3.8. PRECISE DEFINITION OF LIMIT

You might read the article by Dr. Monty C. Kester posted on Blackboard. It helps motivate the definition I give now.

Definition 3.8.1: We say that the limit of a function f at $a \in \mathbb{R}$ exists and is equal to $L \in \mathbb{R}$ iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$

Notice we do not require that the limit point be in the domain of the function. The zero in $0 < |x - a| < \delta$ is precise way of saying that we do not consider the limit point in the limit. All other x that are within δ units of the limit point a are included in the analysis (recall that $|a - b|$ gives the distance from a to b on the number line). If the limit exists then we can choose the δ such that the values $f(x)$ are within ϵ units of the limiting value L.

Example 3.8.1: Prove that

$$\lim_{x \to 3}(2x) = 6.$$

Let us examine what we need to produce. We need to find a δ such that

$$|x - 3| < \delta \implies |2x - 6| < \epsilon$$

The way this works is that ϵ is chosen to start the proof so we cannot adjust ϵ, however the value for δ we are free to choose. But, whatever we choose it must do the needed job, it must make the implication hold true. I usually look at what I want to get in the end and work backwards. We want, $|2x - 6| < \epsilon$. Notice

$$|2x - 6| = |2(x - 3)| = |2||x - 3| = 2|x - 3| < 2\delta$$

If we choose δ such that $2\delta = \epsilon$ then it should work. So we will want to use $\delta = \epsilon/2$ in our proof. Let's begin the proof:

Let $\epsilon > 0$ *choose* $\delta = \epsilon/2$. *If* $x \in \mathbb{R}$ *such that* $0 < |x - 3| < \delta$ *then*

$$|2x - 6| = 2|x - 3| < 2\delta = 2(\epsilon/2) = \epsilon.$$

Therefore, $\lim_{x \to 3}(2x) = 6$.

I put the proof in italics to alert you to the fact that the rest of this jibber-jabber was just to prepare for the proof. Often a textbook will just give the proof and leave it to the reader to figure out how the proof was concocted.

I'll now give a formal proof that the limit is linear. This proof I include to show you how these things are argued, you are responsible for problems more like the easy example unless I specifically say otherwise. If I were to put this on a test I'd warn you it was coming (or it would be a bonus question)

Proposition 3.8.2: If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ then $\lim_{x\to a}[f(x)+g(x)] = L_1 + L_2$. In other words,

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

Proof: Let $\epsilon > 0$ *and assume that* $\lim_{x\to a} f(x) = L_1$ *and* $\lim_{x\to a} g(x) = L_2$.

Clearly $\epsilon/2 > 0$ *thus as* $\lim_{x\to a} f(x) = L_1$ *there exists* $\delta_1 > 0$ *such that* $|x-a| < \delta_1$ *implies* $|f(x) - f(a)| < \epsilon/2$*. Likewise, as* $\lim_{x\to a} g(x) = L_2$ *there exists* $\delta_2 > 0$ *such that* $|x-a| < \delta_2$ *implies* $|g(x) - g(a)| < \epsilon/2$*.*

Define $\delta_3 = min(\delta_1, \delta_2)$*. Suppose* $x \in \mathbb{R}$ *such that* $0 < |x-a| < \delta_3$ *then* $|x-a| < \delta_1$ *and* $|x-a| < \delta_2$ *because* $\delta_3 \leq \delta_1, \delta_2$*. Consider then,*

$$\begin{aligned} |(f+g)(x) - (L_1+L_2)| &= |f(x) - L_1 + g(x) - L_2| \\ &\leq |f(x) - L_1| + |g(x) - L_2| \\ &< \epsilon/2 + \epsilon/2 \\ &< \epsilon \end{aligned}$$

Hence, for each $\epsilon > 0$ *there exists* $\delta_3 > 0$ *such that* $|(f+g)(x) - (L_1+L_2)| < \epsilon$ *whenever* $|x-a| < \delta_3$*. Thus,* $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$*.*

The proof I just gave may leave you with some questions. Such as:

- Where did the $\epsilon/2$ come from ?

- Where did the $\delta_3 = min(\delta_1, \delta_2)$ come from?

Short answer, imagination. Longer answer, we typically work these sort of proofs backwards as in Example 3.8.1.

As I said before, you start with what you want to show then determine how you should use the given data to prove the conclusion. There are a few facts which are helpful in these sorts of arguments. Let's make a collection:

- If $a < b$ and $b < c$ then $a < c$.
- Let $\delta > 0$. If $a < b$ then $a\delta < b\delta$. (preserved inequality)
- Let $\gamma < 0$. If $a < b$ then $\gamma a > \gamma b$. (reversed inequality)
- $|-a| = |a|$
- $|ab| = |a||b|$
- $|a| \geq 0$
- $0 \leq |x - a|$
- Let $\delta > 0$ then $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta$
- The triangle inequality; $|a + b| \leq |a| + |b|$
- $|a - b| \leq |a| + |b|$
- Let $\delta > 0$ then if we add to the denominator of some fraction it makes the fraction smaller: (assuming $b > 0$)

$$\frac{a}{b+\delta} < \frac{a}{b}$$

- Let $\delta > 0$ then if we subtract from the denominator of some fraction it makes the fraction larger: (assuming $b > 0$)

$$\frac{a}{b-\delta} > \frac{a}{b}$$

Now, I doubt we will use all these tricks. In include them here because if you do take a course in real analysis you'll need to know these things. Sadly, not all real analysis books make any attempt to organize or be clear about these basic tools. (I speak from bad experience) Enough about all that let's try some more examples.

Example 3.8.2: (this is the bow-tie proof) Prove that

$$\lim_{x\to 3}(4x-5)=7.$$

For each $\epsilon > 0$ we need to find a δ such that

$$|x-3|<\delta \implies |(4x-5)-7|<\epsilon$$

Observe, given that $|x-3|<\delta$ we have

$$|4x-12| = 4|x-3| < 4\delta$$

If we choose δ such that $4\delta = \epsilon$ then it should work. So we will want to use $\delta = \frac{\epsilon}{4}$ in our proof. Let's begin the proof:

Let $\epsilon > 0$ *choose* $\delta = \epsilon/4$. *If* $x \in \mathbb{R}$ *such that* $|x-3|<\delta$ *then*

$$|f(x)-L| = |(4x-5)-7| = |4x-12| = 4|x-3| < 4\delta = \epsilon.$$

Therefore, $\lim_{x\to 3}(4x-5)=7$.

I put the proof in italics to alert you to the fact that the rest of this jibber-jabber was just to prepare for the proof. Often a textbook will just give the proof and leave it to the reader to figure out how the proof was concocted.

The text also discusses a technical definition for what is meant by limits that go to infinity or negative infinity. We will not cover those this semester. You will have a problem like Example 3.8.1 or 3.8.2 on the first test. It will be worth 10 points.

Example 3.8.3: Prove that

$$\lim_{x \to 0}(x^2) = 0.$$

For each $\epsilon > 0$ we need to find a δ such that

$$0 < |x - 0| < \delta \implies |x^2| < \epsilon$$

Observe, given that $|x| < \delta$ we have

$$|x^2| = |x||x| < \delta\delta = \delta^2$$

If we choose δ such that $\delta^2 < \epsilon$ then it should work. So we will want to use $\delta = \sqrt{\epsilon}$ in our proof. Let's begin the proof:

Let $\epsilon > 0$ *choose* $\delta = \sqrt{\epsilon}$. *If* $x \in \mathbb{R}$ *such that* $|x| < \delta$ *then*

$$|f(x) - L| = |x^2| = |x|^2 < \delta^2 = (\sqrt{\epsilon})^2 = \epsilon.$$

Therefore, $\lim_{x \to 0}(x^2) = 0$.

I put the proof in italics again as to emphasize the distinction between preparing for the proof and stating the proof.