1.6 curvelinear coordinates

Cartesian coordinates are a nice starting point, but they make many simple problems needlessly complex. If a two-dimensional problem has a quantity which only depends on distance from the central point then probably polar coordinates will simplify the equations of the problem. Similarly, if a three dimensional problem possesses a cylindrical symmetry then use cylindrical coordinates. If a three dimensional problem has spherical symmetry then use spherical coordinates.

A coordinate system is called **right-handed** if the unit-vectors which point in the direction of increasing coordinates at each point are related to each other by the right-hand-rule just like the xyz-coordinates. We call this set of unit-vectors the **frame** of the coordinate system. Generally a frame in $\mathbb{R}^n$ is just an assignment of n-vectors at each point in $\mathbb{R}^n$. In linear-algebra langauge, a frame is an assignment of a basis at each point of $\mathbb{R}^n$. Dimensions $n = 2$ and $n = 3$ suffice for our purposes. If y_1, y_2, y_3 denote coordinates with unit-vectors $\widehat{u}_1, \widehat{u}_2, \widehat{u}_3$ in the direction of increasing y_1, y_2, y_3 respective then we say the coordinate system is **right-handed** iff

$$\widehat{u}_1 \times \widehat{u}_2 = \widehat{u}_3, \quad \widehat{u}_2 \times \widehat{u}_3 = \widehat{u}_1, \quad \widehat{u}_3 \times \widehat{u}_1 = \widehat{u}_2.$$

In contrast with the constant frame $\{\widehat{x}, \widehat{y}, \widehat{z}\}$ the frame $\{\widehat{u}_1, \widehat{u}_2, \widehat{u}_3\}$ is usually point-dependent. An assignment of a vector to each point in some space is called a **vector field**. A frame is actually a triple of vector fields which is given over a space. Enough terminology, the equations speak for themselves soon enough.

1.6.1 polar coordinates

Polar coordinates (r, θ) are defined by

$$x = r\cos\theta, \qquad y = r\sin\theta$$

In quadrants I and IV (regions with $x > 0$) we also have

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}\left[\frac{y}{x}\right].$$

In quadrants II and III (regions with $x < 0$) we have

$$r^2 = x^2 + y^2, \qquad \theta = \pi + \tan^{-1}\left[\frac{y}{x}\right].$$

Geometrically it is clear that we can label any point in $\mathbb{R}^2$ either by cartesian coordinates (x, y) or by polar coordinates (r, θ). We may view equations in cartesian or polar form.

Example 1.6.1. *The circle $x^2 + y^2 = R^2$ has polar equation $r = R$.*

Typically in the polar context the angle plays the role of the independent variable. In the same way it is usually customary to write $y = f(x)$ for a graph we try to write $r = f(\theta)$.

Example 1.6.2. *The line $y = mx + b$ has polar equation $r\sin\theta = mr\cos\theta + b$ hence*

$$r = \frac{b}{\sin\theta - m\cos\theta}.$$

Example 1.6.3. *The polar equation $\theta = \pi/4$ translates to $y = x$ for $x > 0$. The reason is that*

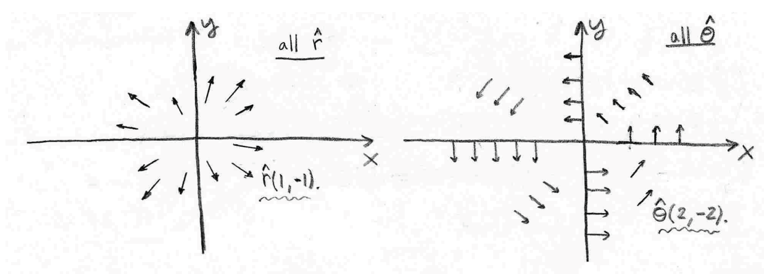
$$\frac{\pi}{4} = \tan^{-1}\left[\frac{y}{x}\right] \quad \Rightarrow \quad \tan\frac{\pi}{4} = \frac{y}{x} \quad \Rightarrow \quad 1 = \frac{y}{x} \quad \Rightarrow \quad y = x$$

and the ray with $\theta = \pi/4$ is found in quadrant I where $x > 0$.

Let us denote unit vectors in the direction of increasing r, θ by $\widehat{r}$, $\widehat{\theta}$ respective. You can derive by geometry alone that

$$\begin{aligned} \widehat{r} &= \cos(\theta)\,\widehat{x} + \sin(\theta)\,\widehat{y} \\ \widehat{\theta} &= -\sin(\theta)\,\widehat{x} + \cos(\theta)\,\widehat{y}. \end{aligned} \tag{1.5}$$

We call $\{\widehat{r},\ \widehat{\theta}\}$ the **frame** of polar coordinates. Notice that these are perpendicular at each point; $\widehat{r} \bullet \widehat{\theta} = 0$.



Example 1.6.4. *If we want to assign a vector to each point on the unit circle such that the vector is tangent and pointing in the counter-clockwise (CCW) direction then a natural choice is $\widehat{\theta}$.*

Example 1.6.5. *If we want to assign a vector to each point on the unit circle such that the vector is pointing radially out from the center then a natural choice is $\widehat{r}$.*

Suppose you have a perfectly flat floor and you pour paint slowly in a perfect even stream then in principle you'd expect it would spread out on the floor in the $\widehat{r}$ direction if we take the spill spot as the origin and the floor as the xy-plane.

1.6.2 cylindrical coordinates

Cylindrical coordinates (r, θ, z) are defined by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

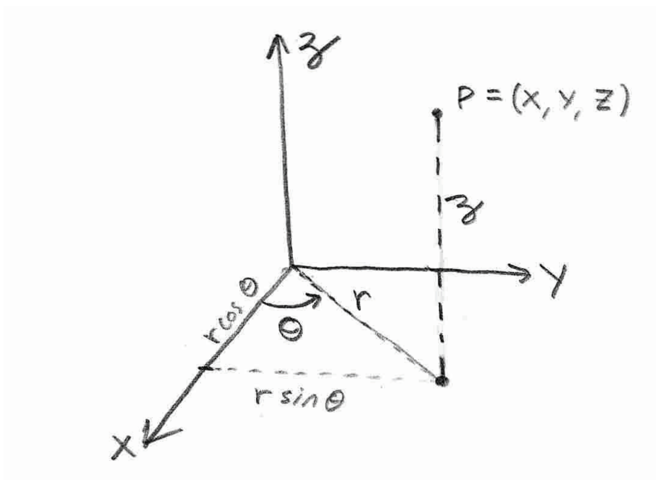
In quadrants I and IV (regions with $x > 0$) we also have

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}\left[\frac{y}{x}\right].$$

In quadrants II and III (regions with $x < 0$) we have

$$r^2 = x^2 + y^2, \qquad \theta = \pi + \tan^{-1}\left[\frac{y}{x}\right].$$

Geometrically it is clear that we can label any point in $\mathbb{R}^3$ either by cartesian coordinates (x, y, z) or by cylindrical coordinates (r, θ, z).



We may view equations in cartesian or cylindrical form.

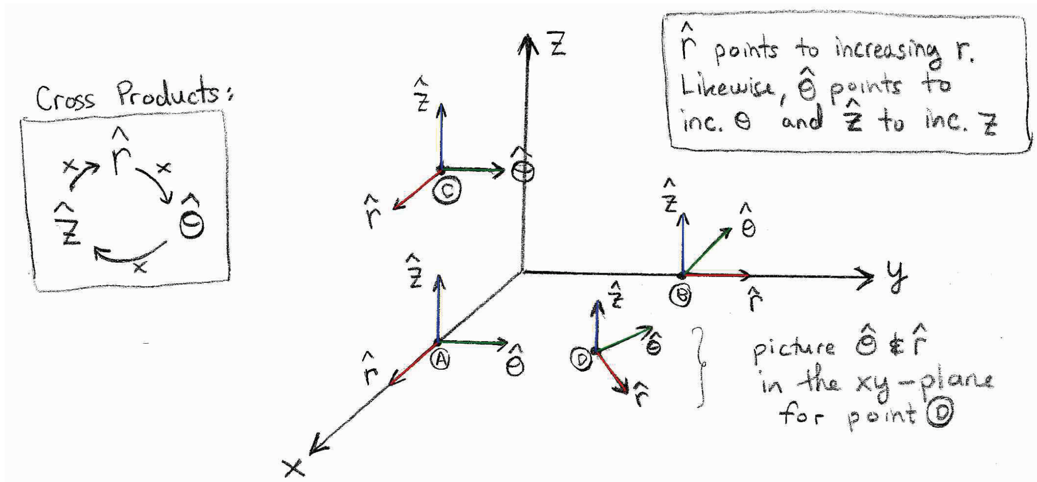
Example 1.6.6. *In cylindrical coordinates the equation $r = 1$ is a cylinder since the z-variable is free. If we denote the unit-circle by $S_1 = \{(x, y) \mid x^2 + y^2 = 1\}$ then the solution set of $r = 1$ has the form $S_1 \times \mathbb{R}$. At each z we get a copy of the circle S_1.*

Example 1.6.7. *The equation $\theta = \pi/4$ is a half-plane which has equation $y = x$ subject to the condition $x > 0$.*

Let us denote unit vectors in the direction of increasing $r,\ \theta,\ z$ by $\widehat{r},\ \widehat{\theta},\ \widehat{z}$ respectively. You can derive by geometry alone that

$$\begin{aligned} \widehat{r} &= \cos(\theta)\,\widehat{x} + \sin(\theta)\,\widehat{y} \\ \widehat{\theta} &= -\sin(\theta)\,\widehat{x} + \cos(\theta)\,\widehat{y} \\ \widehat{z} &= \langle 0, 0, 1\rangle. \end{aligned} \tag{1.6}$$

We call $\{\widehat{r},\ \widehat{\theta},\ \widehat{z}\}$ the **unit-frame** of cylindrical coordinates.



Example 1.6.8. *Suppose we have a line of electric charge smeared along the z-axis with charge density λ. One can easily derive from Gauss' Law that the electric field has the form:*

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_o r}\widehat{r}.$$

Example 1.6.9. *If we have a uniform current $I\,\widehat{z}$ flowing along the z-axis then the magnetic field can be derived from Ampere's Law and has the simple form:*

$$\vec{B} = \frac{\mu_o I}{2\pi r}\widehat{\theta}$$

Trust me when I tell you that the formulas in terms of cartesian coordinates are not nearly as clean.

If we fix our attention to a particular point the cylindrical frame has the same structure as the cartesian fram $\{\,\widehat{x},\,\widehat{y},\,\widehat{z}\}$. In particular, we can show that

$$\widehat{r}\bullet\widehat{r} = 1, \quad \widehat{\theta}\bullet\widehat{\theta} = 1, \quad \widehat{z}\bullet\,\widehat{z} = 1$$

$$\widehat{\theta}\bullet\widehat{r} = 0, \quad \widehat{\theta}\bullet\,\widehat{z} = 0, \quad \widehat{z}\bullet\widehat{r} = 0.$$

We can also calculate either algebraically or geometrically that:

$$\widehat{r}\times\widehat{\theta} = \,\widehat{z}, \quad \widehat{\theta}\times\,\widehat{z} = \widehat{r}, \quad \widehat{z}\times\widehat{r} = \widehat{\theta}$$

Therefore, the cylindrical coordinate system (r, θ, z) is a **right-handed** coordinate system since it provides a right-handed basis of unit-vectors at each point. We can summarize these relations compactly with the notation $\widehat{u}_1 = \widehat{r},\ \widehat{u}_2 = \widehat{\theta},\ \widehat{u}_3 = \,\widehat{z}$ whence:

$$\widehat{u}_i\bullet\widehat{u}_j = \delta_{ij}, \qquad \widehat{u}_i\times\widehat{u}_j = \sum_{k=1}^{3}\epsilon_{ijk}\widehat{u}_k$$

this is the same pattern we saw for the cartesian unit vectors.

1.6.3 spherical coordinates

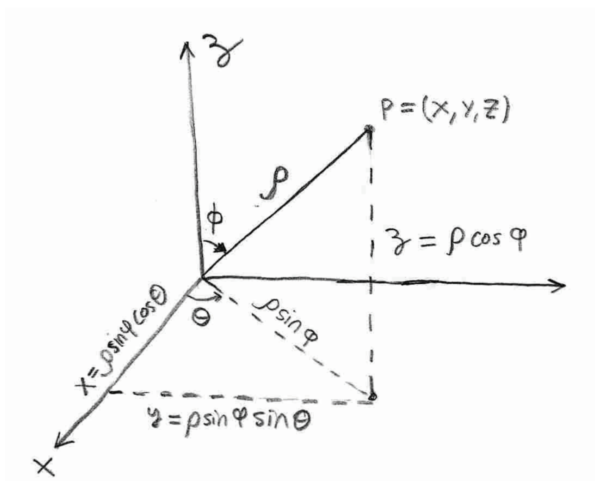
Spherical coordinates[19] (ρ, ϕ, θ) relate to Cartesian coordinates as follows

$$\begin{aligned} x &= \rho\cos(\theta)\sin(\phi) \\ y &= \rho\sin(\theta)\sin(\phi) \\ z &= \rho\cos(\phi) \end{aligned} \tag{1.7}$$

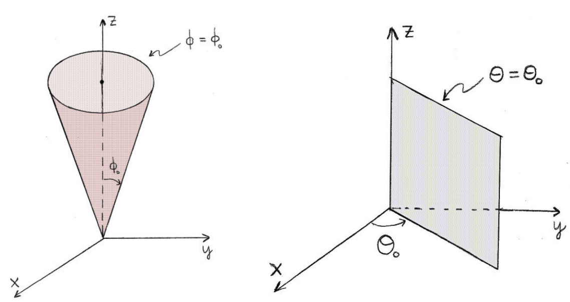
where $\rho > 0$ and $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. We can derive,

$$\begin{aligned} \rho^2 &= x^2 + y^2 + z^2 \\ \tan(\phi) &= \sqrt{x^2 + y^2}/z \\ \tan(\theta) &= y/x. \end{aligned} \tag{1.8}$$

It is clear that any point in $\mathbb{R}^3$ is labeled both by cartesian coordinates (x, y, z) or spherical coordinates (ρ, ϕ, θ).



Also, it is important to distinguish between the geometry of the **polar angle** θ and the **azmuthial angle** ϕ



[19] I'll use notation which is consistent with Stewart, but beware there is a better notation used in physics and engineering where the meaning of ϕ and θ are switched and the spherical radius ρ is instead denoted by r

Example 1.6.10. *The equation* $\sqrt{x^2+y^2+z^2} = R$ *is written as* $\rho = R$ *in spherical coordinates.*

Example 1.6.11. *The plane* $a(x-1)+b(y-2)+c(z-3) = 0$ *has a much uglier form in spherical coordinates. Its:* $a(\rho\cos(\theta)\sin(\phi)-1)+b(\rho\sin(\theta)\sin(\phi)-2)+c(\rho\cos(\phi)-3) = 0$ *hence*

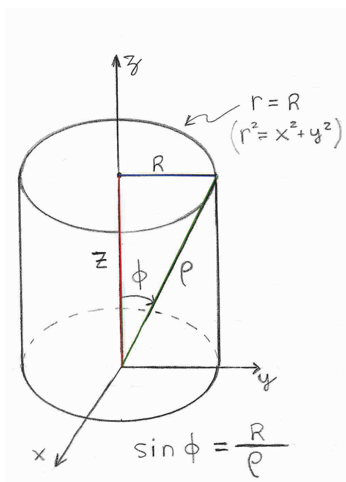
$$\rho = \frac{a+2b+3c}{a\cos(\theta)\sin(\phi)+b\sin(\theta)\sin(\phi)+c\cos(\phi)}.$$

Example 1.6.12. *The equation of a cylinder is* $r = R$ *in cylindrical coordinates. In spherical coordinates it is not as pretty. Note that* $r = R$ *gives* $x^2+y^2 = R^2$ *and*

$$x^2+y^2 = \rho^2\cos^2(\theta)\sin^2(\phi)+\rho^2\cos^2(\theta)\sin^2(\phi) = \rho^2\sin^2(\phi)$$

Thus, the equation of a cylinder in spherical coordinates is $R = \rho\sin(\phi)$.

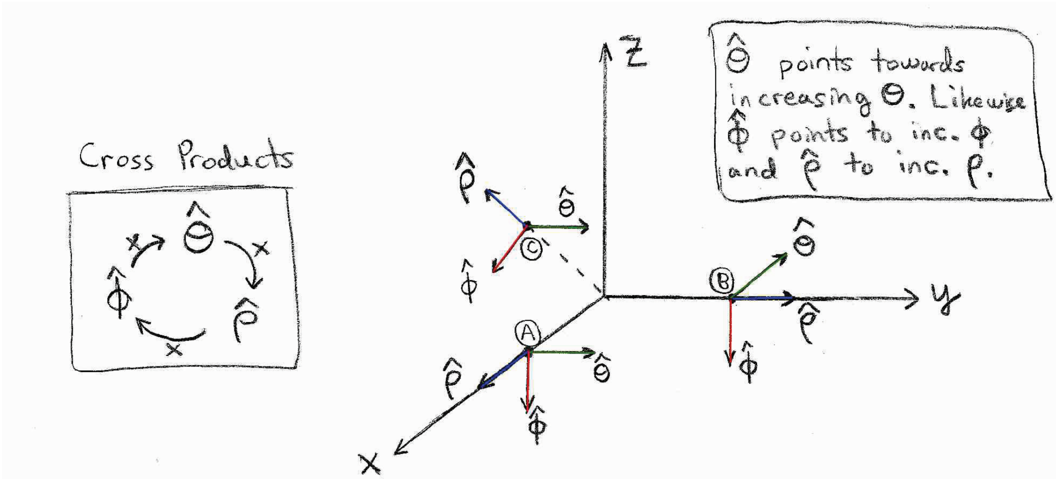
You might notice that the formula above is easily derived geometrically. If you picture a cylinder and draw a rectangle as shown below it is clear that $\sin(\phi) = \frac{R}{\rho}$.



It is important to be proficient in both visualization and calculation. They work together to solve problems in this course, if you get stuck in one direction sometimes the other will help you get free. Let us denote unit vectors in the direction of increasing $\rho,\ \phi,\ \theta$ by $\widehat{\rho},\ \widehat{\phi},\ \widehat{\theta}$ respectively. You can derive by geometry alone that

$$\begin{aligned} \widehat{\rho} &= \sin(\phi)\cos(\theta)\,\widehat{x}+\sin(\phi)\sin(\theta)\,\widehat{y}+\cos(\phi)\,\widehat{z} \\ \widehat{\phi} &= -\cos(\phi)\cos(\theta)\,\widehat{x}-\cos(\phi)\sin(\theta)\,\widehat{y}+\sin(\phi)\,\widehat{z} \\ \widehat{\theta} &= -\sin(\theta)\,\widehat{x}+\cos(\theta)\,\widehat{y}. \end{aligned} \tag{1.9}$$

We call $\{\widehat{\rho},\ \widehat{\phi},\ \widehat{\theta}\}$ the **frame** of spherical coordinates. At each point these unit-vectors point in particular direction.



In constrast to the cartesian frame which is constant[20] over all of $\mathbb{R}^3$.

Example 1.6.13. *Suppose we a point charge q is placed at the origin then by Gauss' Law we can derive*

$$\vec{E} = \frac{q}{4\pi\epsilon_o \rho^2}\widehat{\rho}.$$

This formula makes manifest the spherical direction of the electric field, the absence of the angular unit-vectors says the field has no angular dependence and hence its values depend only on the spherical radius ρ. This is called the **Coulomb field** *or monopole field. Almost the same math applies to gravity. If M is placed at the origin then*

$$\vec{F} = \frac{GmM}{\rho^2}(-\widehat{\rho}).$$

gives the gravitational force $\vec{F}$ of M on m at distance ρ from the origin. The direction of the gravitational field is $-\widehat{\rho}$ which simply says the field points radially inward.

The spherical frame gives us a basis of vectors to build vectors at each point in $\mathbb{R}^3$. More than that, the spherical frame is an orthonormal frame since at any particular point the frame provides an orthonormal set of vectors. In particular, we can show that

$$\widehat{\rho} \bullet \widehat{\rho} = 1, \quad \widehat{\phi} \bullet \widehat{\phi} = 1, \quad \widehat{\theta} \bullet \widehat{\theta} = 1$$

$$\widehat{\phi} \bullet \widehat{\rho} = 0, \quad \widehat{\theta} \bullet \widehat{\rho} = 0, \quad \widehat{\phi} \bullet \widehat{\theta} = 0.$$

We can also calculate either algebraically or geometrically that:

$$\widehat{\theta} \times \widehat{\rho} = \widehat{\phi}, \quad \widehat{\rho} \times \widehat{\phi} = \widehat{\theta}, \quad \widehat{\phi} \times \widehat{\theta} = \widehat{\rho}$$

[20] How do I know the cartesian frame is unchanging? It's not complicated really; $\widehat{x} = \langle 1, 0, 0 \rangle$, $\widehat{y} = \langle 0, 1, 0 \rangle$ and $\widehat{z} = \langle 0, 0, 1 \rangle$.

Therefore, the spherical coordinate system (ρ, ϕ, θ) is a **right-handed** coordinate system since it provides a right-handed basis of unit-vectors at each point. We can summarize these relations compactly with the notation $\widehat{u}_1 = \widehat{\rho},\ \widehat{u}_2 = \widehat{\phi},\ \widehat{u}_3 = \widehat{\theta}$ whence:

$$\widehat{u}_i \bullet \widehat{u}_j = \delta_{ij}, \qquad \widehat{u}_i \times \widehat{u}_j = \sum_{k=1}^{3} \epsilon_{ijk} \widehat{u}_k$$

this is the same pattern we saw for the cartesian unit vectors.

We will return to the polar, cylindrical and spherical coordinate systems as the course progresses. Even now we could consider a multitude of problems based on the combination of the material covered thus-far and it's intersection with curvelinear coordinates. There are other curved coordinate systems beyond these standard three, but I leave those to your imagination for the time being. I do discuss a more general concept of coordinate system in the advanced calculus course. In manifold theory the concept of a coordinate system is refined in considerable depth. We have no need of such abstraction here so I'll behave[21].

[21] I'd guess most calculus text editors would say this whole paragraph is misbehaving, but I have no editor so ha.