2.3 physics of motion

In this section we study kinematics. That is, we study how position, velocity and acceleration are related for physical motions. We do not ask where the force comes from, that is a question for physics. Our starting point is the equation of motion $\vec{F} = m\vec{A}$ which is called Newton's Second Law. Given the force and some initial conditions we can in principle integrate the equations of motion and derive the resulting kinematics. We have already, in the LU calculus sequence, twice studied kinematics. In calculus I for one-dimensional motion, in calculus II for two-dimensional motion. I recycle some examples for our current discussion. However, some comments are added since we now have the proper machinery to break-down vectors along a physical path. Let's see how the preceding section is useful in the analysis of the motion of physical objects. The solution of Newton's equation $\vec{F} = m\vec{A}$ is a path $t \mapsto \vec{r}(t)$. It follows we can analyze the velocity and acceleration of the physical path in terms of the **Frenet Frame** $\{T, N, B\}$. To keep it interesting we'll assume the motion is non-stop and smooth so that the analysis of the last section applies.

In this section the notations $\vec{r}$, $\vec{v}$ and $\vec{a}$ are special and set-apart. I don't use these as abstract variables here with no set meaning. Instead, these are connected as is described in the definition that follows:

Definition 2.3.1. *position, velocity and acceleration.*

The position, velocity and acceleration of an object are vector-valued functions of time and we define them as follows:

1. $\vec{r}(t)$ is the position at time t. (we insist physical paths are parametrized by time)
2. $\vec{v}(t) = \frac{d\vec{r}}{dt}$ is the velocity at time t.
3. $\vec{a}(t) = \frac{d\vec{v}}{dt}$ is the acceleration at time t.

We also define the tangential and normal accelerations of the motion by

$$\vec{a}_T = \vec{a} \bullet \vec{T} \qquad \vec{a}_N = \vec{a} \bullet \vec{N} \qquad \text{note:} \qquad \vec{a} = \vec{a}_T\vec{T} + \vec{a}_N\vec{N}.$$

We know from our study of the geometry of curves that the binormal component of the acceleration is trivial. Acceleration must lie in the osculating plane and as such is perpendicular to the binormal vector which is the normal to the osculating plane. If you're curious, the position vector itself can have nontrivial components in each direction of the Frenet frame whereas the velocity vector clearly has only a tangential component; $\vec{v} = v\vec{T}$.

If we are given the position vector as a function of time then we need only differentiate to find the velocity and acceleration. On the other hand, if we are given the acceleration then we need to integrate and apply initial conditions to obtain the velocity and position.

Example 2.3.2. *Suppose R and ω are positive constants and the motion of an object is observed to follow the path $\vec{r}(t) = \langle R\cos(\omega t), R\sin(\omega t)\rangle = R\langle\cos(\omega t), \sin(\omega t)\rangle$. We wish to calculate the velocity and acceleration as functions of time.*

Differentiate to obtain the velocity

$$\boxed{\vec{v}(t) = R\omega\langle -\sin(\omega t), \cos(\omega t)\rangle.}$$

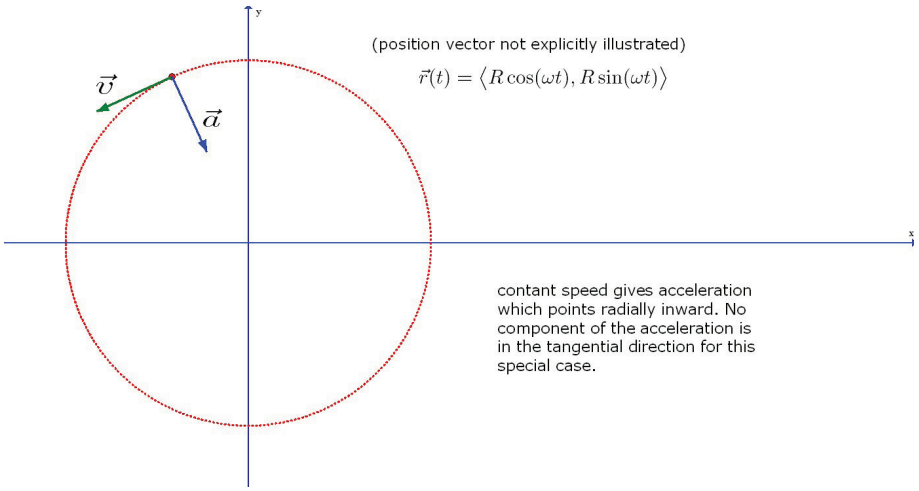
Differentiate once more to obtain the acceleration:

$$\vec{a}(t) = R\omega\langle -\omega\cos(\omega t), -\omega\sin(\omega t)\rangle = \boxed{-R\omega^2\langle\cos(\omega t), \sin(\omega t)\rangle.}$$

Notice we can write that $\vec{a}(t) = -\omega^2\vec{r}(t) = R\omega^2\vec{N}$ in this very special example. This means the acceleration is opposite the direction of the position and it is purely normal. Furthermore, we can calculate

$$r = R, \qquad v = R\omega, \qquad a = R\omega^2$$

Thus the magnitudes of the position, velocity and acceleration are all constant. However, their directions are always changing. Perhaps you recognize these equations as the foundational equations describing constant speed circular motion. This acceleration is called the **centripetal** *or center-seeking acceleration since it points towards the center. Here we imagine attaching the acceleration vector to the object which is traveling in the circle.*



Incidentally, you might wonder how the binormal should be thought of in the example above. We should adjoin a zero to make the vectors three-dimensional and then the cross-product of $\vec{T}\times\vec{N}$ points in the direction given by the right-hand-rule for circles. Curl your right hand around the circle following the motion and your thumb will point in the binormal direction. You can calculate that the binormal is constant:

$$\vec{B} = \vec{T}\times\vec{N} = \langle -\sin(\omega t), \cos(\omega t)\rangle \times \langle -\cos(\omega t), -\sin(\omega t)\rangle = \langle 0, 0, 1\rangle$$

Often when we consider planar motion we omit the third dimension in the vectors since those components are zero throughout the whole discussion. That said, if we wish to properly employ the Frenet Frame analysis then we must think in three dimensions. The next example is also two-dimensional[6].

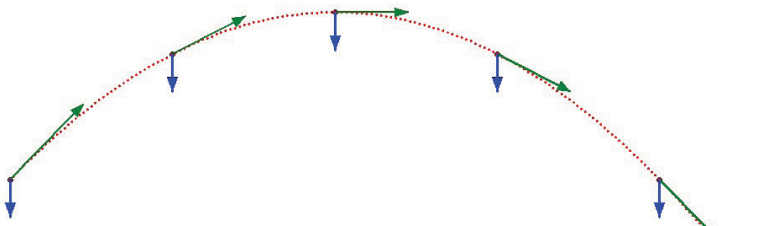
Example 2.3.3. *Suppose that the acceleration of an object is known to be* $\vec{a} = \langle 0, -g \rangle$ *where* g *is a positive constant. Furthermore, suppose that initially the object is at* $\vec{r}_o$ *and has velocity* $\vec{v}_o$*. We wish to calculate the position and velocity as functions of time.*

Integrate the acceleration from 0 *to* t*,*

$$\int_0^t \frac{d\vec{v}}{d\tau} d\tau = \int_0^t a(\tau) d\tau \quad \Rightarrow \quad \vec{v}(t) - \vec{v}(0) = \int_0^t \langle 0, -g \rangle d\tau \quad \Rightarrow \quad \boxed{\vec{v}(t) = \vec{v}_o + \langle 0, -gt \rangle}$$

Integrate the velocity from 0 *to* t*,*

$$\int_0^t \frac{d\vec{r}}{d\tau} d\tau = \int_0^t v(\tau) d\tau \quad \Rightarrow \quad \vec{r}(t) - \vec{r}(0) = \int_0^t \big(\vec{v}_o + \langle 0, -gt \rangle\big) d\tau \quad \Rightarrow \quad \boxed{\vec{r}(t) = \vec{r}_o + t\vec{v}_o + \langle 0, -\frac{1}{2} gt^2 \rangle}$$



$$\vec{r}(t) = \left\langle x_o + v_{ox} t,\ y_o + v_{oy} t - \tfrac{1}{2} g t^2 \right\rangle$$
$$\vec{v}(t) = \left\langle v_{ox},\ v_{oy} - gt \right\rangle$$
$$\vec{a}(t) = \left\langle 0, -g \right\rangle$$

The acceleration is constant for this parabolic trajectory.
The velocity is changing in the vertical direction, but is constant in the x-direction.

I'm curious how the decomposition of the acceleration into normal and tangential components works out for the example above. Maybe I'll make it a homework.

The best understanding of Newtonian Mechanics is given by a combination of both vectors and calculus. We need vectors to phrase the geometry of force addition whereas we need calculus to understand how the position, velocity and acceleration variables change in concert.

[6] all motion generated from Newtonian gravity alone is planar. A more general result states all central force motion lies in a plane, probably a homework of yours

2.3.1 position vs. displacement vs. distance traveled

The position of an object is simply the (x, y, z) coordinates of the object. Usually it is convenient to think of the position as a vector-valued function of time which we denote $\vec{r}(t)$. The displacement is also a vector, however it compares two possibly distinct positions:

Definition 2.3.4. *displacement and distance traveled.*

Suppose $\vec{r}(t)$ is the position at time t of some object.

1. The **displacement** from position $\vec{r}_1$ to position $\vec{r}_2$ is the vector $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$.
2. The **distance travelled** during the interval $[t_1, t_2]$ along the curve $t \mapsto \vec{r}(t)$ is given by
$$s_{12} = \int_{t_1}^{t_2} v(t)dt = \int_{t_1}^{t_2} \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\, dt$$
where $v(t) = ||d\vec{r}/dt||$.

Note that the position is the displacement from the origin. Distance travelled is a **scalar** quantity which means it is just a number or if we think of an endpoint as variable it could be a function.

Definition 2.3.5. *arclength function and speed.*

We define
$$s(t) = \int_{t_1}^{t} v(\tau)d\tau = \int_{t_1}^{t} \sqrt{\frac{dx}{d\tau}^2 + \frac{dy}{d\tau}^2 + \frac{dz}{d\tau}^2}\, d\tau$$
to be the arclength travelled from time t_1 to t along the parametrized curve $t \mapsto \vec{r}(t)$. Futhermore, we define the **speed** to be the instantaneous rate of change in the arclength; speed is ds/dt.

Notice it is simple to show that the speed is also equal to the magnitude of the velocity; $ds/dt = v$. Also, note that we drop the z-terms for a typical two-dimensional problem. If you insist on being three dimensional you can just adjoin a bunch of zeros in the examples below. There are unavoidably three dimensional examples a little later in the section.

Example 2.3.6. *Let $\omega, R > 0$. Suppose $\vec{r}(t) = \langle R\cos(\omega t), R\sin(\omega t)\rangle$ for $t \geq 0$. We can calculated that*
$$\frac{d\vec{r}}{dt} = \langle -R\omega\sin(\omega t), R\omega\cos(\omega t)\rangle \quad \Rightarrow \quad v(t) = \sqrt{(-R\omega\sin(\omega t))^2 + (R\omega\cos(\omega t))^2} = \sqrt{R^2\omega^2} = R\omega.$$
Now use this to help calculate the distance travelled during the interval $[0, t]$
$$s(t) = \int_0^t v(\tau)d\tau = \int_{t_1}^t R\omega d\tau = R\omega\tau\Big|_0^t = R\omega t.$$

In other words, $\Delta s = R\omega\Delta t$. On a circle the arclength subtended Δs divided by the radius R is defined to be the radian measure of that arc which we typically denote $\Delta\theta$. We find that $\Delta\theta = \omega\Delta t$ or if you prefer $\omega = \Delta\theta/\Delta t$.

Circular motion which is not at a constant speed can be obtained mathematically by replacing the constant ω with a function of time. Let's examine such an example.

Example 2.3.7. *Suppose $\vec{r}(t) = \langle R\cos(\theta), R\sin(\theta)\rangle$ for $t \geq 0$ where $\theta_o, \omega_o, \alpha$ are constants and $\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$. To calculate the distance travelled it helps to first calculate the velocity:*

$$\frac{d\vec{r}}{dt} = \langle -R(\omega_o + \alpha t)\sin(\theta), R(\omega_o + \alpha t)\cos(\theta)\rangle$$

Next, the speed is the length of the velocity vector,

$$v = \sqrt{[-R(\omega_o + \alpha t)\sin(\theta)]^2 + [R(\omega_o + \alpha t)\cos(\theta)]^2} = R\sqrt{(\omega_o + \alpha t)^2} = R|\omega_o + \alpha t|.$$

Therefore, the distance travelled is given by the integral below:

$$s(t) = \int_0^t R|\omega_o + \alpha\tau|d\tau$$

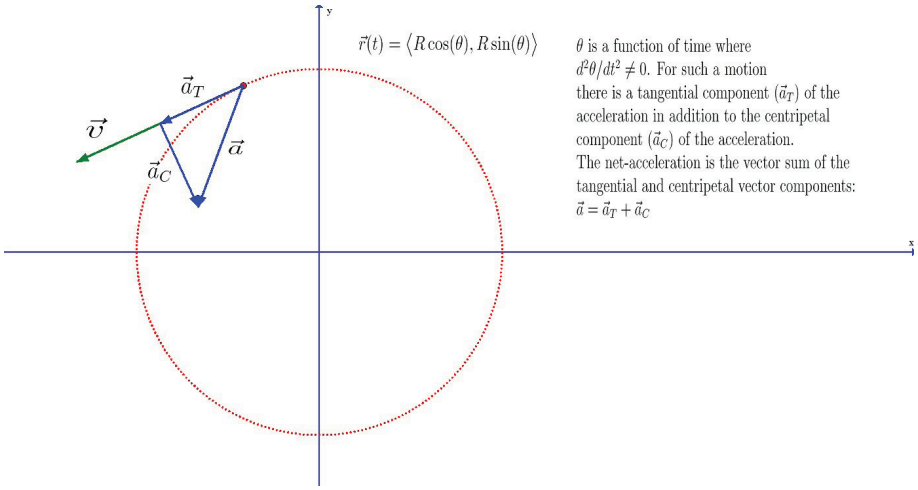
To keep things simple, let's suppose that ω_o, α are given such that $\omega_o + \alpha t \geq 0$ hence $v = R\omega_o + R\alpha t$. To suppose otherwise would indicate the motion came to a stopping point and reversed direction, which is interesting, just not to us here.

$$s(t) = R\int_0^t (\omega_o + \alpha\tau)d\tau = R\omega_o t + \frac{1}{2}R\alpha t^2.$$

Observe that $\theta(t) - \theta_o = (s(t) - s(0))/R$ thus we find that $\Delta\theta = \omega_o t + \frac{1}{2}\alpha t^2$ which is the formula for the angle subtended due to motion at a constant **angular acceleration** *α. In invite the reader to differentiate the position twice and show that*

$$\vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = -\underbrace{R\omega^2\big\langle \cos(\theta(t)), \sin(\theta(t))\big\rangle}_{centripetal} + \underbrace{R\alpha\big\langle -\sin(\theta(t)), \cos(\theta(t))\big\rangle}_{tangential}$$

where $\omega = \omega_o + \alpha t$.



Distance travelled is not always something we can calculate in closed form. Sometimes we need to relegate the calculation of the arclength integral to a numerical method. However, the example that follows is still calculable without numerical assistance. It did require some thought.

Example 2.3.8. *We found that $\vec{a} = \langle 0, -g\rangle$ twice integrated yields a position of $\vec{r}(t) = \vec{r}_o + t\vec{v}_o + \langle 0, -\frac{1}{2}gt^2\rangle$ for some constant vectors $\vec{r}_o = \langle x_o, y_o\rangle$ and $\vec{v}_o = \langle v_{ox}, v_{oy}\rangle$. Thus,*

$$\vec{r}(t) = \left\langle x_o + v_{ox}t, y_o + v_{oy}t - \frac{1}{2}gt^2\right\rangle$$

From which we can differentiate to derive the velocity,

$$\vec{v}(t) = \left\langle v_{ox}, v_{oy} - gt\right\rangle.$$

If you've had any course in physics, or just a proper science education, you should be happy to observe that the zero-acceleration in the x-direction gives rise to constant-velocity motion in the x-direction whereas the gravitational acceleration in the y-direction makes the object fall back down as a consequence of gravity. If you think about $v_{oy} - gt$ it will be negative for some $t\rangle 0$ whatever the initial velocity v_{oy} happens to be, this point where $v_{oy} - gt = 0$ is the turning point in the flight of the object and it gives the top of the parabolic[7] trajectory which is parametrized by $t \to \vec{r}(t)$. Suppose $x_o = y_o = 0$ and calculate the distance travelled from time $t = 0$ to time $t_1 = v_{oy}/g$. Additionally, let us assume $v_{ox}, v_{oy} \geq 0$.

$$\begin{aligned} s = \int_0^{t_1} v(t)dt &= \int_0^{t_1} \sqrt{(v_{ox})^2 + (v_{oy} - gt)^2}dt \\ &= \int_{v_{oy}}^{0} \sqrt{(v_{ox})^2 + (u)^2}\left(\frac{du}{-g}\right) \qquad u = v_{oy} - gt \\ &= \frac{1}{g}\int_0^{v_{oy}} \sqrt{(v_{ox})^2 + (u)^2}du \end{aligned}$$

[7] no, we have not shown this is a parabola, I invite the reader to verify this claim. That is find A, B, C such that the graph $y = Ax^2 + Bx + C$ is the same set of points as $\vec{r}(\mathbb{R})$.

Recall that a nice substitution for an integral such as this is provided by the $\sinh(z)$ *since* $1+\sinh^2(z)=\cosh^2(z)$ *hence a* $u=v_{ox}\sinh(z)$ *subsitution will give*

$$(v_{ox})^2+(u)^2=(v_{ox})^2+(v_{ox}\sinh(z))^2=v_{ox}^2\cosh^2(z)$$

and $du=v_{ox}\cosh(z)dz$ *thus,* $\int\sqrt{(v_{ox})^2+(u)^2}du=\int\sqrt{v_{ox}^2\cosh^2(z)}v_{ox}\cosh(z)dz=\int v_{ox}^2\cosh^2(z)dz$. *Furthermore,* $\cosh^2(z)=\frac{1}{2}(1+\cosh(2z))$ *hence*

$$\int\sqrt{(v_{ox})^2+(u)^2}du=\frac{v_{ox}^2}{2}\left[z+\frac{1}{2}\sinh(2z)\right]+c=\frac{v_{ox}^2}{2}\left[z+\sinh(z)\cosh(z)\right]+c$$

Note $u=v_{ox}\sinh(z)$ *and* $v_{ox}cosh(z)=\sqrt{(v_{ox})^2+(u)^2}$ *hence substituting,*

$$\int\sqrt{(v_{ox})^2+(u)^2}du=\frac{1}{2}\left[v_{ox}^2\sinh^{-1}\left(\frac{u}{v_{ox}}\right)+u\sqrt{v_{ox}^2+u^2}\right]+c$$

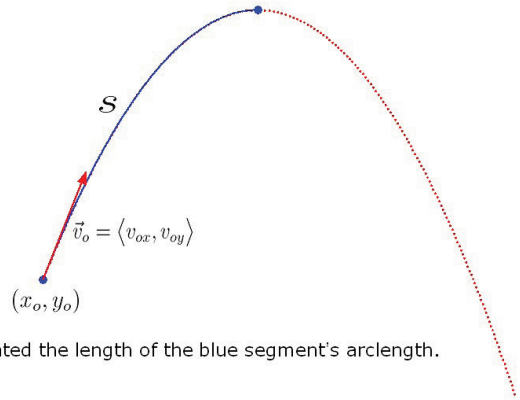
Well, I didn't think that was actually solvable, but there it is. Returning to the definite integral to calculate s *we can use the antiderivative just calculated together with FTC part II to conclude: (provided* $v_{ox}\neq 0$*)*

$$\boxed{s=\frac{1}{2g}\left[v_{ox}^2\sinh^{-1}\left(\frac{v_{oy}}{v_{ox}}\right)+v_{oy}\sqrt{v_{ox}^2+v_{oy}^2}\right]}$$

If $v_{ox}=0$ *then the problem is much easier since* $v(t)=|v_{oy}-gt|=v_{oy}-gt$ *for* $0\leq t\leq t_1=v_{oy}/g$ *hence*

$$s=\int_0^{t_1}v(t)dt=\int_0^{t_1}(v_{oy}-gt)dt=\left[v_{oy}t-\frac{1}{2}gt^2\right]\Big|_0^{v_{oy}/g}=\boxed{\frac{v_{oy}^2}{2g}}$$

Interestingly, this is the formula for the height of the parabola even if $v_{ox}\neq 0$. *The initial* x*-velocity simply determines the horizontal displacement as the object is accelerated vertically by gravity.*



We calculated the length of the blue segment's arclength.

Example 2.3.9. .

Suppose that $\vec{v}(t) = < e^t, t^2, \cos(t) >$ represents the velocity of the Dwight. If the initial position of Dwight is the origin then find the position of Dwight at time t. Also find the acceleration of Dwight at time t.

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \boxed{\langle e^t, 2t, -\sin(t)\rangle = \vec{a}(t)}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} \Rightarrow \int_0^t \vec{v}(u)\,du = \int_0^t \frac{d\vec{r}}{du}\,du = \vec{r}(t) - \vec{r}(0)$$

But, $\vec{r}(0) = \vec{0}$ hence,

$$\vec{r}(t) = \int_0^t \vec{v}(u)\,du = \int_0^t \langle e^u, u^2, \cos(u)\rangle\,du$$

$$= \langle e^u\big|_0^t, \tfrac{1}{3}u^3\big|_0^t, \sin(u)\big|_0^t \rangle$$

$$= \boxed{\langle e^t - 1, \tfrac{1}{3}t^3, \sin(t)\rangle = \vec{r}(t)}$$

Example 2.3.10. .

Let the position be $\vec{r}(t) = \langle t^2+1, t^3, t^2-1\rangle$ assume $t \geq 0$,

velocity $= \vec{r}\,'(t) = \boxed{\vec{v}(t) = \langle 2t, 3t^2, 2t\rangle}$

acceleration $= \vec{r}\,''(t) = \boxed{\vec{a}(t) = \langle 2, 6t, 2\rangle}$

Speed $= |\vec{r}\,'(t)| = |\vec{v}(t)| = \sqrt{4t^2 + 9t^4 + 4t^2} = \boxed{t\sqrt{8+9t^2}}$

Example 2.3.11. .

Find $\vec{v} = \frac{d\vec{r}}{dt}$ and $\vec{r}(t)$ given $\vec{a}(t) = \langle 2, 6t, 12t^2\rangle$ and the initial conditions $\vec{v}(0) = \langle 1, 0, 0\rangle$ and $\vec{r}(0) = \langle 0, 1, -1\rangle$

$$\vec{a} = \frac{d\vec{v}}{dt} \Rightarrow \int_{\vec{v}(0)}^{\vec{v}(t)} d\vec{v} = \int_0^t \vec{a}\,d\tilde{t} \Rightarrow \vec{v}(t) - \vec{v}(0) = \int_0^t \langle 2, 6\tilde{t}, 12\tilde{t}^2\rangle\,d\tilde{t}$$

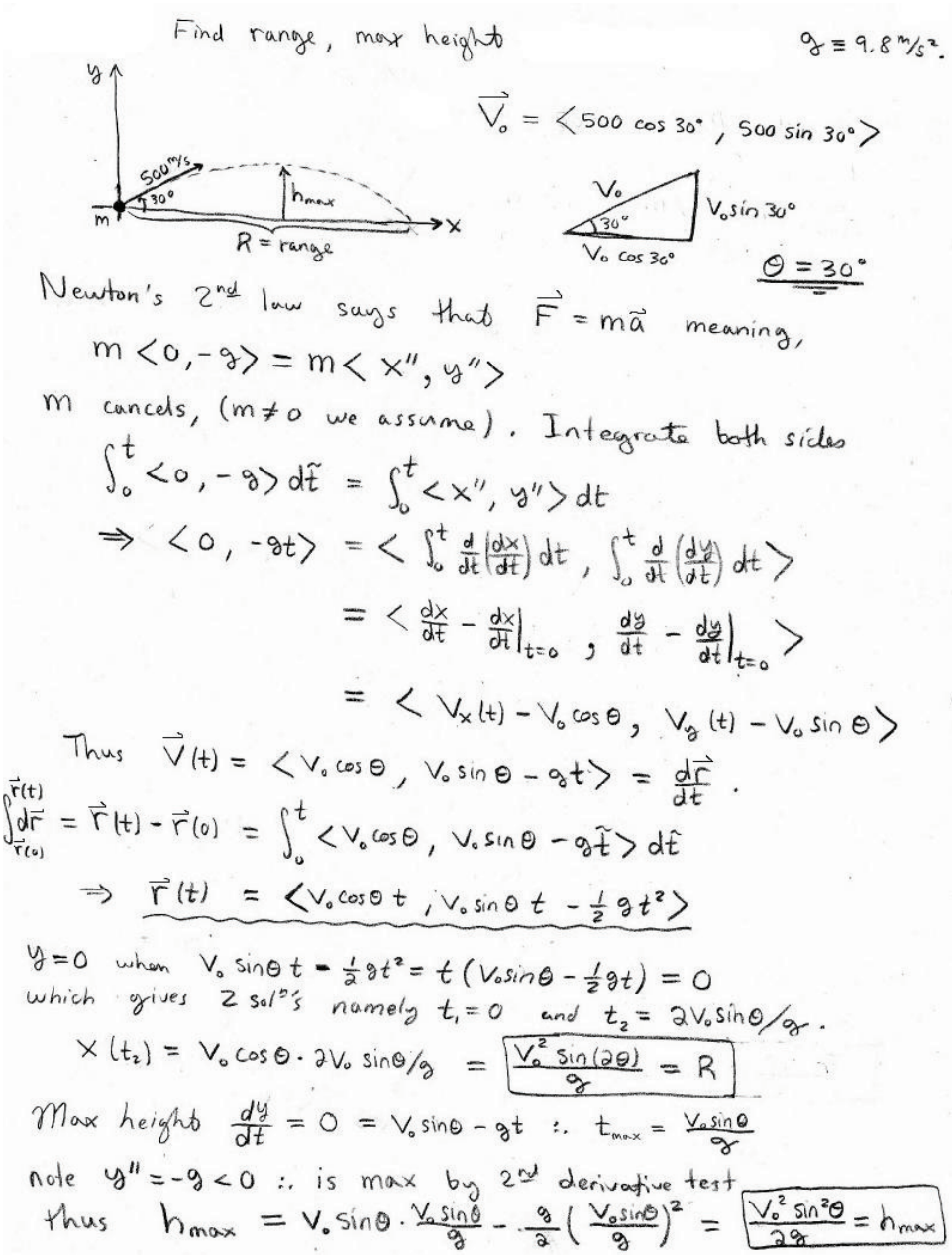
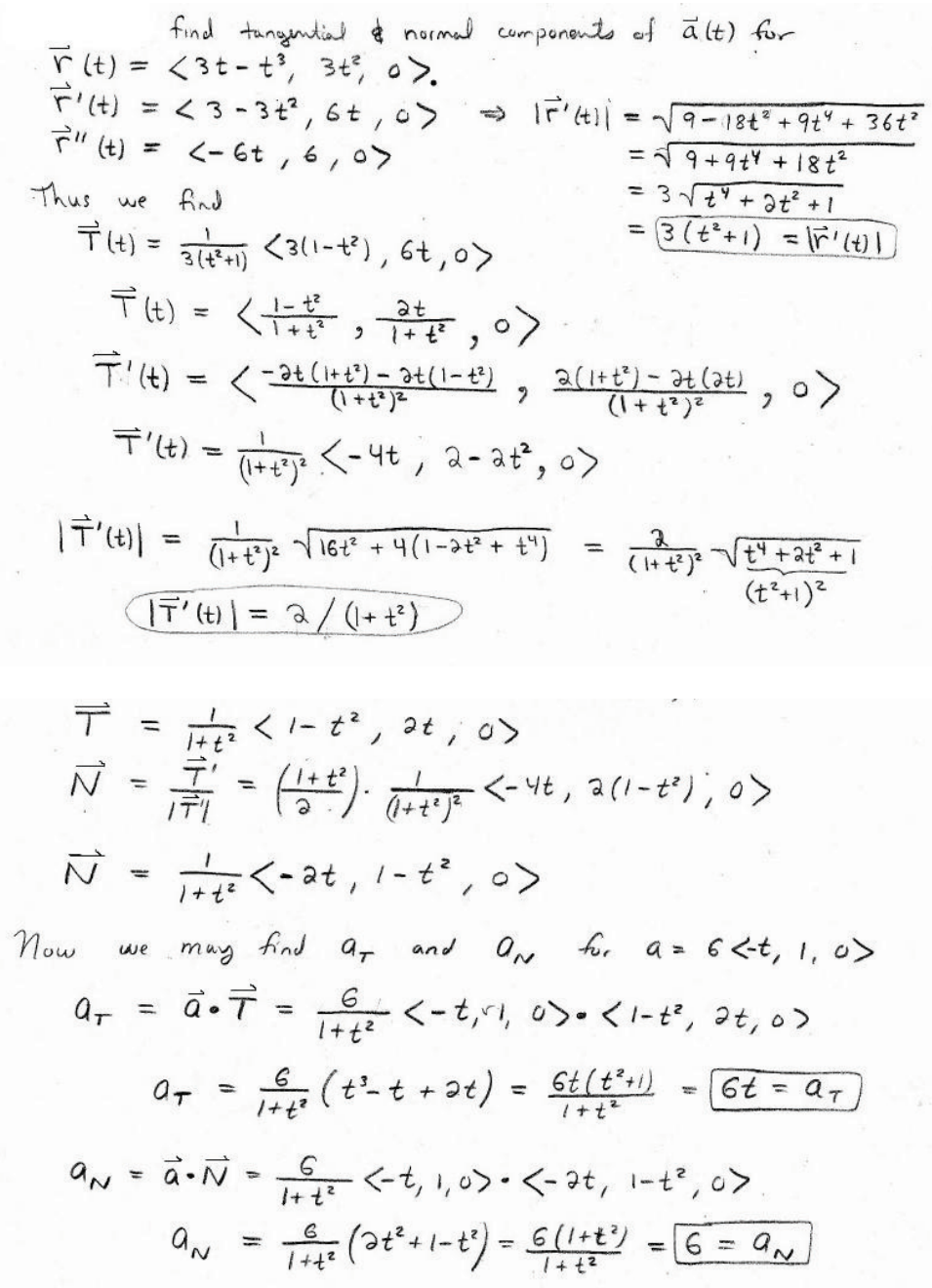
Thus $\vec{v}(t) = \langle 1, 0, 0\rangle + \langle 2t, 3t^2, 4t^3\rangle = \boxed{\langle 2t+1, 3t^2, 4t^3\rangle = \vec{v}}$

$$\vec{v} = \frac{d\vec{r}}{dt} \Rightarrow \int_{\vec{r}(0)}^{\vec{r}(t)} d\vec{r} = \int_0^t \vec{v}\,d\tilde{t} = \int_0^t \langle 2\tilde{t}+1, 3\tilde{t}^2, 4\tilde{t}^3\rangle\,d\tilde{t}$$

$$\Rightarrow \vec{r}(t) = \langle 0, 1, -1\rangle + \langle t^2+t, t^3, t^4\rangle$$

$$\boxed{\vec{r}(t) = \langle t^2+t, t^3+1, t^4-1\rangle}$$

Example 2.3.12. .



Example 2.3.13. .