1.4 curves

A curve is a one-dimensional subset of some space. There are at least three common, but distinct, ways to frame the mathematics of a curve. These viewpoints were already explored in the previous section but I list them once more: we can describe a curve:

1. as a path, that is as a parametrized curve.
2. as a level curve, also known as a solution set.
3. as a graph.

I expect you master all three viewpoints in the two-dimensional context. However, for three or more dimensions we primarily use the parametric viewpoint in this course. Exceptions to this rule are fairly rare: the occasional homework problem where you are asked to find the curve of intersection for two surfaces, or the symmetric equations for a line. In contrast, the parametric decription of a curve in three dimensions is far more natural. Do you want to describe a curve as where two surfaces intersect or would you rather describe a curve as a set of points formed by pasting a copy of the real line through your space? I much prefer the parametric view.

Definition 1.4.1. *vector-valued functions, curves and paths.*

A vector valued function of a real variable is an assignment of a vector for each real number in some domain. It's a mapping $t \mapsto \vec{f}(t) = \langle f_1(t), f_2(t), \dots, f_n(t) \rangle$ for each $t \in J \subset \mathbb{R}$. We say $f_j : J \subseteq \mathbb{R} \to \mathbb{R}$ is the j-th component function of $\vec{f}$. Let $C = \vec{f}(J)$ then C is said to be a **curve which is parametrized by** $\vec{f}$. We can also say that $t \mapsto \vec{f}(t)$ is a **path** in $\mathbb{R}^n$. Equivalently, but not so usefully, we can write the scalar parametric equations for C above as

$$x_1 = f_1(t), \quad x_2 = f_2(t), \ \dots, \ x_n = f_n(t)$$

for all $t \in J$.

When we define a parametrization of a curve it is important to give the formula for the path **and** the domain of the parameter. Note that when I say the word *curve* I mean for us to think about some set of points, whereas when I say the word *path* I mean to refer to the particular mapping whose image is a curve. We may cover a particular curve with infinitely many different paths.

1.4.1 curves in two-dimensional space

We have several viewpoints to consider. Graphs, parametrized curves and level sets.

graphs in the plane

Let's begin by reminding ourselves of the definition of a graph:

Definition 1.4.2. *Graph of a function.*

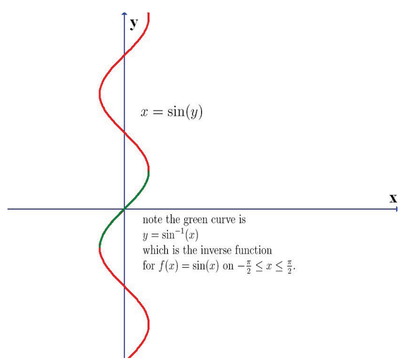
Let $f : dom(f) \to \mathbb{R}$ be a function then

$$graph(f) = \{(x, f(x)) \mid x \in dom(f)\}.$$

We know this is quite restrictive. We must satisfy the vertical line test if we say our curve is the graph of a function.

Example 1.4.3. *To form a circle centered at the origin of radius R we need to glue together two graphs. In particular we solve the equation $x^2 + y^2 = R^2$ for $y = \sqrt{R^2 - x^2}$ or $y = -\sqrt{R^2 - x^2}$. Let $f(x) = \sqrt{R^2 - x^2}$ and $g(x) = -\sqrt{R^2 - x^2}$ then we find $graph(f) \cup graph(g)$ gives us the whole circle.*

Example 1.4.4. *On the other hand, if we wish to describe the set of all points such that $\sin(y) = x$ we also face a similar difficulty if we insist on functions having independent variable x. Naturally, if we allow for functions with y as the independent variable then $f(y) = \sin(y)$ has graph $graph(f) = \{(f(y), y) \mid y \in dom(f)\}$. You might wonder, is this correct? I would say a better question is, "is this allowed?". Different books are more or less imaginative about what is permissible as a function. This much we can say, if a shape fails both the vertical and horizontal line tests then it is not the graph of a single function of x or y.*



Example 1.4.5. *Let $f(x) = mx + b$ for some constants m, b then $y = f(x)$ is the line with slope m and y-intercept b.*

level curves in two-dimensions

Level curves are amazing. The full calculus of level curves is only partially appreciated even in calculus III, but trust me, this viewpoint has many advantages as you learn more. For now it's simple enough:

Definition 1.4.6. *Level Curve.*

A level curve is given by a function of two variables $F : dom(F) \subseteq \mathbb{R}^2 \to \mathbb{R}$ and a constant k. In particular, the set of all $(x, y) \in \mathbb{R}^2$ such that $F(x, y) = k$ is called the level-set of F, but more commonly we just say $F(x, y) = k$ is a level curve.

In an algebra class you might have called this the "graph of an equation", but that terminology is dead to us now. For us, it is a level curve. Moreover, for a particular set of points $C \subseteq \mathbb{R}^2$ we can find more than one function F which produces C as a level set. Unlike functions, for a particular curve there is not just one function which returns that curve. This means that it might be important to give both the level-function F and the level k to specify a level curve $F(x, y) = k$.

Example 1.4.7. *A circle of radius R centered at the origin is a level curve $F(x, y) = R^2$ where $F(x, y) = x^2 + y^2$. We call F the level function (of two variables).*

Example 1.4.8. *To describe $\sin(y) = x$ as a level curve we simply write $\sin(y) - x = 0$ and identify the level function is $F(x, y) = \sin(y) - x$ and in this case $k = 0$. Notice, we could just as well say it is the level curve $G(x, y) = 1$ where $G(x, y) = x - \sin(y) + 1$.*

Note once more this type of ambiguity is one distinction of the level curve langauge, in constrast, the graph $graph(f)$ of a function $y = f(x)$ and the function f are interchangeable. Some mathematicians insist the rule $x \mapsto f(x)$ defines a function whereas others insist that a function is a set of pairs $(x, f(x))$. I prefer the mapping rule because it's how I think about functions in general whereas the idea of a graph is much less useful in general.

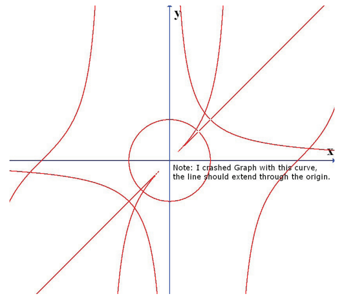
Example 1.4.9. *A line with slope m and y-intercept b can be described by $F(x, y) = mx+b-y = 0$. Alternatively, a line with x-intercept x_o and y-intercept y_o can be described as the level curve $G(x, y) = \frac{x}{x_o} + \frac{y}{y_o} = 1$.*

Example 1.4.10. *Level curves need not be simple things. They can be lots of simple things glued together in one grand equation:*

$$F(x, y) = (x - y)(x^2 + y^2 - 1)(xy - 1)(y - \tan(x)) = 0.$$

Solutions to the equation above include the line $y = x$, the unit circle $x^2+y^2 = 1$, the tilted-hyperbola known more commonly as the reciprocal function $y = \frac{1}{x}$ and finally the graph of the tangent. Some of these intersect, others are disconnected from each other.

It is sometimes helpful to use software to plot equations. However, we must be careful since they are not as reliable as you might suppose. The example above is not too complicated but look what happens with Graph:



Theorem 1.4.11. *any graph of a function can be wriiten as a level curve.*

If $y = f(x)$ is the graph of a function then we can write $F(x, y) = f(x) - y = 0$ hence the graph $y = f(x)$ is also a level curve.

Not much of a theorem. But, it's true. The converse is not true without a lot of qualification. I'll state that theorem (it's called the implicit function theorem) in a future chapter after we've studied partial differentiation.

parametrized curves in two-dimensions

Example 1.4.12. *Suppose $a, b \rangle 0$ and $h, k \in \mathbb{R}$. The parametrization*

$$\vec{r}(t) = \langle h + a\cos(t), k + b\sin(t)\rangle$$

for $t \in [0, 2\pi]$ coveres and the ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Example 1.4.13. *Suppose $a, b \rangle 0$ and $h, k \in \mathbb{R}$. The parametrization*

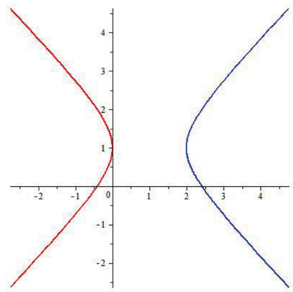
$$\vec{r}_1(t) = \langle h + a\cosh(t), k + b\sinh(t)\rangle$$

for $t \in \mathbb{R}$ coveres one branch of the hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

Note $x = h + a\cosh(t)$ implies $\frac{x-h}{a} = \cosh(t) \geq 1$ therefore it follows $x \geq h + a$. We've covered the right branch. If we wish to cover the left branch of this hyperbola then use:

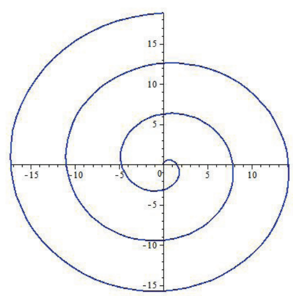
$$\vec{r}_2(t) = \langle h - a\cosh(t), k + b\sinh(t)\rangle.$$



Example 1.4.14. *A spiral can be thought of as a sort of circle with a variable radius. With that in mind I write: for* $t \geq 0$,

$$\vec{r}(t) = \langle t\cos(t), t\sin(t)\rangle$$

to give a spiral whose "radius" is proportional to the angle t *subtended from* $t = 0$.

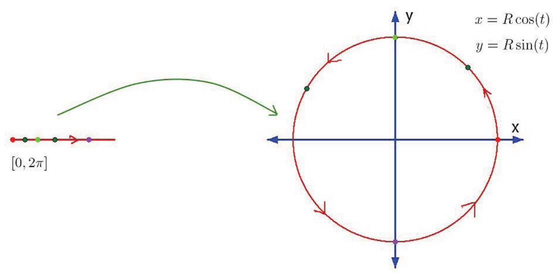


Finding the parametric equations for a curve does require a certain amount of creativity. However, it's almost always some slight twist on the examples I give in this section. The remaining examples I also give in calculus II, I add some detail to emphasize how the parametrization matches the already known identities of certain curves and I add pictures which emphasize the idea that the parametrization pastes a line into $\mathbb{R}^2$.

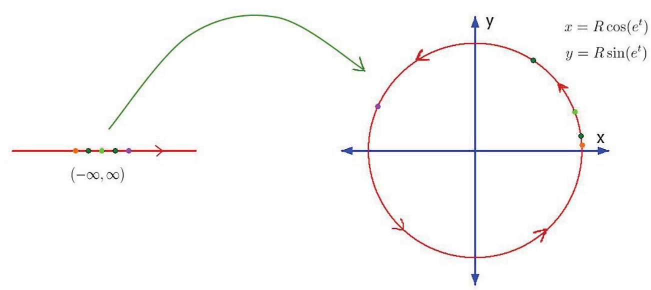
Example 1.4.15. *Let* $x = R\cos(t)$ *and* $y = R\sin(t)$ *for* $t \in [0, 2\pi]$. *This is a parametrization of the circle of radius* R *centered at the origin. We can check this by substituting the equations back into our standard Cartesian equation for the circle:*

$$x^2 + y^2 = (R\cos(t))^2 + (R\sin(t))^2 = R^2(\cos^2(t) + \sin^2(t))$$

Recall that $\cos^2(t) + \sin^2(t) = 1$ *therefore,* $x(t)^2 + y(t)^2 = R^2$ *for each* $t \in [0, 2\pi]$. *This shows that the parametric equations do return the set of points which we call a circle of radius* R. *Moreover, we can identify the parameter in this case as the standard angle from standard polar coordinates.*



Example 1.4.16. *Let $x = R\cos(e^t)$ and $y = R\sin(e^t)$ for $t \in \mathbb{R}$. We again cover the circle at t varies since it is still true that $(R\cos(e^t))^2 + (R\sin(e^t))^2 = R^2(\cos^2(e^t) + \sin(e^t)) = R^2$. However, since range$(e^t) = [1, \infty)$ it is clear that we will actually wrap around the circle infinitly many times. The parametrizations from this example and the last do cover the same set, but they are radically different parametrizations: the last example winds around the circle just once whereas this example winds around the circle ∞-ly many times.*

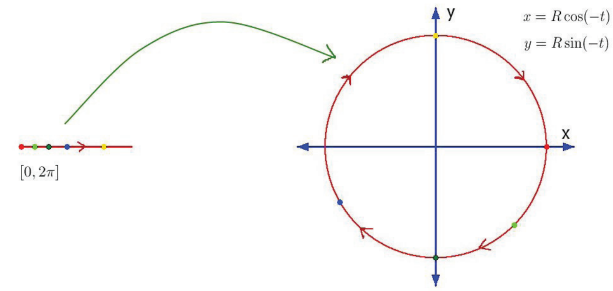


Example 1.4.17. *Let $x = R\cos(-t)$ and $y = R\sin(-t)$ for $t \in [0, 2\pi]$. This is a parametrization of the circle of radius R centered at the origin. We can check this by substituting the equations back into our standard Cartesian equation for the circle:*

$$x^2 + y^2 = (R\cos(-t))^2 + (R\sin(-t))^2 = R^2(\cos^2(-t) + \sin^2(-t))$$

Recall that $\cos^2(-t)+\sin^2(-t) = 1$ therefore, $x(t)^2+y(t)^2 = R^2$ for each $t \in [0, 2\pi]$. This shows that the parametric equations do return the set of points which we call a circle of radius R. Moreover, we can identify the parameter an angle measured CW[12] from the positive x-axis. In contrast, the standard polar coordinate angle is measured CCW from the postive x-axis. Note that in this example we cover the circle just once, but the direction of the curve is opposite that of Example 1.4.15.

[12]CW is an abbreviation for ClockWise,whereas CCW is an abbreviation for CounterClockWise.



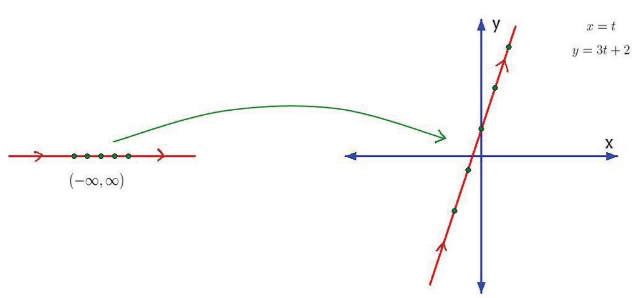
The idea of directionality is not at all evident from Cartesian equations for a curve. Given a graph $y = f(x)$ or a level-curve $F(x, y) = k$ there is no intrinsic concept of direction ascribed to the curve. For example, if I ask you whether $x^2 + y^2 = R^2$ goes CW or CCW then you ought not have an answer. I suppose you could ad-hoc pick a direction, but it wouldn't be natural. This means that if we care about giving a direction to a curve we need the concept of the parametrized curve. We can use the ordering of the real line to induce an ordering on the curve.

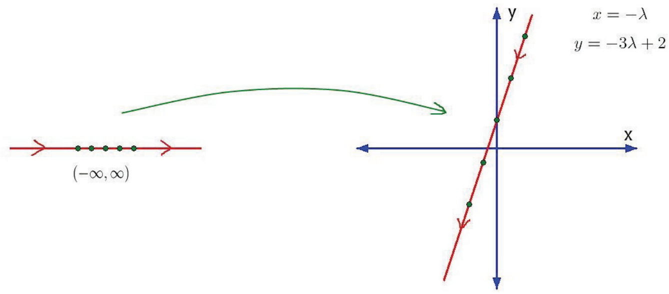
Definition 1.4.18. *oriented curve.*

Suppose $f, g : J \subseteq \mathbb{R} \to \mathbb{R}$ are $1-1$ functions. We say the set $\{(f(t), g(t)) \mid t \in J\}$ is an **oriented curve** and say $t \to (f(t), g(t))$ is a consistently oriented **path** which covers C. If $J = [a, b]$ and $(f(a), g(a)) = p$ and $(f(b), g(b)) = q$ then we can say that C is a curve from to p to q.

I often illustrate the orientation of a curve by drawing little arrows along the curve to indicate the direction. Furthermore, in my previous definition of parametrization I did not insist the parametric functions were $1 - 1$, this means that those parametrizations could reverse direction and go back and forth along a given curve. What is meant by the terms "path", "curve" and "parametric equations" may differ from text to text so you have to keep a bit of an open mind and try to let context be your guide when ambguity occurs. I will try to be uniform in my langauge within this course.

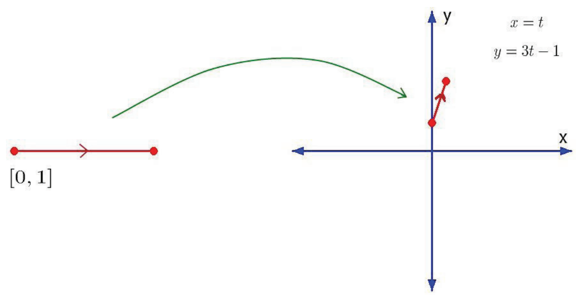
Example 1.4.19. *The line $y = 3x + 2$ can be parametrized by $x = t$ and $y = 3t + 2$ for $t \in \mathbb{R}$. This induces an orientation which goes from left to right for the line. On the other hand, if we use $x = -\lambda$ and $y = -3\lambda + 2$ then as λ increases we travel from right to left on the curve. So the λ-equations give the line the opposite orientation.*

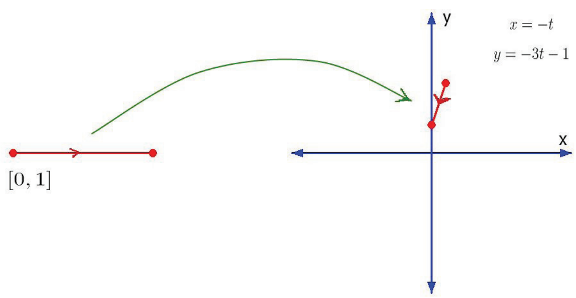




To reverse orientation for $x = f(t), y = g(t)$ for $t \in J = [a, b]$ one may simply replace t by $-t$ in the parametric equations, this gives new equations which will cover the same curve via $x = f(-t), y = g(-t)$ for $t \in [-a, -b]$.

Example 1.4.20. *The line-segment from $(0, -1)$ to $(1, 2)$ can be parametrized by $x = t$ and $y = 3t - 1$ for $0 \leq t \leq 1$. On the other hand, the line-segment from $(1, 2)$ to $(0, -1)$ is parametrized by $x = -t, y = -3t - 1$ for $-1 \leq t \leq 0$.*





The other method to graph parametric curves is simply to start plugging in values for the parameter and assemble a table of values to plot. I have illustrated that in part by plotting the green dots in the domain of the parameter together with their images on the curve. Those dots are the results of plugging in the parameter to find corresponding values for x, y. I don't find that is a very reliable approach in the same way I find plugging in values to $f(x)$ provides a very good plot of $y = f(x)$. That sort of brute-force approach is more appropriate for a CAS system. There are many excellent tools for plotting parametric curves, hopefully I will have some posted on the course website. In addition, the possibility of animation gives us an even more exciting method for visualization of the time-evolution of a parametric curve. In the next chapter we connect the parametric viewpoint with physics and such an animation actually represents the physical motion of some object. My focus in the remainder of this chapter is almost purely algebraic, I could draw pictures to explain, but I wanted the notes to show you that the pictures are not necessary when you understand the algebraic process. That said, the best approach is to do some combination of algebraic manipulation/figuring and graphical reasoning.

converting to and from the parametric viewpoint in 2D

Let's change gears a bit, we've seen that parametric equations for curves give us a new method to describe particular geometric concepts such as orientability or multiple covering. Without the introduction of the parametric concept these geometric ideas are not so easy to describe. That said, I now turn to the question of how to connect parametric descriptions with Cartesian descriptions of a curve. We'd like to understand how to go both ways if possible:

1. how can we find the Cartesian form for a given parametric curve?

2. how can we find a parametrization of a given Cartesian curve?

In case (2.) we mean to include the ideas of level curves and graphs. It turns out that both questions can be quite challenging for certain examples. However, in other cases, not so much: for example any graph $y = f(x)$ is easily recast as the set of parametric equations $x = t$ and $y = f(t)$ for $t \in dom(f)$. For the standard graph of a function we use x as the parameter.

1.4.2 how can we find the Cartesian form for a given parametric curve?

Example 1.4.21. *What curve has parametric equations $x = t$ for $y = t^2$ for $t \in \mathbb{R}$? To find Cartesian equation we eliminate the parameter (when possible)*

$$t^2 = x^2 = y \quad \Rightarrow \quad y = x^2$$

Thus the Cartesian form of the given parametrized curve is simply $y = x^2$.

Example 1.4.22. *Example 15.2.2: Find parametric equations to describe the graph $y = \sqrt{x+3}$ for $0 \leq x < \infty$. We can use $x = t^2$ and $y = \sqrt{t^2+3}$ for $t \in \mathbb{R}$. Or, we could use $x = \lambda$ and $y = \sqrt{\lambda+3}$ for $\lambda \in [0,\infty)$.*

Example 1.4.23. *What curve has parametric equations $x = t$ for $y = t^2$ for $t \in [0,1]$? To find Cartesian equation we eliminate the parameter (when possible)*

$$t^2 = x^2 = y \quad \Rightarrow \quad y = x^2$$

Thus the Cartesian form of the given parametrized curve is simply $y = x^2$, however, given that $0 \leq t \leq 1$ and $x = t$ it follows we do not have the whole parabola, instead just $y = x^2$ for $0 \leq x \leq 1$.

Example 1.4.24. *Identify what curve has parametric equations $x = \tan^{-1}(t)$ and $y = \tan^{-1}(t)$ for $t \in \mathbb{R}$. Recall that range$(\tan^{-1}(t)) = (-\pi/2, \pi/2)$. It follows that $-\pi/2 < x < \pi/2$. Naturally we just equate inverse tangent to obtain $\tan^{-1}(t) = y = x$. The curve is the open line-segment with equation $y = x$ for $-\pi/2 < x < \pi/2$. This is an interesting parameterization, notice that as $t \to \infty$ we approach the point $(\pi/2, \pi/2)$, but we never quite get there.*

Example 1.4.25. *Consider $x = \ln(t)$ and $y = e^t - 1$ for $t \geq 1$. We can solve both for t to obtain*

$$t = e^x = \ln(y+1) \quad \Rightarrow \quad y = -1 + exp(exp(x)).$$

The domain for the expression above is revealed by analyzing $x = \ln(t)$ for $t \geq 1$, the image of $[1, \infty)$ under natural log is precisely $[0, \infty)$; $\ln[1, \infty) = [0, \infty)$.

Example 1.4.26. *Suppose $x = \cosh(t) - 1$ and $y = 2\sinh(t) + 3$ for $t \in \mathbb{R}$. To eliminate t it helps to take an indirect approach. We recall the most important identity for the hyperbolic sine and cosine: $\cosh^2(t) - \sinh^2(t) = 1$. Solve for hyperbolic cosine; $\cosh(t) = x + 1$. Solve for hyperbolic sine; $\sinh(t) = \frac{y-3}{2}$. Now put these together via the identity:*

$$\cosh^2(t) - \sinh^2(t) = 1 \quad \Rightarrow \quad (x+1)^2 - \frac{(y-3)^2}{4} = 1.$$

Note that $\cosh(t) \geq 1$ hence $x + 1 \geq 1$ thus $x \geq 0$ for the curve described above. On the other hand y is free to range over all of $\mathbb{R}$ since hyperbolic sine has range $\mathbb{R}$. You should[13] *recognize the equation as a hyperbola centered at $(-1, 3)$.*

how can we find a parametrization of a given Cartesian curve?

I like this topic more, the preceding bunch of examples, while needed, are boring. The art of parameterizing level curves is much more fun.

Example 1.4.27. Find parametric equations for the circle centered at (h, k) with radius R.

To begin recall the equation for such a circle is $(x-h)^2 + (y-k)^2 = R^2$. Our inspiration is the identity $\cos^(t) + \sin^2(t) = 1$. Let $x - h = R\cos(t)$ and $y - k = R\sin(t)$ thus

$$\boxed{x = h + R\cos(t)} \qquad \textit{and} \qquad \boxed{y = k + R\sin(t).}$$

I invite the reader to verify these do indeed parametrize the circle by explicitly plugging in the equations into the circle equation. Notice, if we want the whole circle then we simply choose any interval for t of length 2π or longer. On the other hand, if you want to select just a part of the circle you need to think about where sine and cosine are positive and negative. For example, if I want to parametrize just the part of the circle for which $x > h$ then I would choose $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

[13] many students need to review these at this point, we use circles, ellipses and hyperbolas as examples in this course. I'll give examples of each in this chapter.

The reason I choose that intuitively is that the parametrization given for the circle above is basically built from polar coordinates[14] centered at (h, k). That said, to be sure about my choice of parameter domain I like to actually plug in some of my proposed domain and make sure it matches the desired criteria. I think about the graphs of sine and cosine to double check my logic. I know that $\cos(-\frac{\pi}{2}, \frac{\pi}{2}) = (0, 1]$ whereas $\sin(-\frac{\pi}{2}, \frac{\pi}{2}) = (-1, 1)$, I see it in my mind. Then I think about the parametric equations in view of those facts,

$$x = h + R\cos(t) \qquad \text{and} \qquad y = k + R\sin(t).$$

I see that x will range over $(h, h + R]$ and y will range over $(k - R, k + R)$. This is exactly what I should expect geometrically for half of the circle. Visualize that $x = h$ is a vertical line which cuts our circle in half. These are the thoughts I think to make certain my creative leaps are correct. I would encourage you to think about these matters. Don't try to just memorize everything, it will not work for you, there are simply too many cases. It's actually way easier to just understand these as a consequence of trigonometry, algebra and analytic geometry.

[14] we will discuss further in a later section, but this should have been covered in at least your precalculus course.

Example 1.4.28. Find parametric equations for the level curve $x^2 + 2x + \frac{1}{4}y^2 = 0$ which give the ellipse a CW orientation.

To begin we complete the square to understand the equation:

$$x^2 + 2x + \frac{1}{4}y^2 = 0 \quad \Rightarrow \quad (x+1)^2 + \frac{1}{4}y^2 = 1.$$

We identify this is an ellipse centered at $(-1, 0)$. Again, I use the pythagorean trig. identity as my guide: I want $(x+1)^2 = \cos^2(t)$ and $\frac{1}{4}y^2 = \sin^2(t)$ because that will force the parametric equations to solve the ellipse equation. However, I would like for the equations to describe CW direction so I replace the t with $-t$ and propose:

$$\boxed{x = -1 + \cos(-t)} \qquad \textit{and} \qquad \boxed{y = 2\sin(-t)}$$

If we choose $t \in [0, 2\pi)$ then the whole ellipse will be covered. I could simplify $\cos(-t) = \cos(t)$ and $\sin(-t) = -\sin(t)$ but I have left the minus to emphasize the idea about reversing the orientation. In the preceding example we gave the circle a CCW orientation.

Example 1.4.29. Find parametric equations for the part of the level curve $x^2 - y^2 = 1$ which is found in the first quadrant.

We recognize this is a hyperbola which opens horizontally since $x = 0$ gives us $-y^2 = 1$ which has no real solutions. Hyperbolic trig. functions are built for a problem just such as this: recall $\cosh^2(t) - \sinh^2(t) = 1$ thus we choose $x = \cosh(t)$ and $y = \sinh(t)$. Furthermore, the hyperbolic sine function $\sinh(t) = \frac{1}{2}\big(e^t - e^{-t}\big)$ is everywhere increasing since it has derivative $\cosh(t)$ which is everywhere positive. Moreover, since $\sinh(0) = 0$ we see that $\sinh(t) \geq 0$ for $t \geq 0$. Choose non-negative t for the domain of the parametrization:

$$\boxed{x = \cosh(t), \qquad y = \sinh(t), \qquad t \in [0, \infty).}$$

Example 1.4.30. Find parametric equations for the part of the level curve $x^2 - y^2 = 1$ which is found in the third quadrant.

Based on our thinking from the last example we just need to modify the solution a bit:

$$\boxed{x = -\cosh(t), \qquad y = \sinh(t), \qquad t \in (-\infty, 0].}$$

Note that if $t \in (-\infty, 0]$ then $-\cosh(t) \leq -1$ and $\sinh(t) \leq 0$, this puts us in the third quadrant. It is also clear that these parametric equations to solve the hyperbola equation since

$$(-\cosh(t))^2 - (\sinh(t))^2 = \cosh^2(t) - \sinh^2(t) = 1.$$

The examples thus far are rather specialized, and in general there is no method to find parametric equations. This is why I said it is an art.

Example 1.4.31. Find parametric equations for the level curve $x^2y^2 = x - 2$.

This example is actually pretty easy because we can solve for $y^2 = \frac{x-2}{x^2}$ *hence* $y = \pm\sqrt{\frac{x-2}{x^2}}$. *We can choose* x *as parameter so the parametric equations are just*

$$x = t \quad \text{and } y = \sqrt{\frac{t-2}{t^2}}$$

for $t \geq 2$. *Or, we could give parametric equations*

$$x = t \quad \text{and } y = -\sqrt{\frac{t-2}{t^2}}$$

for $t \geq 2$. *These parametrizations simply cover different parts of the same level curve.*

Remark 1.4.32. *but... what is* t *?*

If you are at all like me when I first learned about parametric curves you're probably wondering what is t ? You probably, like me, suppose incorrectly that t should be just like x or y. There is a crucial difference between x and y and t. The notations x and y are actually shorthands for the Cartesian coordinate maps $x : \mathbb{R}^2 \to \mathbb{R}$ and $y : \mathbb{R}^2 \to \mathbb{R}$ where $x(a, b) = a$ and $y(a, b) = b$. When I use the notaion $x = 3$ then you know what I mean, you know that I'm focusing on the vertical line with first coordinate 3. On the other hand, if I say $t = 3$ and ask where is it? Then you should say, you question doesn't make sense. The concept of t is tied to the curve for which it is the parameter. There are infinitely many geometric meanings for t. In other words, if you try to find t in the xy-plane without regard to a curve then you'll never find an answer. It's a meaningless question.

On the other hand if we are given a curve and ask what the meaning of t is for that curve then we ask a meaningful question. There are two popular meanings.

1. the parameter s measures the arclength from some base point on the given curve.

2. the parameter t gives the time along the curve.

In case (1.) for an oriented curve this actually is uniquely specified if we have a starting point. Such a parameterization is called the **arclength parametrization** or **unit-speed** parametrization of a curve. These play a fundamental role in the study of the differential geometry of curves. In case (2.) we have in mind that the curve represents the physical trajectory of some object, as t increases, time goes on and the object moves. I tend to use (2.) as my conceptual backdrop. But, keep in mind that these are just applications of parametric curves. In general, the parameter need not be time or arclength. It might just be what is suggested by algebraic convenience.

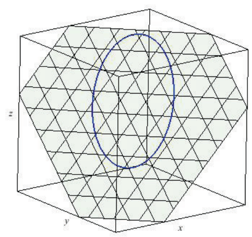
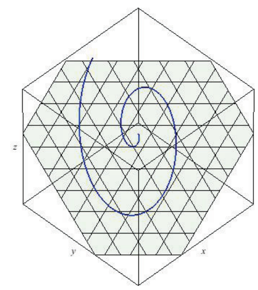
1.4.3 curves in three dimensional space

Other interesting curves can be obtained by feeding a simple curve like a circle into the parameterization of a plane.

Example 1.4.33. *Suppose $\vec{R}(u,v) = \vec{r}_o + u\vec{A} + v\vec{B}$ is the parametrization of a plane S then if we compose $\vec{R}$ with the path $t \mapsto \vec{\gamma}(t) = \langle R\cos(t), R\sin(t)\rangle$ we obtain an ellipse on the plane:*

$$\vec{r}(t) = (\vec{R} \circ \vec{\gamma})(t) = \vec{r}_o + R\cos(t)\vec{A} + R\sin(t)\vec{B}$$

Of course, we could also put a spiral on a plane through much the same device:

The idea of the last example can be used to create many interesting examples. These should suffice for our purposes here. I really just want you to think about what a parametrization does. Moreover, I hope you can find it in your heart to regard the parametric viewpoint as primary. Notice that any curves in three dimensions would require two independent equations in x, y, z. We saw how much of a hassel this was for something as simple as a line. I'd rather not attempt a general treatment of the purely cartesian description of the curves in this section[15] I instead offer a pair of examples to give you a flavor:

Example 1.4.34. *Suppose $x^2 + y^2 + z^2 = 4$ and $x = \sqrt{2}$ then the solution set of this pair of equations defines a curve in $\mathbb{R}^3$. Substutiting $x = \sqrt{3}$ into $x^2 + y^2 + z^2 = 4$ gives $y^2 + z^2 = 1$. The solution set is just a unit-circle in the yz-coordinates placed at $x = \sqrt{3}$. We can parametrize it via:*

$$\vec{r}(t) = \langle \sqrt{3}, \cos t, \sin t\rangle.$$

Example 1.4.35. *Suppose $z = x^2 - y^2$ and $z = 2x$. The solution set is once more a curve in $\mathbb{R}^3$. We can substitute $z = 2x$ into $z = x^2 - y^2$ to obtain $x^2 - y^2 = 2x$ hence $x^2 - 2x - y^2 = 0$ and completing the square reveals $(x-1)^2 - y^2 = 1$. This is the equation of a hyperbola. A natural parametrization is given by $x = 1 + \cosh t$ and $y = \sinh t$ then since $z = 2x$ we have $z = 2 + 2\cosh t$. In total,*

$$\vec{r}(t) = \langle 1 + \cosh t, \sinh t, 2 + 2\cosh t\rangle$$

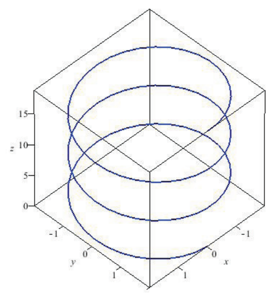
We'll explain the geometry of these calculations in the next section. Basically the idea is just that when two surfaces intersect in $\mathbb{R}^3$ we may obtain a curve.

[15] which is not to say it hasn't been done, in fact, viewing curves as solutions to equations is also a powerful technique, but we focus our efforts in the parametric setting.

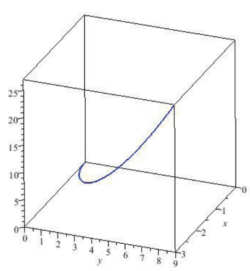
Example 1.4.36. *A helix of radius R which wraps around the z-axis and has a slope of m is given by:*

$$\vec{r}(t) = \langle R\cos(t), R\sin(t), mt\rangle$$

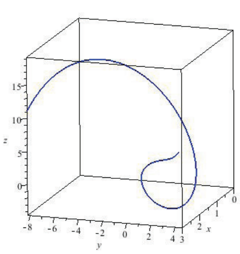
for $t \in [0, \infty)$.



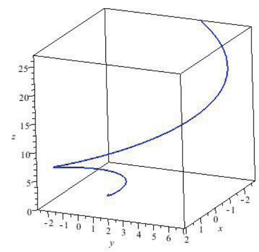
Example 1.4.37. *The curve parametrized by $\vec{r}(t) = \langle t, t^2, t^3$ for $t \geq 0$ has scalar parametric equations $x = t, y = t^2, z = t^3$ and a graph*



Example 1.4.38. *The curve parametrized by $\vec{r}(t) = \langle t, t^2\cos(3t), t^3\sin(3t)\rangle$ for $t \geq 0$ has scalar parametric equations $x = t, y = t^2\cos(t), z = t^3\sin(t)$ and a graph*



Example 1.4.39. *The curve parametrized by $\vec{r}(t) = \langle t\cos(3t), t^2\sin(3t), t^3\rangle$ for $t \geq 0$ has scalar parametric equations $x = t\cos(3t), y = t^2\sin(3t), z = t^3$ and a graph*



We will explore the geometry of curves in the next chapter. We'll find relatively simple calculations which allow us to test how a curve bends within it's plane of motion and bend off it's plane of motion. In other words, we'll find a way to test if a curve lies in a plane and also how it curves away from it's tangential direction. These quantities are called *torsion* and *curvature*. It turns out that these two quantities often classify a curve upto congruence in the sense of high-school geometry. In other words, there is just one circle of radius 1 and we can rotate it and translate it throughout $\mathbb{R}^3$. In this sense all circles in $\mathbb{R}^3$ are the same. We've already seen in this section that parametrization alone does not capture this concept. Why? Well there are many parametrizations of a circle. Are those different circles? I would say not. I say there is a circle and there are many pictures of the circle, some CW, some CCW, but so long as those pictures cover the same curve then they are merely differing perspectives on the same object. That said, these differing pictures are different. They are unique in their assignments of points to parameter values. The problem of the differential geometry of curves is to extract from this infinity of parametrizations some universal data. One seeks a few constants which invariantly characterize the curve independent of the perspective a particular geometer has used to capture it. More generally this is the problem of geometry. How can we classify spaces? What constants can we label a space with unambiguously?

1.5 surfaces

A surface in $\mathbb{R}^3$ is a subset which usually looks two dimensional. There are three main viewpoints; graphs, parametrizations or patches, and level-surfaces. As usual, the parametric view naturally generalizes to surfaces in $\mathbb{R}^n$ for $n > 3$ with relatively little qualification. That said, we almost without exception focus on surfaces in $\mathbb{R}^3$ in this course so I focus our efforts in that direction. This section is introductory in nature, basically this is just show and tell with a little algebra. Your goal should be to learn the names of these surfaces and more importantly to gain a conceptual foothold on the different ways to look at two-dimensional subsets of $\mathbb{R}^3$. Many of the diagrams in this section were created with Maple, others perhaps Mathematica. Ask if interested.

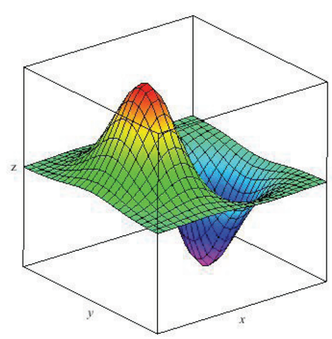
1.5.1 surfaces as graphs

Given a function of two variables it is natural to graph such a function in three-dimensions. In particular, we define:

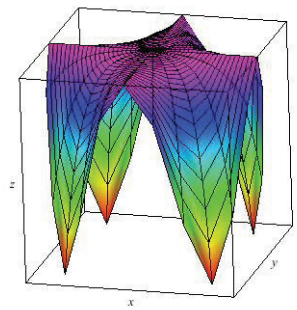
Definition 1.5.1. *graph of a function of two variables.*

Suppose $f : dom(f) \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a function then the set of all (x, y, z) such that $z = f(x, y)$ for some $(x, y) \in dom(f)$ is called the **graph** of f. Moreover, we denote $graph(f) = \{(x, y, f(x, y)) \mid (x, y) \in dom(f)\}$.

Example 1.5.2. *Let $f(x, y) = xe^{-x^2-y^2}$. The graph looks something like:*



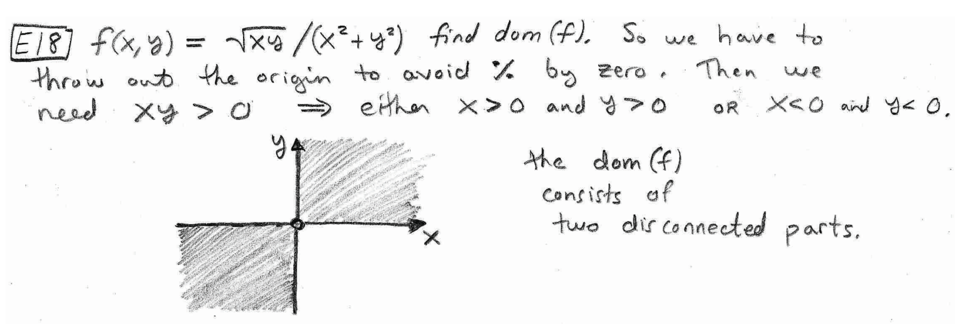
Example 1.5.3. *Let $f(x, y) = -cosh(xy)$. The graph looks something like:*



What is f? Well, many interpretations exist. For example, f could represent the temperature at (x, y). Or, f could represent the mass per unit area close to (x, y), this would make f a mass density function. More generally, if you have a variable which depends by some single-valued rule to another pair of variables then you can find a function in that application. Sometimes college algebra students will ask, but what *is* a function really? With a little imagination the answer is most anything. It could be that f is the cost for making x widgets and y gadgets. Or, perhaps $f(x, y)$ is the grade of a class as a function of x males and y females. Enough. Let's get back to the math, I'll generally avoid cluttering these notes with these silly comments, however, you are free to ask in office hours. Not all such discussion is without merit. Application is important, but is not at all necessary for good mathematics.

We can add, multiple and divide functions of two variables in the same way we did for functions of one variable. Natural domains are also implicit within formulas and points are excluded for much the same reason as in single-variable calculus; we cannot divide by zero, take an even root of a negative number or take a logarithm of a non-positive quantity if we wish to maintain a real output[16]. A typical example is given below:

Example 1.5.4. .



[16]complex variables do give meaning to even roots and logarithms of negative numbers however, division by zero and logarithm of zero continue to lack an arithmetical interpretation.

1.5.2 parametrized surfaces

Definition 1.5.5. *vector-valued functions of two real variables, parametrized surfaces.*

A vector valued function of a two real variables is an assignment of a vector for each pair of real numbers in some domain D of $\mathbb{R}^2$. It's a mapping $(u, v) \mapsto \vec{F}(u, v) = \langle F_1(u, v), F_2(u, v), \dots, F_n(u, v)\rangle$ for each $(u, v) \in D \subset \mathbb{R}^2$. We say $F_j : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the j-th component function of $\vec{F}$. Let $S = \vec{F}(D)$ then S is said to be a **surface parametrized by** $\vec{F}$. Equivalently, but not so usefully, we can write the scalar parametric equations for S above as

$$x_1 = F_1(u, v), \quad x_2 = F_2(u, v), \; \dots, \; x_n = F_n(u, v)$$

for all $(u, v) \in D$. We call $\vec{F}$ a **patch** on S.

When we define a parametrization of a surface it is important to give the formula for the patch **and** the domain D of the parameters. We call D the **parameter space**. Usually we are interested in the case of a surface which is embedded in $\mathbb{R}^3$ so I will focus the examples in that direction. Note however that the parametric equation for a plane actually embeds the plane in $\mathbb{R}^n$ for whatever n you wish, there is nothing particular to three dimensions for the construction of the line or plane parametrizations.

Example 1.5.6. *Suppose $S = \{(x, y, z) \mid z = f(x, y)\}$ where f is a function. Naturally we parametrize this graph via the choice $x = u, y = v$,*

$$\vec{r}(u, v) = \langle u, v, f(u, v)\rangle$$

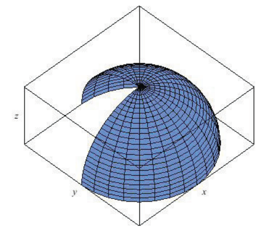
for $(u, v) \in dom(f)$.

As I discussed in the plane section, a graph is given in terms of cartesian coordinates. In the case of surfaces in $\mathbb{R}^3$ you'll often encounter the presentation $z = f(x, y)$ for some function f. This is an important class of examples, however, the criteria that f be a function is quite limiting.

Example 1.5.7. *Let $\vec{r}(u, v) = \langle R\cos(u)\sin(v), R\sin(u)\sin(v), R\cos(v)\rangle$ for $(u, v) \in [0, 2\pi] \times [0, \pi]$. In this case we have scalar equations:*

$$x = R\cos(u)\sin(v), \quad y = R\sin(u)\sin(v), \quad z = R\cos(v).$$

It's easy to show $x^2 + y^2 + z^2 = R^2$ and we should recognize that these are the parametric equations which force $\vec{r}(u, v)$ to land a distance of R away from the origin for each choice of (u, v). Let $S = \vec{r}(D)$ and recognize S is a **sphere** *of radius R centered at the origin. If we restrict the domain of $\vec{r}$ to $0 \le u \le \frac{3\pi}{2}$ and $0 \le v \le \frac{\pi}{2}$ then we select just a portion of the sphere:*

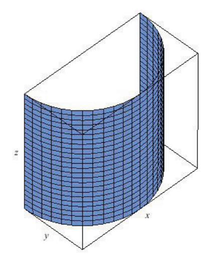


Notice that we could cover the whole sphere with a single patch. We cannot do that with a graph. This is the same story we saw in the two-dimensional case in calculus II. Parametrized curves are not limited by the vertical line test. Graphs are terribly boring in comparison to the geometrical richness of the parametric curve. As an exotic example from 1890, Peano constructed a continuous[17] path from $[0,1]$ which covers all of $[0,1] \times [0,1]$. Think about that. Such curves are called *space filling curves.* There are textbooks devoted to the study of just those curves. For example, see Hans Sagan's *Space Filling Curves.*

Example 1.5.8. *Let $\vec{r}(u,v) = \langle R\cos(u), R\sin(u), v\rangle$ for $(u,v) \in [0,2\pi] \times \mathbb{R}$. In this case we have scalar equations:*

$$x = R\cos(u), \quad y = R\sin(u), \quad z = v.$$

It's easy to show $x^2 + y^2 = R^2$ and z is free to range over all values. This surface is a circle at each possible z. We should recognize that these are the parametric equations which force $\vec{r}(u,v)$ to land on a cylinder of raduius R centered on the z-axis. If we restrict the domain of $\vec{r}$ to $0 \leq u \leq \pi$ and $0 \leq v \leq 2$ then we select a finite half-cylinder:

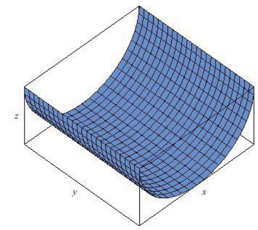


Example 1.5.9. *Let $\vec{r}(u,v) = \langle a\cos(u), v, b\sin(u)\rangle$ for $(u,v) \in [0,2\pi] \times \mathbb{R}$. In this case we have scalar equations:*

$$x = a\cos(u), \quad y = v, \quad z = b\sin(u).$$

It's easy to show $x^2/a^2 + z^2/b^2 = 1$ and y is free to range over all values. This surface is a ellipse at each possible y. We should recognize that these are the parametric equations which force $\vec{r}(u,v)$ to land on an elliptical cylinder centered on the y-axis. If we restrict the domain of $\vec{r}$ to $0 \leq u \leq \pi$ and $0 \leq v \leq 2$ then we select a finite half-cylinder:

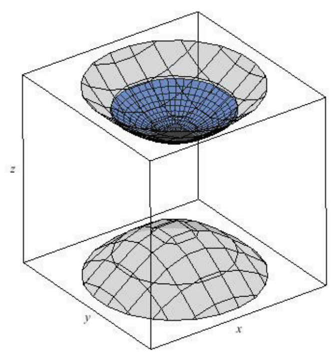
[17] we will define this carefully in a future chapter

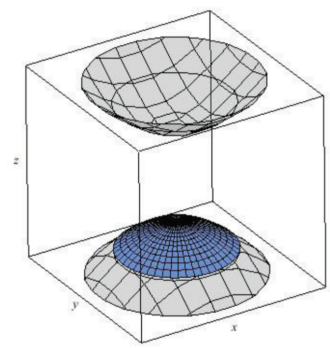


Example 1.5.10. *Let $\vec{r}(u,v) = \langle R\cos(u)\sinh(v), R\sin(u)\sinh(v), R\cosh(v)\rangle$ for $(u,v) \in [0,2\pi] \times \mathbb{R}$. In this case we have scalar equations:*

$$x = R\cos(u)\sinh(v), \quad y = R\sin(u)\sinh(v), \quad z = R\cosh(v).$$

It's easy to show $-x^2 - y^2 + z^2 = R^2$. If we restrict the domain of $\vec{r}$ to $0 \leq u \leq 2\pi$ and $-2 \leq v \leq 2$ then we select a portion of the upper branch:



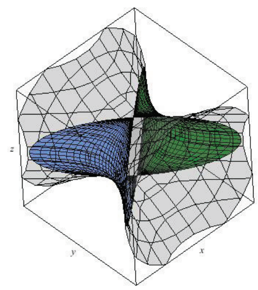


The part of the lower branch which is graphed above is covered by the mapping Let $\vec{r}(u,v) = \langle R\cos(u)\sinh(v), R\sin(u)\sinh(v), -R\cosh(v)\rangle$ for $(u,v) \in [0,2\pi] \times [-2,2]$. The grey shape is where the parametrization will cover if we enlarge the domain of the parameterizations.

Example 1.5.11. *Let $\vec{r}(u,v) = \langle R\cosh(u)\sin(v), R\sinh(u)\sin(v), R\cos(v)\rangle$ for $(u,v) \in \mathbb{R} \times [0,2\pi]$. In this case we have scalar equations:*

$$x = R\cosh(u)\sin(v), \quad y = R\sinh(u)\sin(v), \quad z = R\cos(v).$$

It's easy to show $x^2 - y^2 + z^2 = R^2$. I've plotted $\vec{r}$ with domain restricted to $dom(\vec{r}) = [-1.3, 1.3] \times [0,\pi]$ in blue and $dom(\vec{r}) = [-1.3, 1.3] \times [\pi, 2\pi]$ in green. The grey shape is where the parametrization will go if we enlarge the domain.



1.5.3 surfaces as level sets

Unlike curves, we do not need two equations to fix a surface in $\mathbb{R}^3$. In three dimensional space[18] if we have just one equation in x, y, z that should suffice to leave just two free variables. In a nutshell that is what a surface is. It is a space which has two degrees of freedom. In the parametric set-up we declare those freedoms explicitly from the outset by the construction of the patch in terms of the parameters. In the level set formulation we focus the attention on an equation which defines the surface of interest. We already saw this for a plane; the solutions of $ax + by + cz = d$ fill out a plane with normal $\langle a, b, c\rangle$.

Definition 1.5.12. *level surface in three dimensional space*

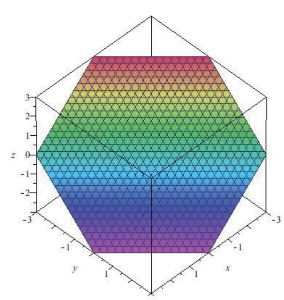
Suppose $F : dom(F) \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a function. Let

$$S = \{(x, y, z) \mid F(x, y, z) = k\}.$$

We say that S is a **level surface** of level k with level function F. In other words, $S = F^{-1}\{k\}$ is a level surface.

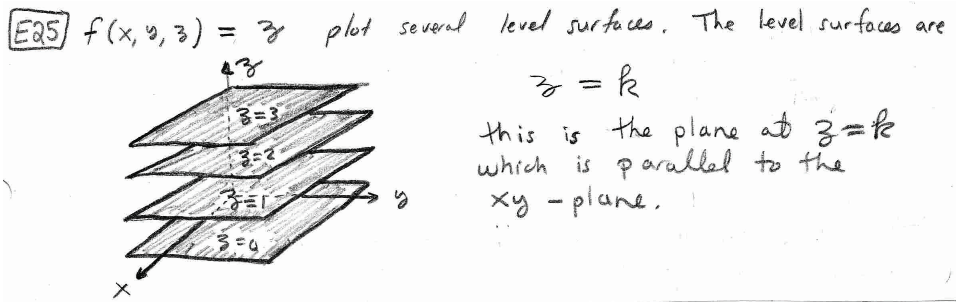
A level surface is a *fiber* of a real-valued function on $\mathbb{R}^3$.

Example 1.5.13. *Let $F(x, y, z) = a(x - x_o) + b(y - y_o) + c(z - z_o)$. Recognize that the solution set of $F(x, y, z) = 0$ is the plane with base-point (x_o, y_o, z_o) and normal $\langle a, b, c\rangle$.*



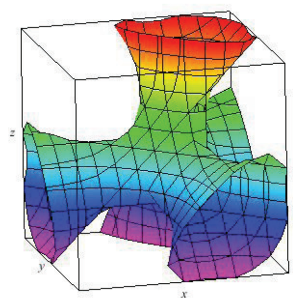
[18]to pick out a two-dimensional surface in $\mathbb{R}^4$ it would take two equations in t, x, y, z, but, we really only care about $\mathbb{R}^3$ so, I'll behave and stick with that case.

Example 1.5.14. *some level surfaces I can plot without fancy CAS programs:*



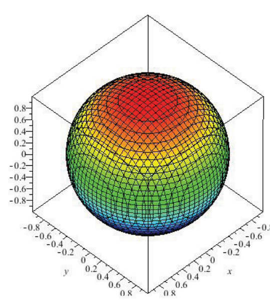
The example below is not such a case:

Example 1.5.15. *This surface has four holes. I have an animation on my website, check it out.*



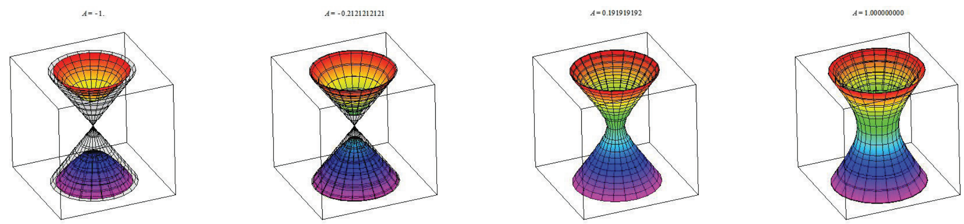
In fact, I don't think I want to parametrize this beast. Wait, I have students, isn't this what homework is for?

Example 1.5.16. *Let* $F(x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2$*. The solution set of* $F(x, y, z) = 1$ *is called an* **ellipsoid** *centered at the origin. In the special case* $a = b = c = R$ *the ellipsoid is a sphere of radius* R*. Here's a special case,* $a = b = c = 1$ *the unit-sphere:*



Example 1.5.17. *Let* $F(x, y, z) = x^2 + y^2 - z^2$*. The solution set of* $F(x, y, z) = 0$ *is called an* **cone** *through the origin. However, the solution set of* $F(x, y, z) = k$ *for* $k \neq 0$ *forms a* **hyperboloid of**

one-sheet *for* $k > 0$ *and a* **hyperboloid of two-sheets** *for* $k < 0$. *The hyperboloids approach the cone as the distance from the origin grows. I plot a few representative cases:*



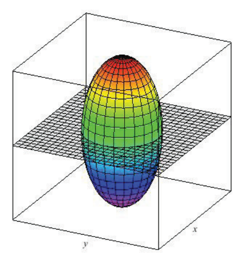
There is an animation on my webage, take a look.

Some of the examples above fall under the general category of a **quadratic surface**. Suppose

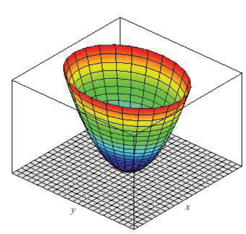
$$Q(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz.$$

For any particular nontrivial selection of constants $a, b, \dots, h, i$ we say the solution of $Q(x, y, z) = k$ is a **quadratic surface**. For future reference let me list the proper terminology. We'd like to get comfortable with these terms.

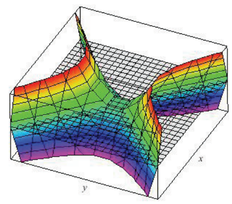
1. a standard **ellipsoid** is the solution set of $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. If $a = b = c$ then we say the ellipsoid is a **sphere**.



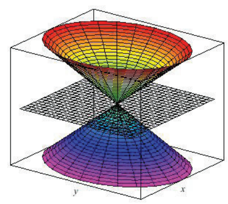
2. a standard **elliptic paraboloid** is the solution set of $z/c = x^2/a^2 + y^2/b^2$. If $a = b$ then we say the paraboloid is **circular**.



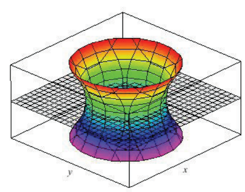
3. a standard **hyperbolic paraboloids** is the solution set of $z/c = y^2/b^2 - x^2/a^2$.



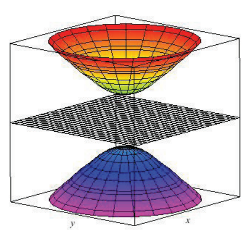
4. a standard **elliptic cone** is the solution set of $z^2/c^2 = x^2/a^2 + y^2/b^2$. If $a = b$ then we say the cone is **circular**.



5. a standard **hyperboloid of one sheet** is the solution set of $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$.



6. a standard **hyperboloid of two sheets** is the solution set of $x^2/a^2 + y^2/b^2 - z^2/c^2 = -1$.



If you study the formulas above you'll notice the absense of certain terms in the general quadratic form: terms such as dxy, exz, fyz, gx, hy are absent. Inclusion of these terms will either shift or rotate the standard equations. However, we need linear algebra to properly construct the rotations from the eigenvectors of the quadratic form. I leave that for Math 321 where we have more toys to play with. You'll have to be content with the standard examples for the majority of this course.

I've inserted the term *standard* because I don't mean to say that every elliptic cone has the same equation as I give. I expect you can translate the standard examples up to an interchange of coordinates, that ought not be too hard to understand. For example, $y = x^2 + 2z^2$ is clearly an elliptical cone. Or $y = x^2 - z^2$ is clearly a hyperbolic paraboloid. Or $x^2 + z^2 - y^2 = 1$ is clearly a hyperboloid of one sheet whereas $-x^2 - z^2 + y^2 = 1$ is a hyperboloid of two sheets. These are the possibilities we ought to anticipate when faced with the level set of some quadratic form. I don't try to memorize all of these, I use the method sketched in the next pair of examples. Basically the idea is simply to slice the graph into planes where we find either circles, hyperbolas, lines, ellipses or perhaps nothing at all. Then we take a few such slices and extrapolate the graph. Often the slices $x = 0$, $y = 0$ and $z = 0$ are very helpful.

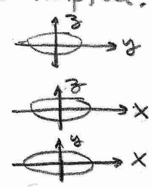
Example 1.5.18. .

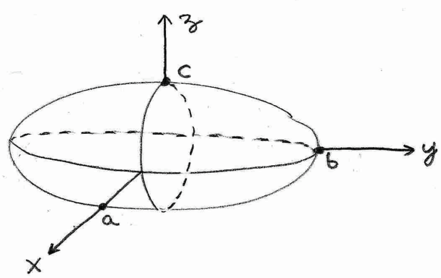
$\boxed{\text{E21}}$ Ellipsoid. The name is appropriate, anyway we slice it we'll get an ellipse. I usually look at the coordinate planes then go from there to gather whatever other data might appear to be helpful.

$x = 0 \Rightarrow y^2/b^2 + z^2/c^2 = 1$

$y = 0 \Rightarrow x^2/a^2 + z^2/c^2 = 1$

$z = 0 \Rightarrow x^2/a^2 + y^2/b^2 = 1$





Assuming $a, b, c > 0$ it looks something like this. Notice the idea is to use the cross-sections to get an idea where the surface is.

Example 1.5.19. .

$\boxed{E22}$ Cone: $z^2 = x^2 + y^2$

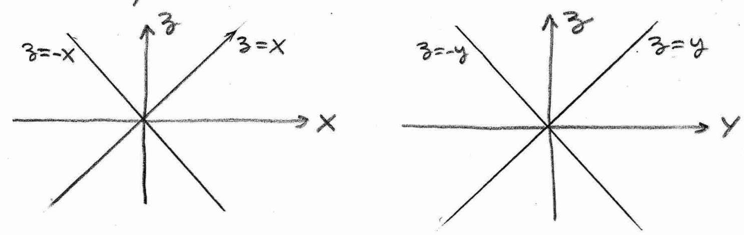
We can figure out somethings algebraically to begin,

$z = 0 = x^2 + y^2 \Rightarrow x = 0$ and $y = 0$ $\therefore$ intersects xy-plane only at the origin.

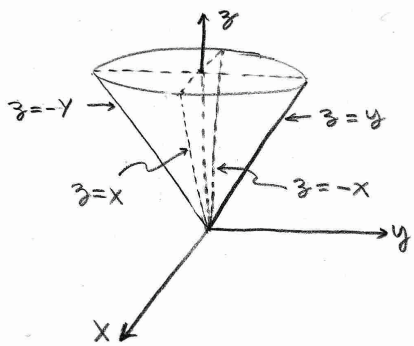
$x = 0 \Rightarrow z^2 = y^2 \Rightarrow z = \pm y$.

$y = 0 \Rightarrow z^2 = x^2 \Rightarrow z = \pm x$.

We can draw cross-sections of the surface with the coordinate planes $x = 0$ and $y = 0$



Then I'll attempt a 3-d rendition, (just for $z \geq 0$)



the trick is to draw the shape you imagine then check it with the cross-section lines (the text calls these "traces")

Example 1.5.20. *Let $F(x, y, z) = x^2 + y^2 - z$. The solution set of $F(x, y, z) = k$ sometimes called a paraboloid. Notice that if we fix a value for z say $z = c$ then $x^2 + y^2 - c = k$ reduces to $x^2 + y^2 = k + c$. If $k + c \geq 0$ then the solution in the $z = c$ plane is a circle or a point. In other words, all horizontal cross-sections of this shape are circles and if $z < -k$ there is no solution. This surface opens up from the vertex at $(0, 0, -k)$.*

What about the geometry of surfaces? How can we classify surfaces? What constants can we label a surface with unambiguously? Here's a simple example: $x^2 + y^2 + z^2 = 1$ and $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 1$ define the same shape at different points in $\mathbb{R}^3$. The problem of the differential geometry of surfaces is to find general invariants which discover this equivalence through mathematical calculation. This is a more difficult problem and we will not treat it in this course. It turns out this geometry begins to provide the concepts needed for Einstein's General Relativity. In any event, we do not currently have a course at LU which does this topic justice. I know of at least two professors who will happily conduct an independent study on this topic once you've mastered linear algebra.

1.5.4 combined concept examples

Example 1.5.21. .

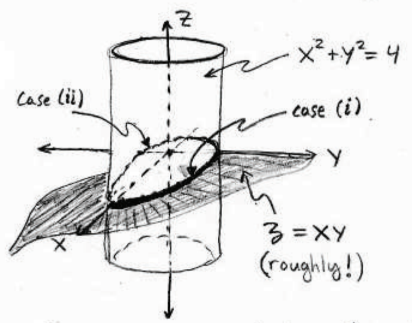
At what point does $r(t) = \langle t, 0, 2t - t^2 \rangle$ intersect the paraboloid $z = x^2 + y^2$? As usual we find intersection point by assuming both eq$^{\text{n}}$'s hold,

$$z = x^2 + y^2 \Rightarrow 2t - t^2 = t^2 + 0^2$$
$$\Rightarrow 2t - 2t^2 = 2t(1-t) = 0$$
$$\Rightarrow \underline{t = 0 \text{ OR } t = 1}$$

the points of intersection are $\boxed{(0,0,0) \text{ and } (1,0,1).}$

Example 1.5.22. .

Find vector function which represents the intersection of the surfaces $x^2 + y^2 = 4$ and $z = xy$. Lets use x as the parameter. Then



$$y = \pm\sqrt{4 - x^2}$$
$$z = x(\pm\sqrt{4-x^2})$$

the question then is (+) or (−) when and where? I'll break it up into cases.

(i) $y \geq 0 \Rightarrow r(x) = \langle x, \sqrt{4-x^2}, x\sqrt{4-x^2} \rangle$, $-2 \leq x \leq 2$.
(ii) $y \leq 0 \Rightarrow r(x) = \langle x, -\sqrt{4-x^2}, -x\sqrt{4-x^2} \rangle$, $-2 \leq x \leq 2$.

I suppose we could paste these together by shifting the parameter on either (i) or (ii). There are other ways, for example,

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 2\sin(2t), \quad 0 \leq t \leq 2\pi$$

Example 1.5.23. .

find curve of intersection of $z = \sqrt{x^2+y^2}$ and the plane $z = 1 + y$. Again use x as parameter,

note $y = z - 1 \Rightarrow z = \sqrt{x^2 + (z-1)^2}$

$$\Rightarrow z^2 = x^2 + (z-1)^2$$
$$\Rightarrow z^2 = x^2 + z^2 - 2z + 1$$
$$\Rightarrow 2z = x^2 + 1$$
$$\Rightarrow \underline{z = \tfrac{1}{2}(x^2+1)}$$
$$y = z - 1 = \tfrac{1}{2}x^2 + \tfrac{1}{2} - 1 = \underline{\tfrac{1}{2}(x^2-1) = y}$$

So if you wish we may introduce t as a parameter then

$\boxed{r(t) = \langle t, \tfrac{1}{2}(t^2-1), \tfrac{1}{2}(t^2+1)\rangle}$ (there are other answers.)

Example 1.5.24. .

Consider the following trajectories, do they collide? For $t \geq 0$

$r_1(t) = \langle t^2, 7t - 12, t^2\rangle$ and $r_2(t) = \langle 4t-3, t^2, 5t-6\rangle$

For vector functions to be equal we need each component to match with the corresponding component. That is,

$x_1 = x_2 \Rightarrow t^2 = 4t - 3 \Rightarrow t^2 - 4t + 3 = (t-1)(t-3) = 0 \therefore \boxed{t = 1 \ \& \ t = 3}$

$y_1 = y_2 \Rightarrow 7t - 12 = t^2 \Rightarrow t^2 - 7t + 12 = (t-3)(t-4) = 0 \therefore \boxed{t = 3 \ \& \ t = 4}$

$z_1 = z_2 \Rightarrow t^2 = 5t - 6 \Rightarrow t^2 - 5t + 6 = (t-3)(t-2) = 0 \therefore \boxed{t = 3 \ \& \ t = 2}$

We find $x_1 = x_2$ at $t = 1$, $y_1 = y_2$ at $t = 4$ and $z_1 = z_2$ at $t = 2$.

But only at $t = 3$ do we get $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$, this means the $\boxed{\text{particles will collide at } t = 3}$

Example 1.5.25. .

the question of collision is $r_1(t) \stackrel{?}{=} r_2(t)$ for some $t \geq 0$?. I leave that to you. The question of intersection is a bit different we should study $r_1(t) \stackrel{?}{=} r_2(s)$, can we find $s, t \geq 0$ so that the positions match-up (for possibly different times)

$$\left.\begin{array}{ll} x: & t = 1 + 2s \\ y: & t^2 = 1 + 6s \\ z: & t^3 = 1 + 14s \end{array}\right\} \underline{1 = t - 2s = t^2 - 6s = t^3 - 14s}$$

note $s = 0$ works if we make $t = 1$.

$\boxed{\text{So yes the paths intersect at } r_1(1) = r_2(0) = \langle 1, 1, 1\rangle.}$

there is one other place they intersect, can you find it?