Calculus I in a Nutshell

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Abstract

This handout is meant to be read as a companion piece to my first lecture in Calculus II. Our jumping off point is integral calculus where it is assumed you have already studied the technique of u-substitution. In some previous math courses there are weeks of review woven throughout the term. Calculus II is not like that. Nearly the entire term is new material for most students. My apologies if your Calculus I course did not already cover what I cover in this handout. I understand some of you have not seen a proof of the FTC and I think it is important for you to see it, hence we essentially begin with the FTC in lecture 1. Then I will review the calculation of definite and indefinite integrals including u-substitution. It is likely I use Mission 1 to help you refresh elements of Calculus I which may be rusty. As always, I am here to help.

I should mention, this handout is in part taken from my more complete set of notes on Calculus I. A relatively recent copy is found at: (click here) on my website www.supermath.info. I'm shifting to handout-based notes this term to try a different approach to my textbook supplements. See the Course Planner for my tentative plans.

1 Brief Review of Differential Calculus

I provide this concise review of the theory of continuity and differentiation from Calculus I to remind the reader what we already know from the previous course. This section is just for reference and review, I do not intend to lecture on this directly.

Definition 1.1. *slope of function, derivative at point, tangent line.*

If the limit below exists then we say f is **differentiable** at $x = a$ and define

$$f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

For f differentiable at $x = a$, the equation of the tangent line is $\boxed{y = f(a) + f'(a)(x - a)}$ and we say the **slope** of f at a is $f'(a)$. The function defined by $x \mapsto f'(x)$ is called the **derivative** of f. We also denote $f'(x) = \frac{df}{dx}$.

We learned four basic rules which allow us to differentiate a multitude of expressions. In particular:

name of property	operator notation	prime notation
Linearity	$\frac{d}{dx}[f+g] = \frac{d}{dx}[f] + \frac{d}{dx}[g]$ $\frac{d}{dx}[cf] = c\frac{d}{dx}[f]$	$(f+g)' = f' + g'$ $(cf)' = cf'$
Product Rule	$\frac{d}{dx}[fg] = \frac{df}{dx}g + f\frac{dg}{dx}$	$(fg)' = f'g + fg'$
Quotient Rule	$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
Chain Rule	$\frac{d}{dx}[f \circ u] = \frac{df}{du}\frac{du}{dx}$	$(f \circ u)'(x) = f'(u(x))u'(x)$

Next, we recall all the basic rules of differentiation for common functions.

$f(x)$	$\frac{df}{dx}$	Comments about $f(x)$	Formulas I use
c	0	constant function	
x	1	line $y = x$ has slope 1	
x^2	$2x$		
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$		
x^n	nx^{n-1}	power rule	
e^x	e^x	the exponential	
5^x	$\ln(5)5^x$		
a^x	$\ln(a)a^x$	an exponential	
$\ln(x)$	$\frac{1}{x}$	the natural log	$\ln(e^x) = x$, $e^{\ln(x)} = x$
$\log x$	$\frac{1}{\ln(10)x}$	log base 10	
$\log_a(x)$	$\frac{1}{\ln(a)x}$	log base a	$\log_a(a^x) = x$, $a^{\log_a(x)} = x$
$\sin(x)$	$\cos(x)$		$\sin^2(x) + \cos^2(x) = 1$
$\cos(x)$	$-\sin(x)$		
$\tan(x)$	$\sec^2(x)$		$\tan^2(x) + 1 = \sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$	reciprocal of cosine	$\sec(x) = 1/\cos(x)$
$\cot(x)$	$-\csc^2(x)$	reciprocal of tangent	$\cot(x) = \cos(x)/\sin(x)$
$\csc(x)$	$-\csc(x)\cot(x)$	reciprocal of sine	$\csc(x) = 1/\sin(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$	inverse sine	$\sin(\sin^{-1}(x)) = x$
$\cos^{-1}(x)$	$\frac{-1}{\sqrt{1-x^2}}$	inverse cosine	$\cos^{-1}(\cos(x)) = x$
$\tan^{-1}(x)$	$\frac{1}{x^2+1}$	inverse tangent	
$\sinh(x)$	$\cosh(x)$	hyperbolic sine	$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$
$\cosh(x)$	$\sinh(x)$	hyperbolic cosine	$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$
$\tanh(x)$	$sech^2(x)$	hyperbolic tangent	
$\sinh^{-1}(x)$	$\frac{1}{\sqrt{1+x^2}}$	inverse sinh	
$\cosh^{-1}(x)$	$\frac{1}{\sqrt{x^2-1}}$	inverse cosh	
$\tanh^{-1}(x)$	$\frac{1}{1-x^2}$	inverse tanh	

It would be healthy to have most of the results above memorized. Or, at least have enough memorized so that you can with relatively little effort derive whatever you forgot.

Example 1.2. *To derive the derivative of* $y = \tan^{-1}(x)$ *note* $\tan y = x$ *hence by implicit differentiation we find* $\sec^2(y)\frac{dy}{dx} = 1$*. Therefore,* $\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$.

Example 1.3. *To derive the derivative of* $y = \tanh(x)$ *note* $\tanh x = \frac{\sinh x}{\cosh x}$ *hence by quotient rule*

$$\frac{d}{dx}\tanh x = \frac{\frac{d}{dx}(\sinh x)\cosh x - \sinh x \frac{d}{dx}\cosh x}{\cosh^2(x)} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = sech^2(x).$$

Beyond direct differentiation, we also will use differential calculus to help find bounds for various expressions. We turn to such analysis to complete our brief review of differential calculus.

Theorem 1.4. *sign of the derivative function* f' *indicates strict increase or decrease of* f*.*

Suppose that f is a function and J is a connected subset of $dom(f)$

(1.) if $f'(x) > 0$ for all $x \in J$ then f is strictly increasing on J

(2.) if $f'(x) < 0$ for all $x \in J$ then f is strictly decreasing on J.

Example 1.5. *Consider* $f(x) = e^{-x^2}$ *over* $1 \leq x \leq 2$*. We wish to* **bound** *this function. In particular, we desire to find real numbers* m *and* M *for which* $m \leq f(x) \leq M$ *for each* $x \in [0,1]$*. To achieve this goal we use the Theorem above. Notice*

$$\frac{df}{dx} = -2xe^{-x^2} < 0$$

whenever $x > 0$ *since* $e^{-x^2} > 0$*. Therefore* f *is strictly decreasing on* $[1,2]$*. By definition of strict decrease we have* $f(1) > f(x) > f(2)$ *for each* $x \in [1,2]$*. Hence* $e^{-4} \leq e^{-x^2} \leq e^{-1}$ *for each* $x \in [1,2]$*.*

Theorem 1.6. *closed interval method.*

If we are given function f which is continuous on a closed interval $[a,b]$ the we can find the absolute minimum and maximum values of the function over the interval $[a,b]$ as follows:

(1.) Locate all critical numbers $x = c$ in (a,b) and calculate $f(c)$.

(2.) Calculate $f(a)$ and $f(b)$.

(3.) Compare values from steps (1.) and (2.) the largest of these values is the absolute maximum, the smallest (or largest negative) value is the absolute minimum of f on $[a,b]$.

The theory above will be useful to us as we seek to bound certain terms as the semester unfolds.

Example 1.7. *Consider* $f(x) = e^{-x^2}$ *over* $-1 \leq x \leq 1$*. We wish to* **bound** *this function. In particular, we desire to find real numbers* m *and* M *for which* $m \leq f(x) \leq M$ *for each* $x \in [-1,1]$*. To achieve this goal we use the Theorem above. Notice*

$$\frac{df}{dx} = -2xe^{-x^2} = 0$$

implies $x = 0$ *is the only critical number since* $e^{-x^2} \neq 0$*. Thus calculate:*

$$f(-1) = e^{-(-1)^2} = e^{-1} \qquad \& \qquad f(0) = e^0 = 1 \qquad \& \qquad f(1) = e^{-(1)^2} = e^{-1}$$

Thus, by the closed interval method we find $e^{-1} \leq e^{-x^2} \leq 1$ *for each* $x \in [-1,1]$*.*

2 Direct Calculation of Area

This section is necessary to properly understand the definition of the Definite Integral. It is unlikely I lecture on this material directly.

Definition 2.1. *partition of* $[a, b]$.

Suppose $a < b$ then $[a, b] \subset \mathbb{R}$. Define $\Delta x = \frac{b-a}{n}$ for $n \in \mathbb{N}$ and let $x_j = a + j\Delta x$ for $j = 0, 1, \dots, n$. In particular, $x_o = a$ and $x_n = b$.

The closed interval $[a, b]$ is a union of n-subintervals of length Δx. Note that the closed interval $[a, b] = [x_o, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n]$.

Definition 2.2. *Given function* f *where* $[a, b] \subseteq dom(f)$ *we define:*

the **Left-Endpoint Rule** (L_n) and **Right-Endpoint Rule** (R_n) via

$$L_n = \sum_{j=0}^{n-1} f(x_j)\Delta x = [f(x_0) + f(x_1) + \cdots + f(x_{n-1})]\Delta x,$$

$$R_n = \sum_{j=1}^{n} f(x_j)\Delta x = [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x,$$

and the **Midpoint Rule** (M_n) using the midpoints by $\bar{x}_k = \frac{1}{2}(x_k + x_{k-1})$:

$$M_n = \sum_{j=1}^{n} f(\bar{x}_j)\Delta x = [f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]\Delta x.$$

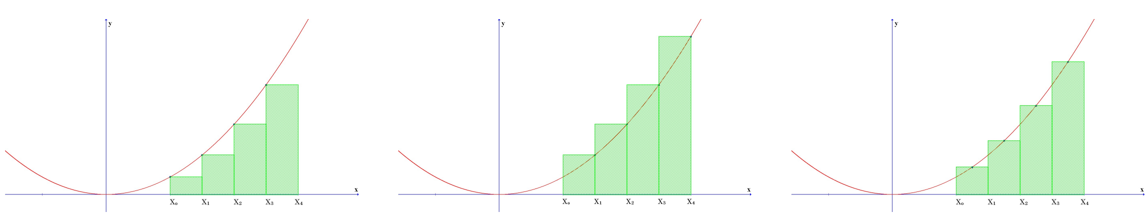
Example 2.3. *Let* $f(x) = x^2$ *and estimate the signed-area bounded by* f *on* $[1, 3]$ *by the left/right-endpoint rules with* $n = 4$. *Note* $\Delta x = \frac{3-1}{4} = 0.5$ *thus* $x_o = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5$ *and* $x_4 = 3$.

$$L_4 = [f(1) + f(1.5) + f(2) + f(2.5)]\Delta x = [1 + 2.25 + 4 + 6.25](0.5) = 6.75$$

$$R_4 = [f(1.5) + f(2) + f(2.5) + f(3)]\Delta x = [2.25 + 4 + 6.25 + 9](0.5) = 10.75$$

$$M_4 = [f(1.25) + f(1.75) + f(2.25) + f(2.75)]\Delta x = [1.5625 + 3.0625 + 5.0625 + 7.5625](0.5) = 8.625$$

Let A *defnote the exact area bounded by* $y = f(x)$ *for* $1 \leq x \leq 3$. *It's clear that* $L_4 < A < R_4$*:*



If we examine a picture which represents the above calculation then it is not immediately obvious whether M_4 *under or over-estimates the area* A. *Notice that the size of the errors will shrink if we increase* n. *In particular, it is intuitively obvious that as* $n \to \infty$ *we will obtain the precise area bounded by the curve. Moreover, we expect that the distinction between* L_n, R_n *and* M_n *should* *vanish as $n \to \infty$. Careful proof of this seemingly obvious claim is beyond the scope of this course.*

Let $f(x) = x^2$ and calculate the signed-area bounded by f on $[1, 3]$ by the right end-point rule. To perform this calculation we need to set up R_n for arbitrary n and then take the limit as $n \to \infty$. Note $x_k = 1 + k\Delta x$ and $\Delta x = 2/n$ thus $x_k = 1 + 2k/n$. Calculate,

$$f(x_k) = \left(1 + \frac{2k}{n}\right)^2 = 1 + \frac{4k}{n} + \frac{4k^2}{n^2}$$

thus[1]

$$\begin{aligned} R_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left[1 + \frac{4k}{n} + \frac{4k^2}{n^2}\right]\frac{2}{n} &= \frac{2}{n}\sum_{k=1}^{n} 1 + \frac{8}{n^2}\sum_{k=1}^{n} k + \frac{8}{n^3}\sum_{k=1}^{n} k^2 \\ &= \frac{2}{n}n + \frac{8}{n^2}\frac{n(n+1)}{2} + \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} \\ &= 2 + 4\left(1 + \frac{1}{n}\right) + \frac{8}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \end{aligned}$$

Note that $\frac{1}{n}$ and $\frac{1}{n^2}$ clearly tend to zero as $n \to \infty$ thus

$$\lim_{n\to\infty} R_n = 2 + 4 + \frac{16}{6} = \frac{26}{3} \approx 8.6667.$$

Challenge: show L_n and M_n also have limit $\frac{26}{3}$ as $n \to \infty$.

There are theorems which bound the error in L_n, R_n or M_n. Furthermore, we could also study the **Trapezoid Rule** or **Simpson's Rule** which approximate the area under $y = f(x)$ using trapezoids and parabolas respectively. A good course in numerical methods will likely include proof that Simpson's Rule is in some sense the optimal method to approximate the area under arbitrary continuous functions. See Section 7.7 in Stewart's Calculus for detailed statements of the error bound theorems.

3 Definition of the Definite Integral

Definition 3.1. *Riemann sum and the definite integral of continuous function on $[a, b]$.*

Suppose that f is continuous on $[a, b]$ suppose $x_k^* \in [x_{k-1}, x_k]$ for all $k \in \mathbb{N}$ such that $1 \leq k \leq n$ then an n-th Riemann sum is defined to be

$$\mathcal{R}_n = \sum_{j=1}^{n} f(x_k^*)\Delta x = [f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]\Delta x.$$

Notice that no particular restriction is placed on the sample points x_k^*. This means a Riemann sum could be a left, right or midpoint rule. This freedom will be important in the proof of the Fundamental Theorem of Calculus I offer in a later section.

[1] we use $\sum_{k=1}^{n} 1 = n$, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ in going from the third to the fourth equalities. These can be proved by the technique of mathematical induction.

Definition 3.2. *definite integrals.*

Suppose that f is continuous on $[a, b]$, the **definite integral** of f from a to b is defined to be $\lim_{n\to} \mathcal{R}_n$ in particular we denote:

$$\int_a^b f(x)\,dx = \lim_{n\to} \mathcal{R}_n = \lim_{n\to\infty} \left[\sum_{j=1}^{n} f(x_k^*)\Delta x\right].$$

The function f is called the **integrand**. The variable x is called the **dummy variable of integration**. We say a is the **lower bound** and b is the **upper bound**. The symbol dx is the **measure**. We also define for $a < b$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \qquad \text{and} \qquad \int_a^a f(x)\,dx = 0.$$

The **signed-area** bounded by $y = f(x)$ for $a \leq x \leq b$ is defined to be $\int_a^b f(x)\,dx$.

The integral above is known as the Riemann-integral. Other definitions are possible[2].

If f is continuous on the intervals $(a_1, a_2), (a_2, a_3), \ldots (a_k, a_{k+1})$ and each discontinuity is a finite-jump discontinuity then the definite integral of f on $[a_1, a_{k+1}]$ is defined to be the sum of the integrals:

$$\int_{a_1}^{a_{k+1}} f(x)\,dx = \sum_{j=1}^{k} \int_{a_j}^{a_{j+1}} f(x)\,dx.$$

Technically this leaves something out since we have only carefully defined integration over a closed interval and here we need the concept of integration over a half-open or open interval. To be careful one has the limit of the end points tending to the points of discontinuity. We discuss this further when we study *improper integration*.

Example 3.3. *Suppose $f(x) = \sin(x)$. Set-up the definite integral from $[0, \pi]$. We choose $\mathcal{R} = R_n$ for convenience. Note $\Delta x = \pi/n$ and the typical sample point is $x_j^* = j\pi/n$. Thus*

$$R_n = \sum_{j=1}^{n} \sin(x_j^*)\Delta x = \sum_{j=1}^{n} \sin\left(\frac{j\pi}{n}\right)\frac{\pi}{n} \quad \Rightarrow \quad \int_0^\pi \sin(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} \sin\left(\frac{j\pi}{n}\right)\frac{\pi}{n}.$$

At this point, most of us would get stuck. That said, once we know FTC II, this problem is easy.

3.1 properties of the definite integral

As we just observed a particular Riemann integral can be very difficult to calculate *directly* even if the integrand is a relatively simple function. That said, there are a number of intuitive properties for the definite integral whose proof is easier in general than the preceding specific case.

[2] the Riemann-Stieltjes integral or Lesbesque are generalizations of this the basic Riemann integral. Riemann-Stieltjes integral might be covered in some undergraduate analysis courses whereas Lesbesque's measure theory is typically a graduate analysis topic.

Proposition 3.4. *algebraic properties of definite integration.*

Suppose f, g are continuous on $[a, b]$ and $a < c < b$, $\alpha \in \mathbb{R}$

$$
\begin{aligned}
&(i.) \ \int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx, \\
&(ii.) \ \int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx, \\
&(iii.) \ \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.
\end{aligned}
$$

Proposition 3.5. *inequality properties of definite integration.*

Suppose f, g are continuous on $[a, b]$ and $m, M \in \mathbb{R}$,

$(i.)$ if $f(x) \geq 0$ for all $x \in [a, b]$ then $\int_a^b f(x)dx \geq 0$,

$(ii.)$ if $f(x) \geq g(x)$ for all $x \in [a, b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$,

$(iii.)$ if $m \leq f(x) \leq M$ for all $x \in [a, b]$ then $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$.

4 Fundamental Theorems of Integral Calculus

In the preceding section we detailed a careful procedure for direct calculation of the signed area between $y = f(x)$ and $y = 0$ for $a \leq x \leq b$. In this section we motivate and prove FTC I and FTC II which make calculation of area a reasonable endeavor for many commonly encountered functions.

4.1 area functions and FTC part I

In that discussion the endpoints a and b were given and fixed in place. We now shift gears a bit. We study **area functions** in this section. The idea of an area function is simply this: if we are given a function f then we can define an area function for f once we pick some base point a. Then $A(x)$ will be defined to be the signed-area bounded by the graph of f over $[a, x]$.

Definition 4.1. *area function.*

Given f and a point a we define the **area function** of f relative to a as follows:

$$A(x) = \int_a^x f(t)\, dt.$$

We say that $A(x)$ is the signed-area bounded by f on $[a, x]$.

We can calculate area without calculus for rectangles and triangles:

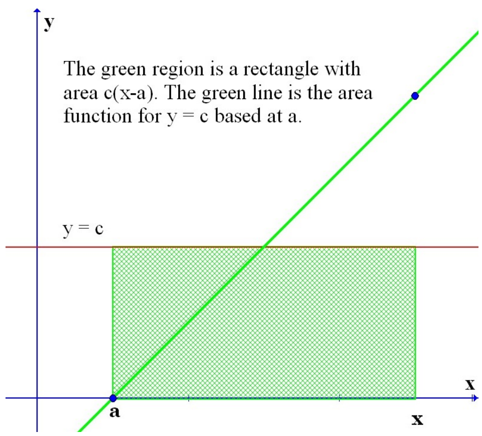
(1.) Area function of rectangle: with length is $(x-a)$ and height c we find

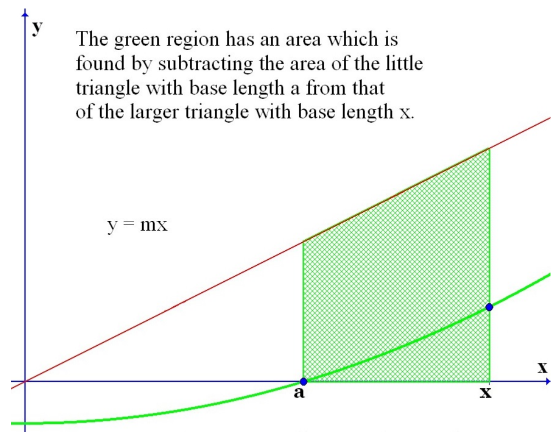
$$A(x) = \int_a^x c\,dt = c(x-a) = cx - ca. \tag{1}$$

(2.) Area function of triangle: We calculate the area bounded by $y = mx$ over $[a, x]$ by subtracting the area of the small triangle over $[0, a]$ from the area of the larger triangle over $[0, x]$

$$A(x) = \int_a^x mt\,dt = \frac{1}{2}mx^2 - \frac{1}{2}ma^2. \tag{2}$$

In both illustrations below, the green line illustrates the area function based at a:



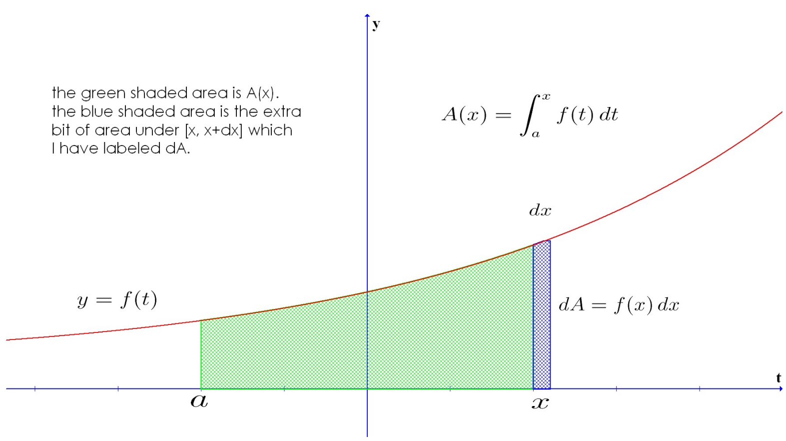


We should notice a pattern:

(1.) $A(x) = \int_a^x c\,dt = cx - ca$ has $\frac{dA}{dx} = c$.

(2.) $A(x) = \int_a^x mt\,dt = \frac{1}{2}mx^2 - \frac{1}{2}ma^2$ has $\frac{dA}{dx} = mx$.

We suspect that if $A(x) = \int_a^x f(t)\,dt$ then $\frac{dA}{dx} = f(x)$. Let's examine an intuitive graphical argument for why this is true for an arbitrary function:



Formally, $dA = A(x+dx) - A(x) = f(x)dx$ hence $dA/dx = f(x)$. This proof made sense to you (if it did) because you believe in Leibniz' notation. We should offer a rigorous proof since this is one of the most important theorems in all of calculus.

Theorem 4.2. *Fundamental Theorem of Calculus part I (FTC I).*

Suppose f is continuous on $[a, b]$ and $x \in [a, b]$ then,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Proof: let $A(x) = \int_a^x f(t)\,dt$ and note that

$$A(x+h) = \int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt = A(x) + \int_x^{x+h} f(t)\,dt$$

Therefore, the difference quotient for the area function is simply as follows:

$$\frac{A(x+h) - A(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

However, note that by continuity of f we can find bounds for f on $J = [x, x+h]$ (if $h > 0$) or $J = [x+h, x]$ (if $h < 0$). By the extreme value theorem, there exist $u, v \in J$ such that $f(u) \leq f(x) \leq f(v)$ for all $x \in J$. Therefore, if $h > 0$, we can apply the inequality properties of definite integrals and find

$$(x+h-x)f(u) \leq \int_x^{x+h} f(t)\,dt \leq (x+h-x)f(v) \quad \Rightarrow \quad f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq f(v)$$

If $h < 0$ then dividing by h reverses the inequalities hence $f(v) \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq f(u)$. Finally, observe that $\lim_{h\to 0} u = x$ and $\lim_{h\to 0} v = x$. Therefore, by continuity of f, $\lim_{h\to 0} f(u) = f(x)$ and $\lim_{h\to 0} f(v) = f(x)$. Remember, $f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq f(v)$ and apply the squeeze theorem to deduce:

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$$

Consequently,

$$\lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = f(x)$$

Which, by definition of the derivative for A, gives $\frac{dA}{dx} = f(x)$. $\square$

The FTC part I is hardly a solution to the area problem. It's just a curious formula. The FTC part II takes this curious formula and makes it useful. It is true there are a few functions defined as area functions hence the differentiation in the FTC I is physically interesting. For example, the Fresnel function can be defined in terms of an integral with a variable bound.

4.2 FTC part II, the standard arguments

We need a word to describe the opposite of taking a derivative:

Definition 4.3. *antiderivative.*

If F is a function for which $\frac{dF}{dx} = f$ then we say F is an **antiderivative** of f.

Examining Equations 1 and 2 we notice the area was given by the difference of the antiderivative of the integrand at the end points:

(1.) $\int_a^x c\,dt = cx - ca$ identifying $f(x) = c$ has antiderivative $F(x) = cx$,
(2.) $\int_a^x mt\,dt = \frac{1}{2}mx^2 - \frac{1}{2}ma^2$ identifying $f(x) = mx$ has antiderivative $F(x) = \frac{1}{2}mx^2$.

FTC II simply says the pattern seen in the above examples is generally true:

Theorem 4.4. *Fundamental Theorem of Calculus part II (FTC II).*

Suppose f is continuous on $[a, b]$ and has antiderivative F then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof: consider the area function based at a: $A(x) = \int_a^x f(t)\,dt$. Since FTC I says that A is an antiderivative of f and $F' = f$ is given we know that $F'(x) = A'(x) = f(x)$. Thus F and A differ by at most a constant $c \in \mathbb{R}$; $F(x) = A(x) + c$. Observe,

$$\begin{aligned} F(b) - F(a) &= (A(b) + c) - (A(a) + c) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt. \end{aligned}$$

Since $\int_a^a f(t)\,dt = 0$. Finally, $\int_a^b f(t)\,dt = \int_a^b f(x)\,dx$ and this completes the proof. $\square$

Definition 4.5. *evaluation bar notation:*

We define $F(x)\Big|_a^b = F(b) - F(a)$.

Example 4.6. *We return to Example 3.3 where we were stuck due to an incalculable summation. We wish to calculate* $\int_0^\pi \sin(x)\,dx$. *Observe that* $F(x) = -\cos(x)$ *has* $F'(x) = \sin(x)$ *hence this is a valid antiderivative for the given integrand* $\sin(x)$. *Apply the FTC part II to find the area:*

$$\int_0^\pi \sin(x)\,dx = F(\pi) - F(0) = -\cos(\pi) + \cos(0) = 2.$$

Obviously this is much easier than calculation from the definition of the Riemann integral.

4.3 FTC part II an intuitive constructive proof

Let me restate the theorem to begin:

FTC II: Suppose f is continuous on $[a,b]$ and has antiderivative F then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof: We seek to calculate $\int_a^b f(x)\,dx$. Use the usual partition for the n-th Riemann sum of f on $[a,b]$; $x_o = a, x_1 = a + \Delta x, \dots, x_n = b$ where $\Delta x = \frac{b-a}{n}$. Suppose that f has an antiderivative F on $[a,b]$. Recall the Mean Value Theorem for $y = F(x)$ on the interval $[x_o, x_1]$ tells us that there exists $x_1^* \in [x_o, x_1]$ such that

$$F'(x_1^*) = \frac{F(x_1) - F(x_o)}{x_1 - x_o} = \frac{F(x_1) - F(x_o)}{\Delta x}$$

Notice that this tells us that $F'(x_1^*)\Delta x = F(x_1) - F(x_o)$. But, $F'(x) = f(x)$ so we have found that $f(x_1^*)\Delta x = F(x_1) - F(x_o)$. In other words, the area under $y = f(x)$ for $x_o \le x \le x_1$ is well approximated by the difference in the antiderivative at the endpoints. Thus we choose the sample points for the n-th Riemann sum by applying the MVT on each subinterval to select x_j^* such that $f(x_j^*)\Delta x = F(x_j) - F(x_{j-1})$. With this construction in mind calculate:

$$\begin{aligned}
\int_a^b f(x)\,dx &= \lim_{n\to\infty}\left(\sum_{j=1}^{n} f(x_j^*)\Delta x\right) \\
&= \lim_{n\to\infty}\left(\sum_{j=1}^{n}\big[F(x_j) - F(x_{j-1})\big]\right) \\
&= \lim_{n\to\infty}\big(F(x_1) - F(x_o) + F(x_2) - F(x_1) + \cdots + F(x_n) - F(x_{n-1})\big) \\
&= \lim_{n\to\infty}\big(F(x_n) - F(x_o)\big) \\
&= \lim_{n\to\infty}\big(F(b) - F(a)\big) \\
&= F(b) - F(a). \square
\end{aligned}$$

This result clearly extends to piecewise continuous functions which have only finite jump discontinuities. We can apply the FTC to each piece and take the sum of those results. This Theorem is amazing. We can calculate the area under a curve based on the values of the antiderivative at the endpoints. Think about that, if $a = 1$ and $b = 3$ then $\int_1^3 f(x)\,dx$ depends only on $F(3)$ and $F(1)$. Doesn't it seem intuitively likely that what value $f(2)$ takes should matter as well? Why don't we have to care about $F(2)$? The values of the function at $x = 2$ certainly went into the calculation of the area, if we calculate a left sum we would need to take values of the function between the endpoints. The cancellation that occurs in the proof is the root of why my naive intuition is bogus.

5 Basic Integration

In this section we study the basic theory of indefinite integration and then apply our results to calculate some definite integrals at the conclusion of this section.

5.1 Indefinite Integration

The key concept to understand non-uniqueness of antiderivatives is simply this:

Proposition 5.1. *antiderivatives differ by at most a constant.*

If f has antiderivatives F_1 and F_2 then there exists $c \in \mathbb{R}$ such that $F_1(x) = F_2(x) + c$.

Proof: Since $\frac{dF_1}{dx} = f(x) = \frac{dF_2}{dx}$ therefore $F_1(x) = F_2(x) + c$. $\square$

To understand the significance of this constant we should consider a physical question.

Example 5.2. *Suppose that the velocity of a particle at position x is measured to be constant. In particular, suppose that $v(t) = \frac{dx}{dt}$ and $v(t) = 1$. The condition $v(t) = \frac{dx}{dt}$ means that x should be an antiderivative of v. For $v(t) = 1$ the form of all antiderivatives is easy enough to guess: $x(t) = t + c$. The value for c cannot be determined unless we are given additional information about this particle. For example, if we also knew that at time zero the particle was at $x = 3$ then we could fit this* **initial data** *to pick a value for c:*

$$x(0) = 0 + c = 3 \quad \Rightarrow \quad c = 3 \quad \Rightarrow \quad \boxed{x(t) = t + 3}.$$

For a given velocity function each antiderivative gives a possible position function. To determine the precise position function we need to know both the velocity and some initial position. Often we are presented with a problem for which we do not know the initial condition so we'd like to have a mathematical device to leave open all possible initial conditions.

Definition 5.3. *indefinite integral.*

If f has an antiderivative F then the **indefinite integral** of f is given by:

$$\int f(x)dx = \{G(x) \mid G'(x) = f(x)\} = \{F(x) + c | c \in \mathbb{R}\}.$$

However, we will customarily drop the set-notation and simply write

$$\int f(x)dx = F(x) + c \ \text{ where } F'(x) = f(x).$$

The indefinite integral includes all possible antiderivatives for the given function.

Remark 5.4. *The indefinite integral is a family of antiderivatives: $\int f(x) = F(x) + c$ where $F'(x) = f(x)$. The following equation shows how indefinite integration is undone by differentiation:*

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

the function f is called the **integrand** *and the variable of indefinite integration is x. Notice the constant is obliterated by the derivative in the equation above. Leibniz' notation intentionally makes* *you think of cancelling the dx's as if they were tiny quantities. Newton called them fluxions. In fact calculus was sometimes called the theory of fluxions in the early 19-th century. Newton had in mind that dx was the change in x over a tiny time, it was a fluctuation with respect to a time implicit. We no longer think of calculus in this way because there are easier ways to think about foundations of calculus. That said, it is still an intuitive notation and if you are careful not to overextend intuition it is a powerful mnemonic. For example, the chain rule $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$. Is the chain rule just from multiplying by one? No. But, it is a nice way to remember the rule.*

Now we turn to theorems which help us calculate indefinite integrals systematically. The theorems we cover here are at the base of the first quarter of Calculus II.

Proposition 5.5. *linearity of indefinite integration.*

Suppose f, g are functions with antiderivatives and $c \in \mathbb{R}$ then

$$\int [f(x) + g(x)]dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

Proof: Suppose $\int f(x)\,dx = F(x) + c_1$ and $\int g(x)\,dx = G(x) + c_2$ note that

$$\frac{d}{dx}[F(x) + G(x)] = \frac{d}{dx}[F(x)] + \frac{d}{dx}[G(x)] = f(x) + g(x)$$

hence $\int [f(x) + g(x)]dx = F(x) + G(x) + c_3 = \int f(x)\,dx + \int g(x)\,dx$ where the constant c_3 is understood to be included in either the $\int f(x)\,dx$ or the $\int g(x)\,dx$ integral. $\square$

Proposition 5.6. *power rule for integration. suppose $n \in \mathbb{R}$ and $n \neq -1$ then*

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c.$$

Proof: $\frac{d}{dx}[\frac{1}{n+1}x^{n+1}] = \frac{n+1}{n+1}x^{n+1-1} = x^n$. Note that $n + 1 \neq 0$ since $n \neq -1$. $\square$

Note that the special case of $n = -1$ stands alone. You should recall that $\frac{d}{dx}\ln(x) = \frac{1}{x}$ provided $x > 0$. In the case $x < 0$ then by the chain rule applied to the positive case: $\frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$. Observe then that for all $x \neq 0$ we have $\frac{d}{dx}\ln|x| = \frac{1}{x}$. Therefore the proposition below follows:

Proposition 5.7. *reciprocal function is special case.*

$$\int \frac{1}{x}\,dx = \ln|x| + c.$$

Note that it is common to move the differential into the numerator of such expressions. We could just as well have written that $\int \frac{dx}{x} = \ln|x| + c$. I leave the proof of the propositions in the remainder of this section to the reader. They are not difficult.

Proposition 5.8. *exponential functions. suppose $a > 0$ and $a \neq 1$,*

$$\int a^x\, dx = \frac{1}{\ln(a)} a^x + c \qquad \text{in particular:} \qquad \int e^x\, dx = e^x + c$$

The exponential function has base $a = e$ and $\ln(e) = 1$ so the formulas are consistent.

Proposition 5.9. *trigonometric functions.*

$$\int \sin(x)\, dx = -\cos(x) + c \qquad \int \cos(x)\, dx = \sin(x) + c$$
$$\int \sec^2(x)\, dx = \tan(x) + c \qquad \int \sec(x)\tan(x)\, dx = \sec(x) + c$$
$$\int \csc^2(x)\, dx = -\cot(x) + c \qquad \int \csc(x)\cot(x)\, dx = -\csc(x) + c.$$

You might notice that many trigonometric functions are missing. For now we are simply making a list of the basic antiderivatives that stem from reading basic derivative rules backwards.

Proposition 5.10. *hyperbolic functions.*

$$\int \sinh(x)\, dx = \cosh(x) + c \qquad \int \cosh(x)\, dx = \sinh(x) + c$$
$$\int \text{sech}^2(x)\, dx = \tanh(x) + c \qquad \int \text{sech}(x)\text{tanh}(x)\, dx = -\text{sech}(x) + c$$
$$\int \text{csch}^2(x)\, dx = -\text{coth}(x) + c \qquad \int \text{csch}(x)\text{coth}(x)\, dx = -\text{csch}(x) + c$$

It is neat to see the parallel between the circular functions and hyperbolic functions. To fully appreciate the interplay you need to take the course in Complex Analysis.

Proposition 5.11. *special algebraic and rational functions*

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c \qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + c.$$
$$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}(x) + c \qquad \int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x) + c.$$
$$\int \frac{dx}{1-x^2} = \tanh^{-1}(x) + c.$$

I should mention there are fascinating expressions which recast the inverse hyperbolic functions as the composite of a logarithm and an algebraic function. In particular:

$$\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right) \qquad \text{for } x \geq 1.$$
$$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right) \qquad \text{for } x \in \mathbb{R}.$$
$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \text{for } |x| < 1.$$

Example 5.12.

$$\int dx = \int x^0 dx = \boxed{x + c}$$

Example 5.13.

$$\int \left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right) dx = \int x^{\frac{1}{2}} dx + \int x^{\frac{-1}{3}} dx = \boxed{\frac{2}{3}x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{2}{3}} + c}$$

Example 5.14.

$$\int \sqrt{13x^7}\, dx = \int \sqrt{13}\sqrt{x^7}\, dx = \sqrt{13} \int x^{7/2} dx = \boxed{\frac{2\sqrt{13}}{9}x^{9/2} + c}$$

Example 5.15.

$$\int \frac{dx}{3x^2} = \frac{1}{3}\int x^{-2} dx = \frac{-1}{3}x^{-1} = \boxed{\frac{-1}{3x} + c}$$

Example 5.16.

$$\int \frac{2x dx}{x^2} = 2\int \frac{dx}{x} = 2\ln|x| + c = \boxed{\ln(x^2) + c}$$

Note that $|x| = \pm x$ thus $|x|^2 = (\pm x)^2 = x^2$ so it was logical to drop the absolute value bars after bringing in the factor of two by the property $\ln(A^c) = c\ln(A)$.

Example 5.17.

$$\int 3e^{x+2} dx = 3\int e^2 e^x dx = 3e^2 \int e^x dx = 3e^2(e^x + c_1) = \boxed{3e^{x+2} + c}$$

Example 5.18.

$$\int \frac{2x^3 + 3}{x} dx = \int \left(\frac{2x^3}{x} + \frac{3}{x}\right) dx = 2\int x^2 dx + 3\int \frac{dx}{x} = \boxed{\frac{2}{3}x^3 + 3\ln|x| + c}$$

Example 5.19.

$$\begin{aligned} \int (x+2)^2 dx &= \int (x^2 + 4x + 4) dx \\ &= \int x^2 dx + 4\int x dx + 4\int dx \\ &= \boxed{\frac{1}{3}x^3 + 2x^2 + 4x + c} \end{aligned}$$

Example 5.20.

$$\int (2^x + 3\cosh(x)) dx = \int 2^x dx + 3\int \cosh(x) dx = \boxed{\frac{1}{\ln(2)}2^x + 3\sinh(x) + c}$$

Example 5.21.

$$\int \frac{x^2}{1+x^2} dx = \int \frac{1 + x^2 - 1}{1 + x^2} dx = \int [1 - \frac{1}{1+x^2}] dx = \boxed{x - \tan^{-1}(x) + c}$$

Example 5.22.

$$\begin{aligned}\int \sin(x+3)dx &= \int \big[\sin(x)\cos(3)+\sin(3)\cos(x)\big]\,dx \\ &= \cos(3)\int \sin(x)dx + \sin(3)\int \cos(x)dx \\ &= -\cos(3)[\cos(x)+c_1] + \sin(3)[\sin(x)+c_2] \\ &= \sin(3)\sin(x) - \cos(3)\cos(x) + c \\ &= \boxed{-\cos(x+3)+c}\end{aligned}$$

Incidentally, we find a better way to do this later with the technique of u-substitution.

Example 5.23.

$$\int \frac{1}{\cos^2(x)}dx = \int \sec^2(x)\,dx = \boxed{\tan x + c}$$

Example 5.24.

$$\int \frac{dx}{x^2+\cos^2(x)+\sin^2(x)} = \int \frac{dx}{x^2+1} = \boxed{\tan^{-1}(x)+c}$$

Example 5.25.

$$\int \frac{\sqrt{x^2-1}}{(x+1)(x-1)}dx = \int \frac{\sqrt{x^2-1}}{x^2-1}dx = \int \frac{dx}{\sqrt{x^2-1}} = \boxed{\cosh^{-1}(x)+c}$$

5.2 examples of definite integration

In each example below we use FTC II; $\int_a^b f(x)dx = F(b) - F(a)$ where $F' = f$.

Example 5.26.

$$\int_1^9 \frac{dx}{\sqrt{5x}} = \frac{1}{\sqrt{5}}\int_1^9 \frac{dx}{\sqrt{x}} = \frac{2\sqrt{x}}{\sqrt{5}}\bigg|_1^9 = \frac{2\sqrt{9}}{\sqrt{5}} - \frac{2\sqrt{1}}{\sqrt{5}} = \boxed{\frac{4}{\sqrt{5}}.}$$

Example 5.27.

$$\int_0^1 2^x\,dx = \frac{1}{\ln(2)}2^x\bigg|_0^1 = \frac{1}{\ln(2)}(2^1-2^0) = \boxed{\frac{1}{\ln(2)}.}$$

Example 5.28. *Let a, b be constants,*

$$\int_a^b \sinh(t)\,dt = \cosh(t)\bigg|_a^b = \boxed{\cosh(b)-\cosh(a).}$$

Example 5.29.

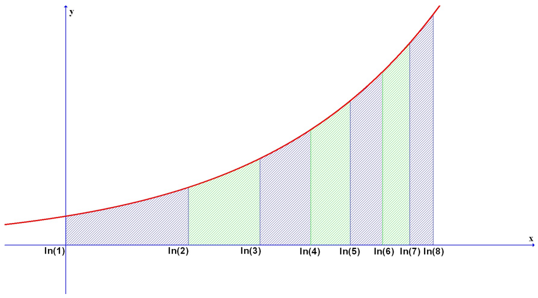
$$\int_{-4}^{-2} \frac{dx}{x} = \ln|x|\bigg|_{-4}^{-2} = \ln|-2| - \ln|-4| = \ln(2) - \ln(4) = \boxed{\ln(1/2).}$$

If we had neglected the absolute value function in the antiderivative then we would have obtained an incorrect result. The absolute value bars are important for this integral. Note the answer is negative here because $y = 1/x$ is under the x-axis in the region $-4 \leq x \leq -2$.

Example 5.30. *Let $n > 0$ and consider,*

$$\int_{\ln(n)}^{\ln(n+1)} e^x \, dx = e^{\ln(n+1)} - e^{\ln(n)} = n + 1 - n = \boxed{1.}$$

This is an interesting result. I've graphed a few examples of it below. Notice how as n increases the distance between $\ln(n)$ and $\ln(n+1)$ decreases, yet the exponential increases such that the bounded area still works out to one-unit.



Remark 5.31. *To calculate the area bounded by $y = f(x)$ for $a \leq x \leq b$ we calculate*

$$Area = \int_a^b |f(x)| \, dx.$$

Example 5.32. *Calculate the area bounded by $y = \cos(x)$ on $0 \leq x \leq \frac{5\pi}{2}$.*

$$\begin{aligned}
\int_0^{5\pi/2} |\cos(x)| dx &= \int_0^{\frac{\pi}{2}} \cos(x) dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(x) dx + \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \cos(x) dx \\
&= \sin(x)\Big|_0^{\frac{\pi}{2}} - \sin(x)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \sin(x)\Big|_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \\
&= \sin\left(\frac{\pi}{2}\right) - \sin(0) - \left(\sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right) + \sin\left(\frac{5\pi}{2}\right) - \sin\left(\frac{3\pi}{2}\right) \\
&= \boxed{5.}
\end{aligned}$$

Example 5.33. *Calculate the area bounded by $f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ -e^x & x > 1 \end{cases}$ over the interval $[0, \ln 4]$.*

$$\begin{aligned}
\int |f(x)| \, dx &= \int_0^1 |x^2| dx + \int_1^{\ln 4} |-e^x| dx \\
&= \int_0^1 x^2 dx + \int_1^{\ln 4} e^x dx \\
&= \frac{1}{3}x^3\Big|_0^1 + e^x\Big|_1^{\ln 4} \\
&= \frac{1}{3} + e^{\ln 4} - e^1 \\
&= \boxed{\frac{13}{3} - e}
\end{aligned}$$

6 u-substitution

The integrations we have done up to this point have been elementary. Basically all we have used is linearity of integration and our basic knowledge of differentiation. We made educated guesses as to what the antiderivative was for a certain class of rather special functions. Integration requires that you look ahead to the answer before you get there. For example, $\int \sin(x)\,dx$. To reason this out we think about our basic derivatives, we note that the derivative of $\cos(x)$ gives $-\sin(x)$ so we need to multiply our guess by -1 to fix it. We conclude that $\int \sin(x)\,dx = -\cos(x) + c$. The logic of this is essentially educated guessing. You might be a little concerned at this point. Is that all we can do? Just guess? Well, no. There is more. But, those basic guesses remain, They form the basis for all elementary integration theory.

The new idea we look at in this section is called "u-substitution". It amounts to the reverse chain rule. The goal of a properly posed u-substitution is to change the given integral to a new integral which is elementary. Typically we go from an integration in x which seems incalculable to a new integration in u which is elementary. For the most part we will make direct substitutions, these have the form $u = g(x)$ for some function g however, this is not strictly speaking the only sort of substitution that can be made. Implicitly defined substitutions such as $x = f(\theta)$ play a critical role in many interesting integrals, we will deal with those more subtle integrations when we cover the technique of trigonometric and hyperbolic substitution.

Finally, I should emphasize that when we do a u-substitution we must be careful to convert each and every part of the integral to the new variable. This includes both the integrand($f(x)$) and the measure(dx) in an indefinite integral $\int f(x)\,dx$. Or the integrand($f(x)$), measure(dx) and upper and lower bounds a, b in a definite integral $\int_a^b f(x)\,dx$. I will forego a careful proof in this handout, please ask if interested (I have proof in my *Calculus !* notes).

6.1 u-substitution in indefinite integrals

Example 6.1.

$$\int xe^{x^2}\,dx = \int xe^u \frac{du}{2x} \qquad \boxed{\textit{let } u = x^2,\ \frac{du}{dx} = 2x \textit{ and } dx = \frac{du}{2x}}$$

$$= \frac{1}{2}\int e^u du \qquad \textit{see how all the x's cancelled, this has to happen.}$$

$$= \frac{1}{2}e^u + c \qquad \textit{not done yet.}$$

$$= \boxed{\frac{1}{2}e^{x^2} + c} \qquad \textit{differentiate to check if in doubt.}$$

Example 6.2. *Let a, b be constants. If $a \neq 0$ then,*

$$\int (ax+b)^{13}dx = \int u^{13}\frac{du}{a} \qquad \boxed{\textit{let } u = ax+b,\ \frac{du}{dx} = a \textit{ and } dx = \frac{du}{a}}$$

$$= \frac{1}{14a}u^{14} + c \qquad \textit{not done yet.}$$

$$= \boxed{\frac{1}{14a}(ax+b)^{14} + c}$$

Example 6.3.

$$\int 5^{\frac{x}{3}}\, dx = \int 5^u (3du)$$
$$= \frac{3}{\ln(5)} 5^u + c$$
$$= \boxed{\frac{3}{\ln(5)} 5^{\frac{x}{3}} + c.}$$

$\boxed{\textit{let } u = \frac{x}{3},\ \frac{du}{dx} = \frac{1}{3} \textit{ and } dx = 3du}$

Example 6.4.

$$\int \tan x dx = \int \frac{\sin x\, dx}{\cos x} = \int \frac{-du}{u}$$
$$= -\ln|u| + c$$
$$= -\ln|\cos x| + c$$
$$= \boxed{\ln|\sec x| + c.}$$

$\boxed{\textit{let } u = \cos x \textit{ then } \sin x\, dx = -du}$

In case you're wondering, $-\ln|\cos x| = \ln|\cos x|^{-1} = \ln\left(\frac{1}{|\cos x|}\right) = \ln\left|\frac{1}{\cos x}\right| = \ln|\sec x|$.

Example 6.5.

$$\int \frac{2x}{1+x^2} dx = \int \frac{du}{u}$$
$$= \ln(|u|) + c$$
$$= \boxed{\ln(1+x^2) + c.}$$

$\boxed{\textit{let } u = 1 + x^2,\ \frac{du}{dx} = 2x \textit{ and } 2xdx = du}$

Notice that $x^2 + 1 > 0$ *for all* $x \in \mathbb{R}$ *thus* $|x^2+1| = x^2+1$. *We should only drop the absolute value bars if we have good reason.*

Example 6.6.

$$\int \frac{dx}{x+b} = \int \frac{du}{u}$$
$$= \ln|u| + c$$
$$= \boxed{\ln|x+b| + c.}$$

$\boxed{\textit{let } u = x + b \textit{ thus } du = dx}$

Example 6.7. *suppose* $x > 0$.

$$\int \frac{x^2 dx}{\sqrt{x^2 - x^4}} = \int \frac{x^2 dx}{x\sqrt{1-x^2}}$$
$$= \int \frac{xdx}{\sqrt{1-x^2}}$$
$$= \int \frac{-du}{2\sqrt{u}}$$
$$= \frac{-1}{2} 2\sqrt{u} + c$$
$$= \boxed{-\sqrt{1-x^2} + c.}$$

$\boxed{\textit{let } u = 1 - x^2 \textit{ thus } -du/2 = xdx}$

Example 6.8.

$$\int \sin(3\theta)d\theta = \int \sin(u)\frac{du}{3} \qquad \boxed{\textit{let } u = 3\theta \textit{ thus } d\theta = \frac{du}{3}}$$
$$= \frac{-1}{3}\cos(u) + c$$
$$= \boxed{\frac{-1}{3}\cos(3\theta) + c.}$$

Example 6.9. *Suppose $x > 0$, use a $u = \ln x$ substitution for which $du = \frac{dx}{x}$ to calculate:*

$$\int \frac{\ln(x)dx}{x} = \int u\,du = \frac{1}{2}u^2 + c = \boxed{\frac{1}{2}(\ln(x))^2 + c}$$

Example 6.10. *Let $u = \sin^{-1}(z)$ then $du = \frac{dz}{\sqrt{1-z^2}}$ hence*

$$\int \frac{dz}{\sin^{-1}(z)\sqrt{1-z^2}} = \int \frac{du}{u} = \ln|u| + c = \boxed{\ln(\sin^{-1}(z)) + c}$$

Example 6.11.

$$\int t\cos(t^2+\pi)dt = \frac{1}{2}\int \cos(u)du \qquad \boxed{\textit{let } u = t^2+\pi \textit{ thus } tdt = \frac{du}{2}}$$
$$= \frac{1}{2}\sin(u) + c$$
$$= \boxed{\frac{1}{2}\sin(t^2+\pi) + c.}$$

Example 6.12. *suppose $a \neq 0$*

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{dx}{\frac{x^2}{a^2}+1}$$
$$= \frac{1}{a^2}\int \frac{adu}{u^2+1} \qquad \boxed{\textit{let } u = \frac{x}{a} \textit{ thus } adu = dx}$$
$$= \frac{1}{a}\tan^{-1}(u) + c$$
$$= \boxed{\frac{1}{a}\tan^{-1}\left[\frac{x}{a}\right] + c.}$$

Example 6.13. *suppose $a \neq 0$*

$$\int \cos(ae^x+3)e^x dx = \frac{1}{a}\int \cos(u)du \qquad \boxed{\textit{let } u = ae^x+3 \textit{ thus } du/a = e^x dx}$$
$$= \frac{1}{a}\sin(u) + c$$
$$= \boxed{\frac{1}{a}\sin(ae^x+3) + c.}$$

6.2 u-substitution in definite integrals

There are two ways to do these. I expect you understand both methods.

(1.) Find the antiderivative via u-substitution and then use the FTC to evaluate in terms of the given upper and lower bounds in x. (see Example 6.14 below)

(2.) Do the u-substitution and change the bounds all at once, this means you will use the FTC and evaluate the upper and lower bounds in u. (see Example 6.15 below)

I will deduct points if you write things like a definite integral is equal to an indefinite integral (just leave off the bounds during the u-substitution). The notation is not decorative, it is necessary and important to use correct notation.

Example 6.14. *We previously calculated that* $\int t\cos(t^2+\pi)dt = \frac{1}{2}\sin(t^2+\pi)+c$. *We can use this together with the FTC to calculate the following definite integral:*

$$\begin{aligned}\int_0^{\sqrt{\frac{\pi}{2}}} t\cos(t^2+\pi)dt &= \frac{1}{2}\sin(t^2+\pi)\Big|_0^{\sqrt{\frac{\pi}{2}}} \\ &= \frac{1}{2}\sin(\tfrac{\pi}{2}+\pi) - \frac{1}{2}\sin(\pi) \\ &= \boxed{\frac{-1}{2}.}\end{aligned}$$

This illustrates method (1.) we find the antiderivative off to the side then calculate the integral using the FTC in the x-variable. Well, the t-variable here. This is a two-step process. In the next example I'll work the same integral using method (2.). In contrast, that is a one-step process but the extra step is that you need to change the bounds in that scheme. Generally, some problems are easier with both methods. Also, sometimes you may be faced with an abstract question which demands you understand method 2.).

Example 6.15.

$$\begin{aligned}\int_0^{\sqrt{\frac{\pi}{2}}} t\cos(t^2+\pi)dt &= \frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} \cos(u)du \\ &= \frac{1}{2}\sin(u)\Big|_{\pi}^{\frac{3\pi}{2}} \\ &= \frac{1}{2}\sin(\tfrac{3\pi}{2}) - \frac{1}{2}\sin(\pi) \\ &= \boxed{\frac{-1}{2}.}\end{aligned}$$

$\boxed{\textit{let } u = t^2+\pi \textit{ thus } tdt = \frac{du}{2}}$

$\boxed{\textit{also } u\left(\sqrt{\tfrac{\pi}{2}}\right) = \frac{3\pi}{2} \textit{ and } u(0) = \pi}$

Example 6.16.

$$\begin{aligned}\int_{4\pi^2}^{9\pi^2} \frac{\sin(\sqrt{x})dx}{\sqrt{x}} &= \int_{2\pi}^{3\pi} \sin(u)(2du) \\ &= -2\cos(u)\Big|_{2\pi}^{3\pi} \\ &= -2\cos(3\pi) + 2\cos(2\pi) \\ &= \boxed{4.}\end{aligned}$$

$\boxed{\textit{let } u = \sqrt{x} \textit{ thus } 2du = \frac{dx}{\sqrt{x}}}$

$\boxed{\textit{also } u(9\pi^2) = \sqrt{9\pi^2} = 3\pi \textit{ and } u(4\pi^2) = \sqrt{4\pi^2} = 2\pi}$