4.2.1 directional derivatives and the gradient in $\mathbb{R}^2$

Now that we have a little experience in partial differentiation let's return to the problem of the directional derivative. We saw that

$$D_{\langle a,b\rangle}f(x_o, y_o) = \langle\ f_x(x_o, y_o),\ \ f_y(x_o, y_o)\ \rangle \bullet \langle a, b\rangle$$

for the particular example we considered. Is this always true? Is it generally the case that we can build the directional derivative in the $\langle a, b\rangle$-direction from the partial derivatives? If you just try most functions that come to the nonpathological mind then you'd be tempted to agree with this claim. However, many counter-examples exist. We only need one to debunk the claim.

Example 4.2.21. *Suppose that*

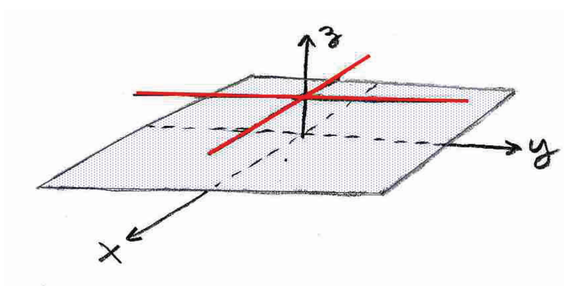
$$f(x, y) = \begin{cases} x + 1 & y = 0 \\ y + 1 & x = 0 \\ 0 & xy \neq 0 \end{cases}$$

Clearly $f_x(0, 0) = 1$ *and* $f_y(0, 0) = 1$ *however the directional derivative is given by*

$$D_{\langle a,b\rangle}f(0, 0) = \lim_{t\to 0} \frac{f(ta, tb) - f(0, 0)}{t} = \lim_{t\to 0} \frac{-1}{t}$$

which diverges. The directional derivative in any non-coordinate direction does not exist since the function jumps from 0 *to* 1 *at the origin along any line except the axes.*

Example 4.2.22. *This example is even easier: let* $f(x, y) = \begin{cases} 1 & y = 0 \\ 1 & x = 0 \\ 0 & xy \neq 0 \end{cases}$. *In this case I can graph the function and it is obvious that* $f_x(0, 0) = 0$ *and* $f_y(0, 0) = 0$ *yet all the directional derivatives in non-coordinate directions fail to exist.*



We can easily see the discontinuity of the function above is the source of the trouble. It is sometimes true that a function is discontinuous and the formula holds. However, the case which we really want to consider, the type of functions for which the derivatives considered are most meaningful, are called **continuously differentiable**. You might recall from single-variable calculus that when a function is differentiable at a point but the derivative function is discontinuous it led to bizzare features for the linearization. That continues to be true in the multivariate case.

Definition 4.2.23.

A function $f : dom(f) \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be **continuously differentiable** at (x_o, y_o) iff the partial derivative functions $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous at (x_o, y_o). We say $f \in C^1(x_o, y_o)$. If all the second-order partial derivatives of f are continuous at (x_o, y_o) then we say $f \in C^2(x_o, y_o)$. If continuous partial derivatives of arbitrary order exist at (x_o, y_o) then we say f is **smooth** and write $f \in C^\infty(x_o, y_o)$.

The continuity of the partial derivative functions implicitly involves multivariate limits and this is what ultimately makes this criteria quite strong.

Proposition 4.2.24.

Suppose f is continuously differentiable at (x_o, y_o) then the directional derivative at (x_o, y_o) in the direction of the unit vector $\langle a, b\rangle$ is given by:

$$D_{\langle a,b\rangle} f(x_o, y_o) = \langle\ f_x(x_o, y_o),\ \ f_y(x_o, y_o)\ \rangle \bullet \langle a, b\rangle$$

Proof: delayed until the next section. $\square$

At this point it is useful to introduce a convenient notation which groups all the partial derivatives together in a particular vector of functions.

Definition 4.2.25.

If the partial derivatives of f exist then we define

$$\nabla f = \langle f_x,\ f_y\rangle = \widehat{x}\frac{\partial f}{\partial x} + \widehat{y}\frac{\partial f}{\partial y}.$$

we also use the notation $grad(f)$ and call this the **gradient** of f.

The upside-down triangle ∇ is also known as *nabla*. Identify that $\nabla = \widehat{x}\partial_x + \widehat{y}\partial_y$ is a vector of operators, it takes a function f and produces a vector field ∇f. This is called the **gradient vector field** of f. We'll think more about that after the examples. For a continuously differentiable function we have the following beautiful formula for the directional derivative:

$$\boxed{D_{\langle a,b\rangle} f(x_o, y_o) = (\nabla f)(x_o, y_o) \bullet \langle a, b\rangle.}$$

This is the formula I advocate for calculation of directional derivatives. This formula most elegantly summarizes how the directional derivative works. I'd make it the definition, but the discontinuous[3] counter-Example 4.2.21 already spoiled our fun.

[3] I don't mean to say there are no continuous counter examples,I'd wager there are examples of continuous functions whose partial derivatives exist but are discontinuous. Then the formula fails because some non-coordinate directions fail to possess a directional derivative.

Example 4.2.26. *Suppose $f(x,y) = x^2 + y^2$. Then*

$$\nabla f = \langle 2x, 2y \rangle.$$

Calculate the directional derivative of f at (x_o, y_o) in the $\langle a,b \rangle$-direction:

$$D_{\langle a,b \rangle} f(x_o, y_o) = \langle 2x_o, 2y_o \rangle \bullet \langle a, b \rangle = 2x_o a + 2y_o b.$$

It is often useful to write $D_{\langle a,b \rangle} f(x_o, y_o) = (\nabla f)(x_o, y_o) \bullet \langle a, b \rangle$ in terms of the angle θ between the $\nabla f(x_o, y_o)$ and $\langle a, b \rangle$:

$$\boxed{D_{\langle a,b \rangle} f(x_o, y_o) = ||(\nabla f)(x_o, y_o)|| \cos\theta.}$$

With this formula the following are obvious:

1. $(\theta = 0)$ when $\langle a, b \rangle$ is parallel to $(\nabla f)(x_o, y_o)$ the direction $\langle a, b \rangle$ points towards **maximum increase** in f

2. $(\theta = \pi)$ when $\langle a, b \rangle$ is antiparallel to $(\nabla f)(x_o, y_o)$ the direction $\langle a, b \rangle$ points towards **maximum decrease** in f

3. $(\theta = \pi/2)$ when $\langle a, b \rangle$ is perpendicular to $(\nabla f)(x_o, y_o)$ the direction $\langle a, b \rangle$ points towards where f remains **constant**.

Example 4.2.27. Problem: *if $f(x,y) = x^2 + y^2$. Then in what direction(s) is(are) f* **(a.)** *increasing the most at $(2,3)$,* **(b.)** *decreasing the most at $(2,3)$,* **(c.)** *not increasing at $(2,3)$?*

Solution of (a.): *f increases most in the $(\nabla f)(2,3)$-direction. In particular, $(\nabla f)(2,3) = \langle 4, 6 \rangle$. If you prefer a unit-vector then rescale $\langle 4, 6 \rangle\rangle$ to $\widehat{u} = \frac{1}{\sqrt{13}}\langle 2, 3 \rangle$. The magnitude $||(\nabla f)(2,3)|| = \sqrt{13}$ is the rate of increase in the $\widehat{u} = \frac{1}{\sqrt{13}}\langle 2, 3 \rangle$-direction.*

Solution of (b.): *f decreases most in the $-(\nabla f)(2,3)$-direction. In particular, $-(\nabla f)(2,3) = \langle -4, -6 \rangle$. If you prefer a unit-vector then rescale $\langle -4, -6 \rangle\rangle$ to $\widehat{u} = \frac{1}{\sqrt{13}}\langle -2, -3 \rangle$. The rate of decrease is also $\sqrt{13}$ in magnitude.*

Solution of (c.): *f is constant in directions which are perpendicular to $(\nabla f)(2,3)$. A unit-vector which is perpendicular to $(\nabla f)(2,3) = \langle 4, 6 \rangle$ satisfied two conditions:*

$$(\nabla f)(2,3) \bullet \langle a, b \rangle = 4a + 6b = 0 \qquad \text{and} \qquad a^2 + b^2 = 1$$

These are easily solved by solving the orthogonality condition for $b = -\frac{2}{3}a$ and substituting it into the unit-length condition:

$$1 = a^2 + b^2 = a^2 + \frac{4}{9}a^2 = \frac{13}{9}a^2 \quad \Rightarrow \quad a^2 = \frac{9}{13} \quad \Rightarrow \quad a = \pm\frac{3}{\sqrt{13}} \quad \Rightarrow b = \mp\frac{2}{\sqrt{13}}.$$

Therefore, we find f is constant in either the $\langle 3/\sqrt{13}, -2/\sqrt{13} \rangle$ or the $\langle -3/\sqrt{13}, 2/\sqrt{13} \rangle$ direction.

Example 4.2.28. **Problem:** *find a point* (x_o, y_o) *at which the function* $f(x,y) = x^2 + y^2$ *is constant in all directions.*

Solution: *We need to find a point* (x_o, y_o) *at which* $(\nabla f)(x_o, y_o)$ *is perpendicular to all unit-vectors. The only vector which is perpendicular to all other vectors is the zero vector. We seek solutions to* $(\nabla f)(x_o, y_o) = \langle 2x_o, 2y_o \rangle = \langle 0, 0 \rangle$. *The only solution is* $x_o = 0$ *and* $y_o = 0$. *Apparently the graph* $z = f(x,y)$ *levels out at the origin since* $f(x,y)$ *stays constant in all directions near* $(0,0)$.

Definition 4.2.29.

We say (x_o, y_o) is a **critical point** of f if $(\nabla f)(x_o, y_o)$ does not exist or $(\nabla f)(x_o, y_o) = \langle 0, 0 \rangle$.

The term critical point is appropriate here since these are points where the function f *may* have a local maximum or minimum. Other possibilities exist and we'll spend a few lectures this semester developing tools to carefully discern what the geometry is near a given critical point.

Example 4.2.30. .

Let $f(x,y) = y^2/x$ and consider $P = (1,2)$ and the unit vector $\hat{u} = \langle 2/3, \sqrt{5}/3 \rangle$ find

a.) $\nabla f = \langle f_x, f_y \rangle$ thus,

$$\boxed{\nabla f = \langle -y^2/x^2, \ 2y/x \rangle}$$

b.) $\nabla f(1,2) = \langle -4/1, \ 4/1 \rangle = \boxed{\langle -4, 4 \rangle = \nabla f(1,2)}$

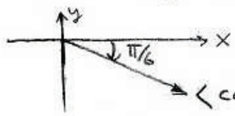
c.)
$$\begin{aligned}(D_{\hat{u}} f)(P) &= (\nabla f)(P) \cdot \hat{u} \\ &= \langle -4, 4 \rangle \cdot \langle 2/3, \sqrt{5}/3 \rangle \\ &= -8/3 + 4\sqrt{5}/3 \\ &= \boxed{\tfrac{1}{3}\left(4\sqrt{5} - 8\right)}.\end{aligned}$$

Example 4.2.31. .

Let $f(x,y) = -\sqrt{5x-4y}$ find Df at $(4,1)$ in $\theta = -\pi/6$ direction

$$(\nabla f)(x,y) = \left\langle \frac{5}{2\sqrt{5x-4y}}, \frac{-2}{\sqrt{5x-4y}} \right\rangle \Rightarrow (\nabla f)(4,1) = \left\langle \frac{5}{2\sqrt{16}}, \frac{-2}{\sqrt{16}} \right\rangle$$

Thus $(\nabla f)(4,1) = \langle 5/8, -1/2 \rangle$. Now lets find the unit vector in the $\theta = -\pi/6$ direction, here θ is the usual polar coordinate.



$$\langle \cos(-\pi/6), \sin(-\pi/6) \rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{-1}{2} \right\rangle = \vec{u}$$

$$\begin{aligned} D_{\vec{u}} f(4,1) &= (\nabla f)(4,1) \bullet \left\langle \frac{\sqrt{3}}{2}, \frac{-1}{2} \right\rangle \\ &= \langle 5/8, -1/2 \rangle \bullet \langle \sqrt{3}/2, -1/2 \rangle \\ &= \boxed{\frac{5\sqrt{3}}{16} + \frac{1}{4}} \end{aligned}$$

Example 4.2.32. .

Let $f(x,y) = 5xy^2 - 4x^3y$.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 5y^2 - 12x^2y, 10xy - 4x^3 \rangle$$

$$(\nabla f)(1,2) = \langle 20 - 12(1)(2), 10(1)(2) - 4(1) \rangle = \langle -4, 16 \rangle$$

The rate of change of f at the point $P = (1,2)$ in the $\vec{u} = \langle \frac{5}{13}, \frac{12}{13} \rangle$ direction is the directional derivative. we should check that $\vec{u} = \hat{u}$ notice $|\vec{u}| = \sqrt{\frac{1}{13^2}(5^2 + 12^2)} = \sqrt{\frac{1}{169}(169)} = 1 \therefore \vec{u}$ is unit vector. Thus,

$$\begin{aligned} (D_{\vec{u}} f)(P) &= (\nabla f)(P) \bullet \vec{u} \\ &= \langle -4, 16 \rangle \bullet \langle 5/13, 12/13 \rangle \\ &= \frac{1}{13}(-20 + 192) = \boxed{172/13} \end{aligned}$$

Example 4.2.33. .

Let $f(x,y) = \ln(x^2+y^2)$ find $(D_{\hat{v}}f)(2,1)$ for $\vec{v} = \langle -1, 2\rangle$.

Notice that $|\vec{v}| = \sqrt{5}$ thus $\hat{v} = \frac{1}{\sqrt{5}}\langle -1, 2\rangle$. You can check $|\hat{v}| = 1$.

$$\nabla f = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$$

$$(\nabla f)(2,1) = \left\langle \frac{4}{5}, \frac{2}{5} \right\rangle$$

$$(\nabla f)(2,1) \cdot \left(\frac{1}{\sqrt{5}}\langle -1, 2\rangle\right) = \frac{1}{\sqrt{5}}\frac{1}{5}\langle 4, 2\rangle \cdot \langle -1, 2\rangle = \frac{1}{5\sqrt{5}}(-4+4) = 0$$

Thus we find $\boxed{(D_{\hat{v}} f)(2,1) = 0}$

Example 4.2.34. .

Find directions in which $f(x,y) = x^2 + \sin(xy)$ has directional derivative at $(1,0)$ with value 1. That is find $\hat{u}$ such that $(D_{\hat{u}} f)(1,0) = 1$. For our convenience let us define a, b unknowns such that $\hat{u} = \langle a, b\rangle$. $(a^2+b^2=1)$

$$\nabla f = \left\langle \frac{\partial}{\partial x}[x^2+\sin(xy)], \frac{\partial}{\partial y}[x^2+\sin(xy)] \right\rangle$$

chain-rule.

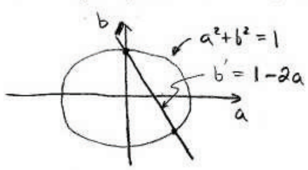
$$= \left\langle 2x + \cos(xy)\frac{\partial}{\partial x}[xy], \cos(xy)\frac{\partial}{\partial y}[xy] \right\rangle$$

$$= \langle 2x + y\cos(xy), x\cos(xy)\rangle$$

Now $(\nabla f)(1,0) = \langle 2, 1\rangle$. We wish to study $(D_{\hat{u}} f)(1,0) = 1$, that is,

$$(\nabla f)(1,0) \cdot \langle a, b\rangle = \langle 2, 1\rangle \cdot \langle a, b\rangle = 2a + b = 1$$

The eq$^{\underline{n}}$ $2a+b$ has ∞ly many sol$^{\underline{n}}$'s But we also demand that $a^2+b^2=1$ since we wish to find the directions in which $(D_{\hat{u}} f)(1,0) = 1$.



- you can see we get two sol$^{\underline{n}}$'s from the graph.
- algebraically we find them as follows,

$$1 = a^2+b^2 = a^2 + (1-2a)^2 \quad : \text{substituting}$$
$$= a^2 + 1 - 4a + 4a^2$$
$$= 5a^2 - 4a + 1$$
$$\therefore\ 5a^2 - 4a = a(5a-4) = 0$$
$$\underline{a = 0 \quad \text{or} \quad a = 4/5}$$

Thus $\hat{u} = \langle a, b\rangle = \langle a, 1-2a\rangle$ should be $\boxed{\langle 0, 1\rangle \text{ or } \langle 4/5, 3/5\rangle}$.

4.2.2 gradient vector fields

We've seen that the value of ∇f at a particular point reveals both the magnitude and the direction of the change in the function f. The gradient vector field is simply the vector field which a differentiable function f generates through the gradient operation.

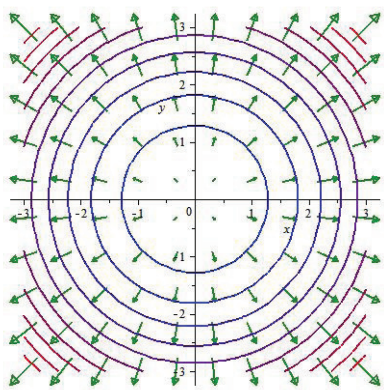
Definition 4.2.35.

If f is differentiable on $U \subseteq \mathbb{R}^2$ then ∇f defines the gradient vector field on U. We assign to each point $\vec{p} \in U$ the vector $\nabla f(\vec{p})$.

Example 4.2.36. *Let $f(x, y) = x^2 + y^2$. We calculate,*

$$\nabla f(x, y) = \langle \partial_x(x^2 + y^2), \partial_y(x^2 + y^2) \rangle = \langle 2x, 2y \rangle$$

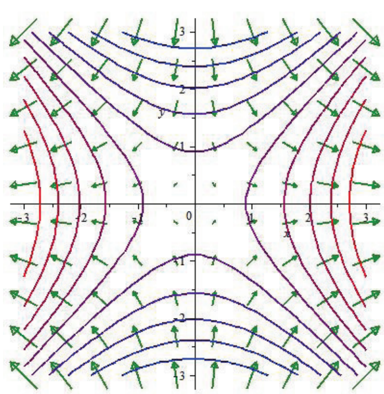
This gradient vector field is easily described; at each point $\vec{p}$ we attach the vector $2\vec{p}$.



Example 4.2.37. *Let $f(x, y) = x^2 - y^2$. We calculate,*

$$\nabla f(x, y) = \langle \partial_x(x^2 - y^2), \partial_y(x^2 - y^2) \rangle = \langle 2x, -2y \rangle$$

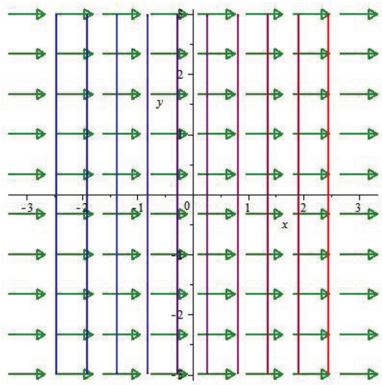
This gradient vector field is not so easily described, however, most CAS will provide nice plots if you are willing to invest a little time.



Example 4.2.38. *Let $f(x, y) = x$. We calculate,*

$$\nabla f(x, y) = \langle \partial_x(x), \partial_y(x) \rangle = \langle 1, 0 \rangle = \widehat{x}$$

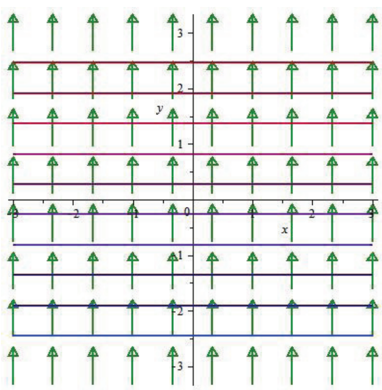
Therefore, $\nabla x = \widehat{x}$. Interesting. The gradient operation reproduces the unit-vector in the direction of increasing x.



Example 4.2.39. *Let $f(x, y) = y$. We calculate,*

$$\nabla f(x, y) = \langle \partial_x(y), \partial_y(y) \rangle = \langle 0, 1 \rangle = \widehat{y}$$

Therefore, $\nabla y = \widehat{y}$. Interesting. The gradient operation reproduces the unit-vector in the direction of increasing y.



Naturally, we are tempted to derive other unit-vector-fields by this method. In the examples above we were a bit lucky, generally when you take the gradient of a coordinate function you'll need to normalize it. But, this is a very nice **algebraic** method to derive the frame of a non-cartesian coordinate system. In particular, if y_1, y_2 are coordinates then there exist differentiable functions f_1, f_2 such that $y_1 = f_1(x, y)$ and $y_2 = f_2(x, y)$ we can calculate the unit-vectors

$$\widehat{y_1} = \frac{\nabla f_1}{||\nabla f_1||} \qquad \text{and} \qquad \widehat{y_2} = \frac{\nabla f_2}{||\nabla f_2||}.$$

Let's see how this method produces the frame for polar coordinates. I initially claimed it could be derived from geometry alone. That is true, but this is also nice:

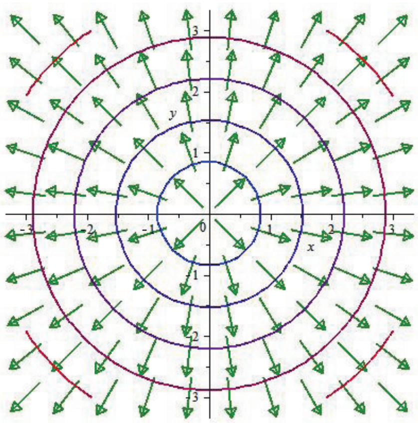
Example 4.2.40. *Consider polar coordinates r, θ, these were defined by $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\big[y/x\big]$ for $x > 0$. Calculate,*

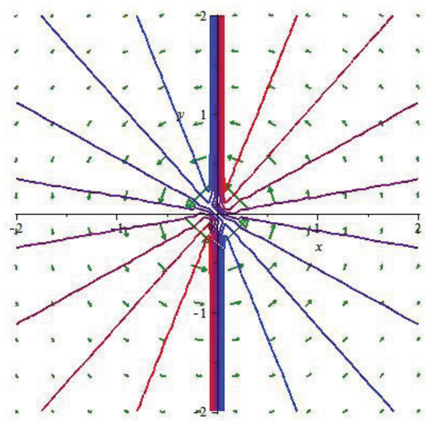
$$\nabla r = \left\langle \frac{\partial}{\partial x}\sqrt{x^2+y^2},\ \frac{\partial}{\partial y}\sqrt{x^2+y^2} \right\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}},\ \frac{y}{\sqrt{x^2+y^2}} \right\rangle = \left\langle \frac{x}{r},\ \frac{y}{r} \right\rangle$$

But, $x = r\cos\theta$ and $y = r\sin\theta$ thus we derive $\nabla r = \langle \cos\theta, \sin\theta\rangle$. Since $||\nabla r|| = 1$ we find $\boxed{\widehat{r} = \langle \cos\theta, \sin\theta\rangle.}$ *The unit-vector in the direction of increasing θ is likewise calculated,*

$$\nabla\theta = \left\langle \frac{\partial}{\partial x}\tan^{-1}\big[y/x\big],\ \frac{\partial}{\partial y}\tan^{-1}\big[y/x\big] \right\rangle = \left\langle \frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2} \right\rangle = \left\langle \frac{-y}{r^2},\ \frac{x}{r^2} \right\rangle.$$

In this case we find $\nabla\theta = \frac{1}{r}\langle -\sin\theta, \cos\theta\rangle$. Gradients and level curves of r and θ are plotted below[4]*:*





The gradient of θ is not a unit-vector so we have to normalize. Since $||\nabla\theta|| = \frac{1}{r}$ we derive $\boxed{\widehat{\theta} = \langle -\sin\theta, \cos\theta\rangle.}$

This is a very nice calculation for coordinates which are not easy to visualize.

Another nice application of the gradient involves level curves. Consider this: a level curve is the set of points which solves $f(x, y) = k$ for some value k. If we consider a point (x_o, y_o) on the level curve $f(x, y) = k$ then the gradient vector $(\nabla f)(x_o, y_o)$ will be perpendicular to the tangent line of the level curve. Remember that when $\theta = \pi/2$ we find a direction in which $f(x, y)$ stays constant near (x_o, y_o). What does this mean? Let's summarize it:

The gradient vector field ∇f is perpendicular to the level curve $f(x, y) = k$.

If you are less than satisfied with my geometric justification for this claim then you'll be happy to hear we can prove it with a simple calculation. However, we need a chain-rule which we have yet to justify. Therefore, further justification is postponed until a later section. That said, let's look at a few examples to appreciate the power of this statement:

[4] notice how the software chokes on $x = 0$

Example 4.2.41. *Suppose* $V(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ *represents the voltage due to a point-charge at the origin. Electrostatics states that the electric field* $\vec{E} = -\nabla V$*. Geometrically this has a simple meaning; the electric field points along the normal direction to the level-curves of the voltage function. In other words, the electric field is normal to the equipotential lines. What is an "equipotential line", it's a line on which the voltage assumes a constant value. This is nothing more than a level-curve of the voltage function. For the given potential function, using* $r = \sqrt{x^2+y^2}$*,*

$$\nabla V = \langle \partial_x(1/r), \partial_y(1/r)\rangle = \langle (-1/r^2)\partial_x r,\ (-1/r^2)\partial_y r\rangle = \frac{-1}{r^2}\langle \partial_x r,\ \partial_y r\rangle = -\frac{1}{r^2}\widehat{r}.$$

Equipotentials $V = V_o = 1/r$ *are simply circles* $r = 1/V_o$ *and the electric field is a purely radial field* $\vec{E} = \frac{1}{r^2}\widehat{r}$*.*

Example 4.2.42. *Consider the ellipse* $f(x,y) = x^2/a^2 + y^2/b^2 = k$*. At any point on the ellipse the vector field*

$$\nabla f = \frac{2x}{a^2}\,\widehat{x} + \frac{2y}{b^2}\,\widehat{y}$$

points in the normal direction to the ellipse.

Example 4.2.43. *Consider the hyperbolas* $g(x,y) = x^2y^2 = k$*. At any point on the hyperbolas the vector field*

$$\nabla g = 2xy^2\,\widehat{x} + 2x^2y\,\widehat{y}$$

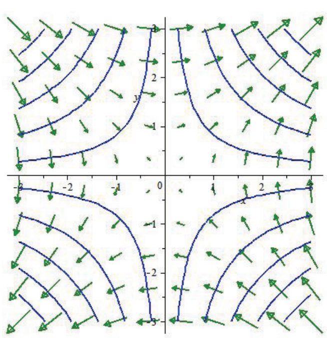
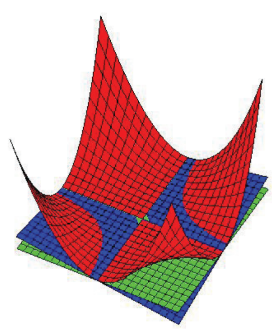
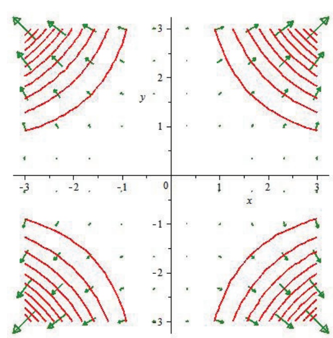
points in the normal direction to the hyperbola. Notice that for $k > 0$ *we have* $y^2 = k/x^2$ *hence* $y = \pm\sqrt{k}/x$*. When* $k = 0$ *we find solutions* $x = 0$ *and* $y = 0$*. The gradient vector field is identically zero on the coordinate axes in this case. I plot it after the next example for the sake of side-by-side comparison*

Example 4.2.44. *Suppose we have a level curve* $f(x,y) = xy = k$*. This either gives a hyperbola* $(k \neq 0)$ *or the coordinate axes* $(k = 0)$*. The gradient vector field is a bit more descriptive in this case:*

$$\nabla f = y\,\widehat{x} + x\,\widehat{y}.$$

In this case the exceptional solution $x = 0$ *has* $\nabla f\big|_{x=0} = y\,\widehat{x}$ *and* $y = 0$ *has* $\nabla f\big|_{y=0} = x\,\widehat{y}$*. The origin* $(0,0)$ *is the only critical point for* f *in this example.*

I plot ∇f on the left and ∇g on the right together with a few level curves. The picture in the middle has $z = x^2y^2$ in red and $z = xy$ in blue with $z = 0$ in green for reference.

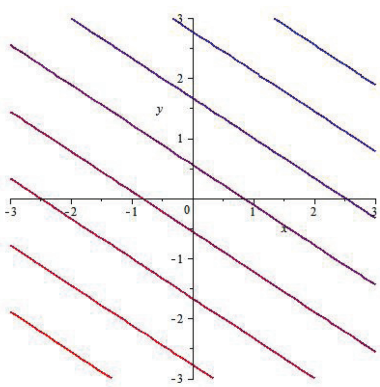




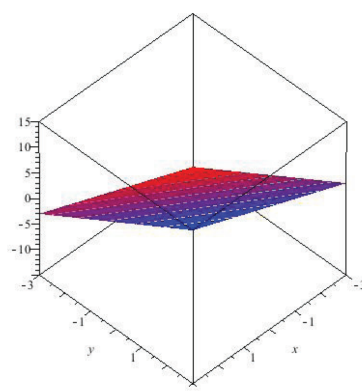
The last pair of examples goes to show that a given set of points can be described by many different level-functions. In particular notice that $xy = 1$ is covered by $x^2y^2 = 1$ but the level functions $f(x, y) = xy$ and $g(x, y) = x^2y^2$ change to other levels in rather distinct fashions. Just compare the gradient vector fields. Or, use a CAS[5] to graph $z = f(x, y)$ and $z = g(x, y)$. Those graphs will intersect along the curve $(x, 1/x, 1)$ for $x > 0$. Do they intersect anywhere else?

4.2.3 contour plots

Perhaps you've studied a *topographical map* before. The topographical map uses a two-dimensional chart to plot a three-dimensional landscape. We can make a similar diagram for graphs of the form $z = f(x, y)$. To form such a plot we simply imagine projecting the graph at a few representative z-values down or up to the xy-plane. This is an invaluable tool since we have much better two-dimensional visualization than we do three. Few people can draw excellent three dimesnional perspective, but the contour plot requires no understanding of perspective. We just slice and project. Moreover, we can use the gradient vector field as a sort of *compass*[6]. The gradient vector field in the domain of $f(x, y)$ points toward higher contours. I use the term *higher* with the idea of traveling from $f(x, y) = k_1$ to $f(x, y) = k_2$ where $k_1 < k_2$. If $f(x, y)$ was actually the altitude function then the term upward would be literally accurate. Usually the term has nothing to do with an actual height, that's just a mental picture for us to help think through the math.

Example 4.2.45. *Suppose $f(x, y) = 2x + 3y$. The graph $z = f(x, y)$ is the plane $z = 2x + 3y$. Contours are level curves of the form $2x + 3y = k$. These contours are simply lines with x-intercept $k/2$ and y-intercept $k/3$. See the plot and graph below to appreciate how the contour plot and graph complement one another. Also, note there is no critical point in this example and the gradient vector field $\nabla f = \langle 2, 3\rangle$ is constant in the domain of f.*



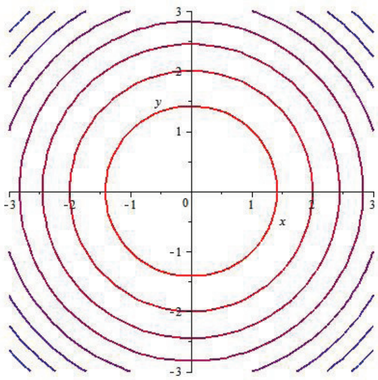


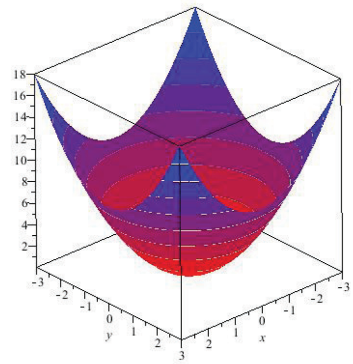
Example 4.2.46. *Suppose $f(x, y) = x^2 + y^2$. The graph $z = f(x, y)$ is the quadratic surface known as a paraboloid. Contours are level curves of the form $x^2 + y^2 = k$. These solutions of $x^2 + y^2 = k$ form circles of radius $\sqrt{k}$ for $k > 0$ and a solitary point $(0, 0)$ for $k = 0$. There are no contours*

[5]I used Maple to create these graphs, of course you could use Mathematica or any other plotting tool, I have links to free ones on my website... I do expect you use something to aid your visualization.

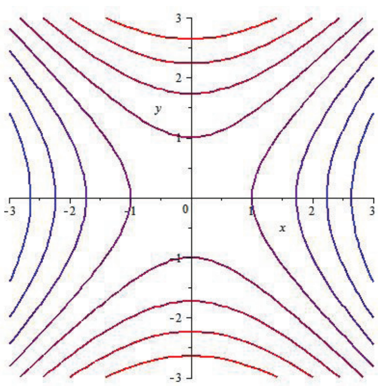
[6]thanks to Dr. Monty Kester for this particular slogan

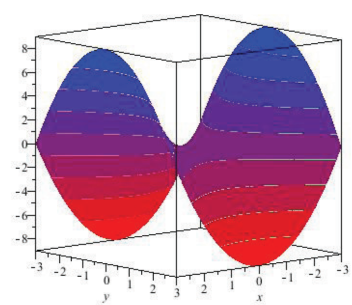
with $k < 0$. Once more see how the graph a contour plot complement one another. Furthermore, observe that $\nabla f = \langle 2x, 2y \rangle$ is zero at the origin which is the only critical point. It's clear from the contours or the graph that $f(0,0)$ is a local minimum for f. In fact, it's clear it is the global minimum for the function.





Example 4.2.47. *Suppose $f(x,y) = x^2 - y^2$. The graph $z = f(x,y)$ is the quadratic surface known as a hyperboloid. Contours are level curves of the form $x^2 - y^2 = k$. These solutions of $x^2 - y^2 = k$ form hyperbolas which open up/down for $k < 0$ and open left/right for $k > 0$. If $k = 0$ the $x^2 - y^2 = 0$ yields the special case $y = \pm x$, these are asymptotes for all the hyperbolas from $k \neq 0$. Once more see how the graph and contour plot complement one another. Furthermore, observe that $\nabla f = \langle 2x, -2y \rangle$ is zero at the origin which is the only critical point. It's clear from the contours or the graph that $f(0,0)$ is a not a local minimum or maximum for f. This sort of critical point is called a* **saddle point**.





Example 4.2.48. *Suppose $f(x, y) = \cos(x)$. The graph $z = f(x, y)$ is sort-of a wavy plane. Contours are solutions of the level curve equation $\cos(x) = k$. In this case y is free, however we only find non-empty solution sets for $-1 \leq k \leq 1$. For a particular $k \in [-1, 1]$ we have the level-curve $\{(x, y) \mid \cos(x) = k\}$. Note that the cosine curve will reach k twice for each 2π interval in x. Let me pick on a few special values,*

$$k = 0, \quad \textit{solve} \quad \cos(x) = 0, \quad \textit{to obtain} \;\; x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

The $k = 0$ contours are of the form $x = \frac{\pi}{2}(2n-1)$ for $n \in \mathbb{Z}$, there are infinitely many such contours and they are disconnected from one another. Another case which is easy to think through without a calculator,

$$k = 1/2, \quad \textit{solve} \quad \cos(x) = 1/2, \quad \textit{to obtain} \;\; x = -\frac{\pi}{3} + 2\pi n, \quad \textit{or} \;\; x = \frac{\pi}{3} + 2\pi n$$

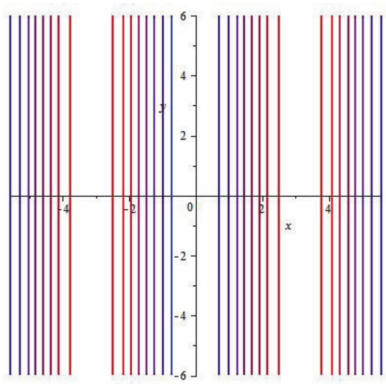
for $n \in \mathbb{Z}$. Once more the level-curves are vertical lines. Continuing, study $k = 1$,

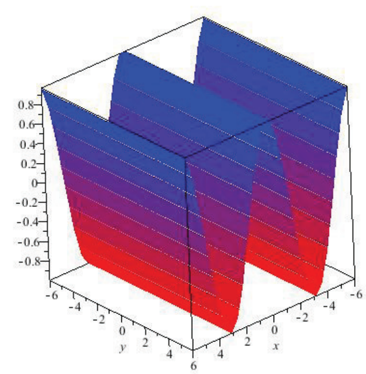
$$k = 1, \quad \textit{solve} \quad \cos(x) = 1, \quad \textit{to obtain} \;\; x = 2\pi n, \textit{for} \;\; n \in \mathbb{Z}.$$

Likewise:

$$k = -1, \quad \textit{solve} \quad \cos(x) = -1, \quad \textit{to obtain} \;\; x = (2n-1)\pi, \textit{for} \;\; n \in \mathbb{Z}.$$

Observe the gradient $\nabla f = \langle -\sin(x), 0\rangle$ is zero along the $k = \pm 1$ contours. The points on $k = 1$ give a local maximum whereas the points on $k = -1$ give local minima for f. This is a special sort of critical point since they are not isolated, no matter how close we zoom in there are always infinitely many critical points in a neighborhood of a given critical point.





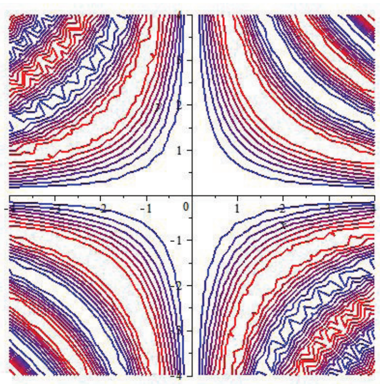
Example 4.2.49. *Suppose $f(x,y) = \cos(xy)$. Calculate $\nabla f = \langle -y\sin(xy), -x\sin(xy)\rangle$ it follows that solutions of $xy = n\pi$ for $n \in \mathbb{Z}$ give critical points of f. Contours are given by the level-curves $\cos(xy) = k$ which have nonempty solutions for $k \in [-1,1]$. For example, note that $\cos(xy) = 1$ has solution $xy = 2j\pi$ for some $j \in \mathbb{Z}$. In particular,*

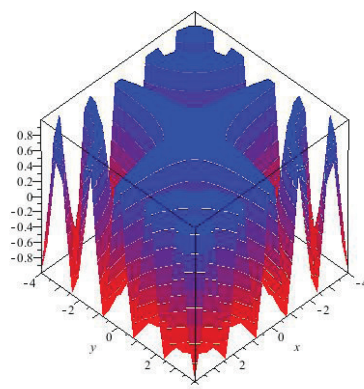
$$xy = 0,\ xy = \pm 2\pi,\ xy = \pm 4\pi,\ \dots \quad \Rightarrow \quad y = 0,\quad x = 0,\ y = \pm\frac{2\pi}{x},\quad y = \pm\frac{4\pi}{x},\ \dots$$

On the other hand, $\cos(xy) = -1$ has solution $xy = (2m-1)\pi$ for some $m \in \mathbb{Z}$. In particular,

$$xy = \pm\pi,\quad xy = \pm 3\pi,,\quad xy = \pm 5\pi \quad \dots \quad \Rightarrow \quad y = \pm\frac{\pi}{x},\quad y = \pm\frac{3\pi}{x},\quad y = \pm\frac{5\pi}{x},\ \dots$$

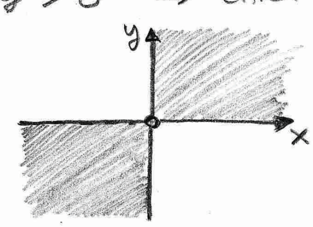
The contours are simply a family of hyperbolas which take the coordinate axes as asymptotes. This is a great example to see both why contour plots help us visualize the graph which we'd rather not illustrate three-dimensionally. Of course we can use a CAS to directly picture $z = f(x,y)$, but such pictures rarely yield the same sort of detailed information a well-drawn contour plot.





Example 4.2.50. *Nice CAS (in this section I used Maple, but all mature CAS's have built-in contour tools) plots are a luxury we don't always have. Notice we can do much just with hand-drawn sketches. I trade color-coding for explicit level labels.*

$\boxed{E18}$ $f(x,y) = \sqrt{xy}/(x^2+y^2)$ find dom(f). So we have to throw out the origin to avoid $\%$ by zero. Then we need $xy > 0 \Rightarrow$ either $x > 0$ and $y > 0$ OR $x < 0$ and $y < 0$.



the dom(f) consists of two disconnected parts.