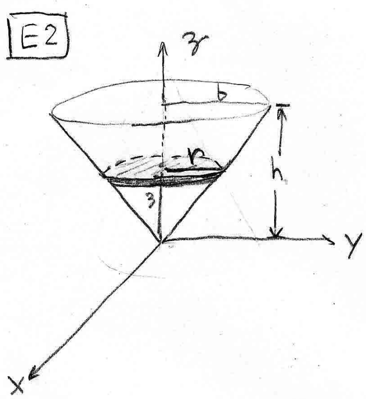
$$dV = A(z)\,dz = \pi b^2 dz$$

$$V = \int dV = \int_0^h \pi b^2 dz = \boxed{\pi b^2 h = V}$$

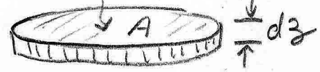
This is the simplest case, the cross-section is constant along the integration axis.

Example 7.4.2:

E2



$A = \pi r^2 = \pi\left(\frac{bz}{h}\right)^2$ (see argument below)



Notice that r should depend linearly on z thus $r = mz + b$.

$r(z=0) = 0$ } clear from
$r(z=h) = b$ } picture.

Thus $0 = b$ and $b = mh \therefore m = b/h$

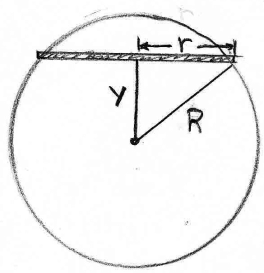
giving that $\boxed{r = \frac{b}{h} z}$.

The volume of each tiny slice is found to be $dV = \frac{\pi b^2}{h^2} z^2 dz$.
So the total volume is found by adding these up,

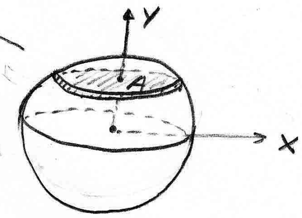
$$V = \int_0^h \frac{\pi b^2}{h^2} z^2 dz = \frac{\pi b^2}{3h^2} z^3 \Big|_0^h = \boxed{\frac{\pi b^2 h}{3} = V_{cone}}$$

Example 7.4.3:

E3 Find the volume of a sphere of radius R



view of same slice as it intersects (xy)-plane.



$$r^2 + y^2 = R^2$$

$$r^2 = R^2 - y^2$$

$$A = \pi r^2 = \pi(R^2 - y^2)$$

$$dV = A\,dy = \pi(R^2 - y^2)\,dy$$

We then find the total volume by summing the volumes of the disks from $y = -R$ up to $y = R$,

$$V = \int_{-R}^{R} \pi(R^2 - y^2)\,dy$$

$$= \pi\left(R^2 y - \tfrac{1}{3}y^3\right)\Big|_{-R}^{R}$$

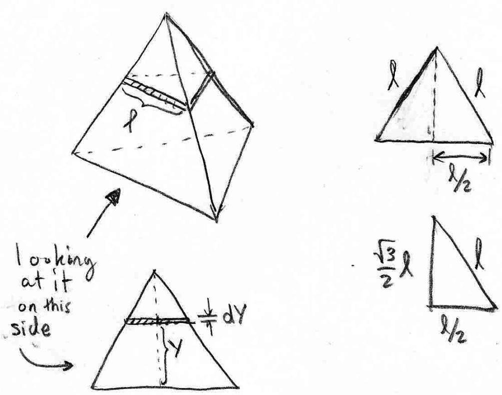
$$= \pi\left[\left(R^3 - \tfrac{1}{3}R^3\right) - \left(-R^3 + \tfrac{1}{3}R^3\right)\right]$$

$$= \boxed{\tfrac{4}{3}\pi R^3 = V_{sphere}}$$

Example 7.4.3b:

Remark: for many shapes the length of a side (or the radius) varies linearly as we go up the shape. So if we know the length of the side (or radius) for two slices we can find the length of the side (or radius) for any slice. An example to clear up what I mean,

$\boxed{E3}$: find volume of pyramid of height h with triangular base with side a,



what's the area of the triangle? look at the pictures to see

$$A = \frac{1}{2}\ell\left(\frac{\sqrt{3}}{2}\ell\right) = \frac{\sqrt{3}}{4}\ell^2$$

The volume from the slice would be $dV = A\,dY = \frac{\sqrt{3}}{4}\ell^2 dY$. Notice that ℓ is a function of Y. More importantly it is a linear function of Y.

$Y = 0 \Rightarrow \ell = a$ (at the base)

$Y = h \Rightarrow \ell = 0$ (at the top)

But we know $\ell(Y) = mY + b$. Plug in the data,

$\ell(h) = 0 = mh + b \Rightarrow m = -b/h$

$\ell(0) = a = m(0) + b \Rightarrow \underline{a = b} \Rightarrow \underline{m = -a/h}$

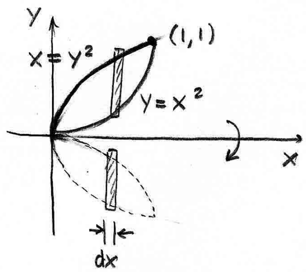
Thus $\ell(Y) = \left(-\frac{a}{h}\right)Y + a = \frac{a}{h}(h - Y)$

$$V = \int_0^h \frac{\sqrt{3}}{4}\ell^2 dY = \int_0^h \frac{\sqrt{3}}{4}\frac{a^2}{h^2}(h^2 - 2hY + Y^2)\,dY$$

$$= \frac{\sqrt{3}a^2}{4h^2}\left(h^2Y - hY^2 + \frac{1}{3}Y^3\right)\Big|_0^h = \boxed{\frac{a^2 h}{4\sqrt{3}}}$$

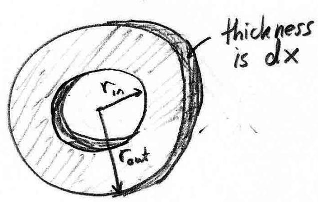
Example 7.4.4:

E4 Find volume of solid obtained by rotating the region bounded by $y = x^2$ and $y^2 = x$ about the x-axis.



intersection points have $y^2 = y^2$ meaning $x = (x^2)^2 = x^4$

$\Rightarrow x^4 - x = x(x^3 - 1) = 0 \Rightarrow \underline{x = 0 \text{ or } x = 1}$



← this is a typical washer in the volume pictured

$r_{in} = x^2$

$r_{out} = \sqrt{x}$

the area of the annulus is given by $A = \pi(r_{out}^2 - r_{in}^2) = \pi(x - x^4)$

thus the volume of a typical washer is,

$$dV = \pi(x - x^4)\,dx$$

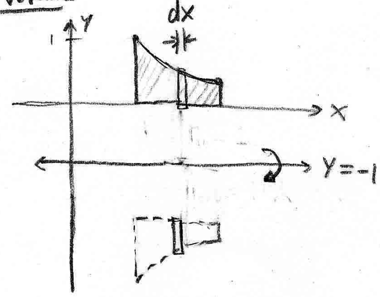
Which allows us to find the total volume V,

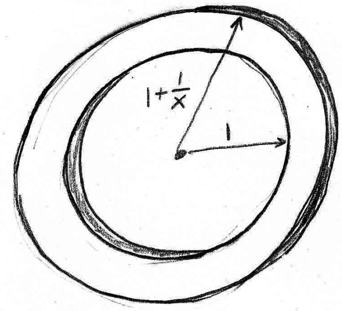
$$\begin{aligned} V &= \int_0^1 \pi(x - x^4)\,dx \\ &= \pi\left(\tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\right|_0^1 \\ &= \pi\left(\tfrac{1}{2} - \tfrac{1}{5}\right) \\ &= \boxed{\frac{3\pi}{10}} \end{aligned}$$

Example 7.4.5:

E5 bound region by $Y = 1/x$, $Y = 0$, $X = 1$, $X = 3$ and rotate around $Y = -1$

- Find its Volume

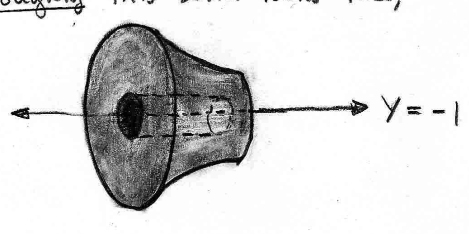




$$dV = \pi(r_{out}^2 - r_{in}^2)\,dx$$
$$= \pi\left(\left(1+\frac{1}{x}\right)^2 - 1\right)dx$$
$$= \pi\left(1 + 2/x + 1/x^2 - 1\right)dx$$
$$= \pi\left(\frac{2}{x} + \frac{1}{x^2}\right)dx$$

$$V = \int_1^3 \pi\left(\frac{2}{x} + \frac{1}{x^2}\right)dx = \left(2\pi\ln(x) - \frac{\pi}{x}\right|_1^3 =$$
$$= \left(2\pi\ln(x) - \frac{\pi}{x}\right|_1^3$$
$$= 2\pi\ln(3) - \frac{\pi}{3} + \pi$$
$$= \boxed{2\pi(\ln(3) + 1/3)}$$

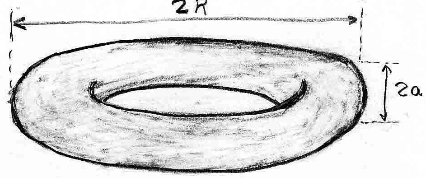
Roughly this solid looks like,



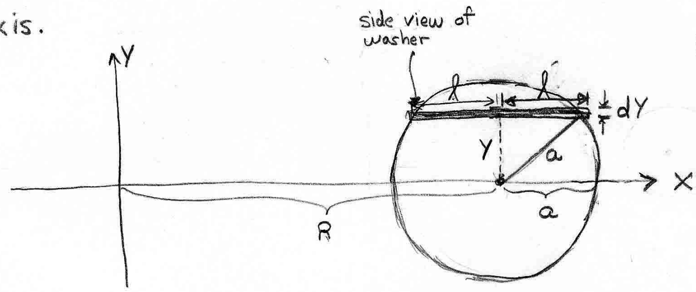
Example 7.4.6: (would provide some assistance with integration if on test)

145

EG Find volume of a donut. Ok, we call these tasty objects a "torus" in math. Let the torus have big radius R and little radius a.

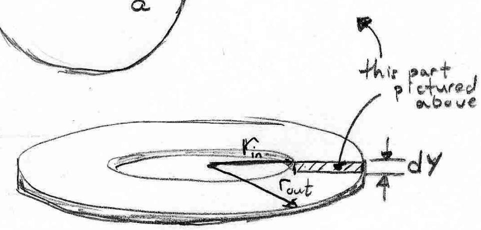


We can see that this shape can be obtained by rotating a circle of radius a centered at $X = R$ around the y-axis.



By pythagorean th^{m}

$\ell = \sqrt{a^2 - Y^2}$



$$r_{in} = R - \ell = R - \sqrt{a^2 - Y^2}$$

$$r_{out} = R + \ell = R + \sqrt{a^2 - Y^2}$$

Find the volume of a typical washer,

$$dV = \pi r_{out}^2\, dY - \pi r_{in}^2\, dY$$

$$= \pi\left[(R+\ell)^2 - (R-\ell)^2\right] dY$$

$$= \pi\left[(\cancel{R^2} + 2R\ell + \cancel{\ell^2}) - (\cancel{R^2} - 2R\ell + \cancel{\ell^2})\right] dY$$

$$= 4\pi R \ell\, dY$$

$$= 4\pi R \sqrt{Y^2 - a^2}\, dY$$

(you can see a washer at each Y from $Y = -a$ all the way upto $Y = a$)

$$V = \int_{-a}^{a} 4\pi R \sqrt{Y^2 - a^2}\, dY$$

$$= \int_{-\pi/2}^{\pi/2} 4\pi R (a\cos\theta) a \cos\theta\, d\theta$$

notice this subst. is geometrically pleasing.

$Y = a\sin\theta$ $\quad$ $Y = a \rightarrow \theta = \pi/2$

$dY = a\cos\theta\, d\theta$ $\quad$ $Y = -a \rightarrow \theta = -\pi/2$

$\sqrt{Y^2 - a^2} = a\cos\theta$

$$= 4\pi R a^2 \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\, d\theta$$

$$= 2\pi R a^2 \left(\underset{\pi}{\theta} + \underset{0}{\tfrac{1}{2}\sin(2\theta)}\right)\Bigg|_{-\pi/2}^{\pi/2} = \boxed{2\pi^2 R a^2 = V}$$

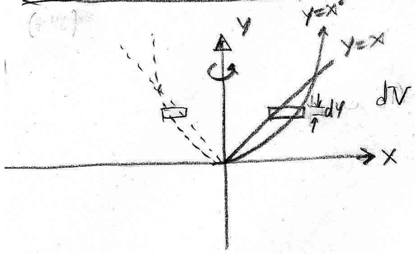
Remark: intuitively nice

$V = (2\pi R)(\pi a^2)$

Example 7.4.7:

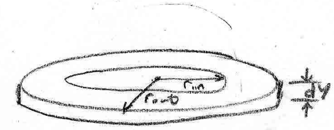
(146)

E7 FIND VOLUME OF SOLID FORMED BY ROTATING REGION PICTURED AROUND Y-AXIS



$x_{out}^2 - x_{in}^2$

$$dV = \pi(y - y^2)\,dy$$



$$dV = \pi(r_{out}^2 - r_{in}^2)\,dy$$

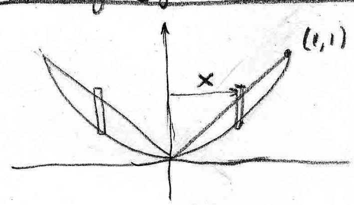
the dy suggests that we must find r_{in} and r_{out} as functions of y.

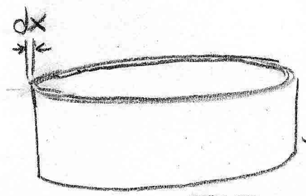
$$V = \int_0^1 \pi(y - y^2)\,dy$$

$$= \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right|_0^1$$

$$= \boxed{\pi/6}$$

E7 using "Cylindrical Shells"





$Y_{TOP} - Y_{BOTTOM}$

need to write in terms of x

$$\underbrace{dV = 2\pi x(x - x^2)\,dx}_{\text{volume of infinitesimal cylinder pictured.}}$$

$$V = \int_0^1 2\pi x(x - x^2)\,dx$$

$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right|_0^1$$

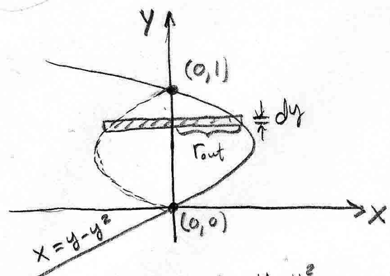
$$= 2\pi\left[\frac{1}{3} - \frac{1}{4}\right]$$

$$= \boxed{\pi/6}$$

Example 7.4.8:

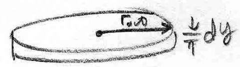
E8

Rotate $x = y - y^2$, $x = 0$ about the y-axis



$r_{out} = x = y - y^2$

Notice that $x = y - y^2 = y(1-y)$ showing $x = 0$ when $y = 0$ or $y = 1$ thus my graph. Rememeber the real graph is 3-d but, I usually just draw the intersection of the volume with the (xy)-plane. That picture is enough to figure out how r_{out} depends on x & y.



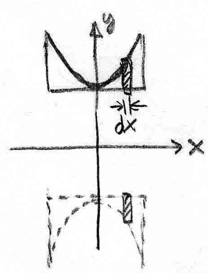
has Volume

$$dV = \pi(y - y^2)^2 dy = \pi(y^2 - 2y^3 + y^4)dy$$

$$V = \int_0^1 \pi(y^2 - 2y^3 + y^4)dy$$
$$= \pi\left(\tfrac{1}{3}y^3 - \tfrac{2}{4}y^4 + \tfrac{1}{5}y^5\right|_0^1$$
$$= \pi\left(\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5}\right) = \pi\left(\tfrac{10-15+6}{30}\right) = \boxed{\tfrac{\pi}{30} = 0.105}$$

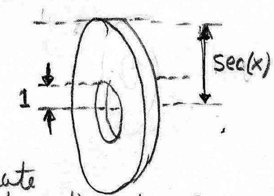
Example 7.4.9:

E9



$y = \sec(x)$, $x = 1$ & $x = -1$ } rotate this region around x-axis & find volume.

$$dV = \pi(r_{out}^2 - r_{in}^2)dx = \pi(\sec^2(x) - 1)dx$$



Because of symmetry can calculate half the volume $(0 \le x \le 1)$ and multiply by 2.

$$V = 2\int_0^1 \pi(\sec^2(x) - 1)dx$$
$$= 2\pi\left(\tan(x) - x\right|_0^1$$
$$= 2\pi\left[(\tan(1) - 1) - (\tan(0) - 0)\right]$$
$$= \boxed{2\pi(\tan(1) - 1) = 3.502}$$

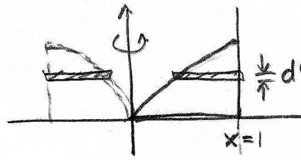
Example 7.4.10:

E10 Bound region by,

$y = x^{2/3}$, $x = 1$, $y = 0$

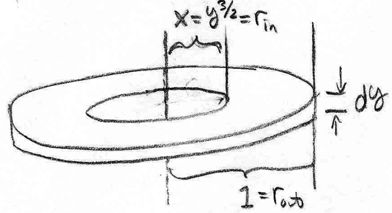
about the y-axis

146c



$y = x^{2/3}$

$\Rightarrow \underline{x = y^{3/2}}$



$$dV = \pi\left(1 - (y^{3/2})^2\right)dy$$
$$= \pi(1 - y^3)\,dy$$

Area of annulus

$A = \pi(r_{out}^2 - r_{in}^2)$

$dV = A\,dy$

$$V = \int_0^1 \pi(1-y^3)\,dy$$
$$= \pi\left(y - \tfrac{1}{4}y^4\right)\Big|_0^1$$
$$= \pi\left(1 - \tfrac{1}{4}\right) = \boxed{\tfrac{3\pi}{4} = 2.356}$$

Example 7.4.11:

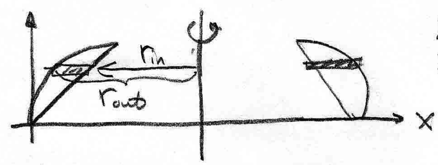
E11 Rotate $\left\{\begin{array}{l} y = x \\ y = \sqrt{x} \end{array}\right\}$ about $x = 2$.

Notice $x = \sqrt{x} \Rightarrow x^2 = x$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0$ & $x = 1$

(intersection pts.)



$x_R = y$

$x_L = y^2$

$r_{out} = 2 - x_L = 2 - y^2$

$r_{in} = 2 - x_R = 2 - y$

$$dV = \pi\left(r_{out}^2 - r_{in}^2\right)dy$$
$$= \pi\left[(2-y^2)^2 - (2-y)^2\right]dy$$
$$= \pi\left[\cancel{4} - 4y^2 + y^4 - (\cancel{4} - 4y + y^2)\right]dy$$
$$= \pi\left[y^4 - 5y^2 + 4y\right]dy$$

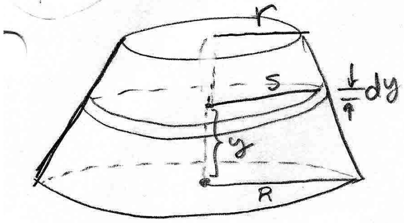
$$\Rightarrow V = \int_0^1 \pi(y^4 - 5y^2 + 4y)\,dy = \pi\left(\tfrac{1}{5}y^5 - \tfrac{5}{3}y^3 + 2y^2\right)\Big|_0^1 = \pi\left(\tfrac{1}{5} - \tfrac{5}{3} + 2\right)$$
$$= \pi\left(\frac{3 - 25 + 30}{15}\right)$$
$$= \boxed{\frac{8\pi}{15} = 1.6755}$$

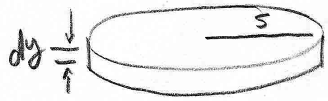
Example 7.4.12:

E12

height of shape is h.

146d





$A = \pi s^2$

s depends linearly on y

$$s = my + b$$

$$s(0) = m(0) + b = b = R$$

$$s(h) = mh + R = r$$

$$\Rightarrow mh = r - R \quad \therefore m = \frac{r-R}{h}$$

$$\therefore s = \left(\frac{r-R}{h}\right)y + R$$

$$dV = \pi s^2 dy$$

$$= \pi\left[\left(\frac{r-R}{h}\right)y + R\right]^2 dy$$

$$= \pi\left[\left(\frac{r-R}{h}\right)^2 y^2 + 2R\left(\frac{r-R}{h}\right)y + R^2\right]dy$$

$$V = \pi\int_0^h \left[\left(\frac{r-R}{h}\right)^2 y^2 + 2R\left(\frac{r-R}{h}\right)y + R^2\right]dy$$

$$= \pi\left[\left(\frac{r-R}{h}\right)^2 \frac{1}{3}y^3 + 2R\left(\frac{r-R}{h}\right)\frac{1}{2}y^2 + R^2 y\,\right|_0^h$$

$$= \pi\left[\frac{1}{3}\left(\frac{r-R}{h}\right)^2 h^3 + R\left(\frac{r-R}{h}\right)h^2 + R^2 h\right]$$

$$= \pi h\left[\frac{1}{3}(r-R)^2 + Rr - \cancel{R^2} + \cancel{R^2}\right]$$

$$= \frac{1}{3}\pi h\left[(r-R)^2 + 3Rr\right]$$

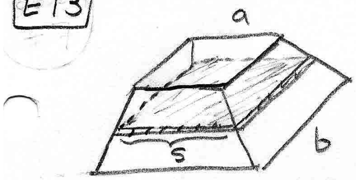
$$= \frac{1}{3}\pi h\left[r^2 - 2Rr + R^2 + 3Rr\right]$$

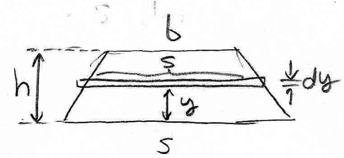
$$= \boxed{\frac{1}{3}\pi h\left[r^2 + Rr + R^2\right] = V}$$

Example 7.4.13:

E13

146e





s depends linearly on y thus

$s = my + B$

From picture

$s(0) = b$

$s(h) = a$

Continuing

$$\left.\begin{aligned} s(0) &= m(0) + B = B = b \\ s(h) &= m(h) + b = a \\ m &= \frac{a-b}{h} \end{aligned}\right\} \Rightarrow \boxed{s = \left(\frac{a-b}{h}\right)y + b}$$

The area of the cross-section is just $A = s^2$ (square) thus

$$dV = s^2 dy = [my + b]^2 dy = [m^2y^2 + 2mby + b^2]dy$$

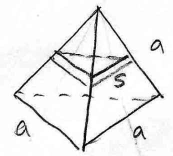
Therefore,

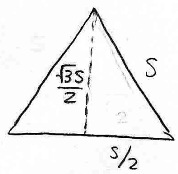
$$\begin{aligned} V &= \int_0^h [m^2y^2 + 2mby + b^2]dy \\ &= \left(\tfrac{1}{3}m^2y^3 + \tfrac{2mb}{2}y^2 + b^2y\right|_0^h \\ &= \tfrac{1}{3}m^2h^3 + mbh^2 + b^2h \\ &= \tfrac{1}{3}\left(\tfrac{a-b}{h}\right)^2h^3 + \left(\tfrac{a-b}{h}\right)bh^2 + b^2h \\ &= h\left(\tfrac{1}{3}(a-b)^2 + (a-\cancel{b})b + \cancel{b^2}\right) \\ &= \tfrac{1}{3}h\left((a-b)^2 + 3ab\right) \\ &= \tfrac{1}{3}h\left(a^2 - 2ab + b^2 + 3ab\right) \\ &= \boxed{\tfrac{1}{3}h\left(a^2 + ab + b^2\right)} \end{aligned}$$

I figure why write $\frac{a-b}{h}$ when I could just write m instead.

Example 7.4.14:

E14 (146 f)





Area of equilateral Δ with side s is just

$$A = \frac{1}{2}(s)\left(\frac{\sqrt{3}}{2}s\right) = \frac{\sqrt{3}}{4}s^2$$

S depends linearly on y : $S = my + b$

(Base) $S(0) = m(0) + b = b = a$

(Vertex) $S(h) = mh + a = 0$

$m = -a/h$

$$\underline{S = \frac{-a}{h}y + a}$$

The volume dV of the slice is $A\,dy$ thus,

$$dV = \frac{\sqrt{3}}{4}(my + a)^2 dy = \frac{\sqrt{3}}{4}(m^2y^2 + 2may + a^2)dy$$

Now integrate to find total volume,

$$V = \int_0^h \frac{\sqrt{3}}{4}(m^2y^2 + 2may + a^2)dy$$

$$= \frac{\sqrt{3}}{4}\left(\frac{1}{3}m^2y^3 + may^2 + a^2y\right|_0^h$$

$$= \frac{\sqrt{3}}{4}\left(\frac{1}{3}m^2h^3 + mah^2 + a^2h\right)$$

$$= \frac{\sqrt{3}}{4}\left(\frac{1}{3}\frac{a^2}{h^2}h^3 - \frac{a}{h}ah^2 + a^2h\right)$$

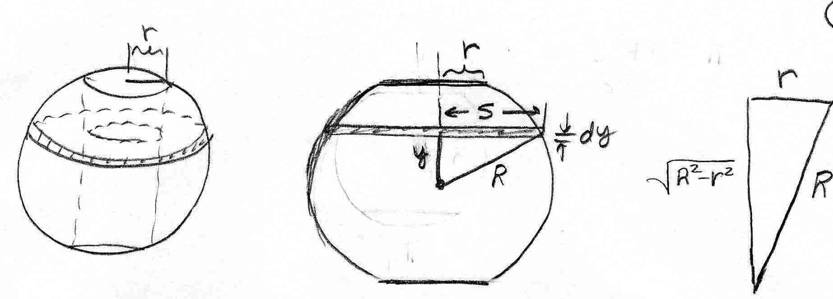
$$= \frac{\sqrt{3}}{4}\left(\frac{1}{3}a^2 - \cancel{a^2} + \cancel{a^2}\right)h$$

$$= \boxed{\frac{\sqrt{3}}{12}a^2h = V}$$

Example 7.4.15:

E15

146g



$$dV = \pi(r_{out}^2 - r_{in}^2)\,dy$$
$$= \pi(s^2 - r^2)\,dy$$
$$= \pi(R^2 - y^2 - r^2)\,dy$$
$$= \pi(R^2 - r^2 - y^2)\,dy$$

$$y^2 + s^2 = R^2$$
$$s^2 = R^2 - y^2$$

Notice that
$-\sqrt{R^2-r^2} \le y \le \sqrt{R^2-r^2}$
we add volume of washers in this range
By symmetry can just integrate from zero $\longrightarrow \sqrt{R^2-r^2}$ and double it.

$$V = 2\int_0^{\sqrt{R^2-r^2}} \pi(R^2 - r^2 - y^2)\,dy$$
$$= 2\pi\left((R^2-r^2)y - \frac{1}{3}y^3\right|_0^{\sqrt{R^2-r^2}}$$
$$= 2\pi\left((R^2-r^2)\sqrt{R^2-r^2} - \frac{1}{3}\left(\sqrt{R^2-r^2}\right)^3\right)$$
$$= 2\pi\left((R^2-r^2)^{3/2} - \frac{1}{3}(R^2-r^2)^{3/2}\right)$$
$$= \boxed{\frac{4\pi}{3}(R^2-r^2)^{3/2} = V}$$

Notice when $r = 0$ we get $V = \frac{4}{3}\pi R^3$ which is a good thing.