Chapter 12

Curves

In this short chapter we analyze the concept of a curve from the three major viewpoints prevalent in analytic geometry. For now, we use only Cartesian coordinates[1]. The three views of a curve are:

1. a graph of a function; $graph(f) = \{(x, y) \mid y = f(x), x \in dom(f)\}$
2. a level curve; $C_k = \{(x, y) \mid F(x, y) = k\}$
3. a parametrized curve; $C = \{(x(t), y(t)) \mid t \in J \subseteq \mathbb{R}\}$

These views are not mutually exclusive and each has their advantange and disadvantage. We desire you understand all three in this course. Experiment and question is key, you have to discover these concepts for yourself. I'll tell you what I think, but don't stop with my comments. Think. Ask your own questions.

12.1 graphs

Let's begin by reminding ourselves of the definition of a graph:

Definition 12.1.1. *Graph of a function.*

Let $f : dom(f) \to \mathbb{R}$ be a function then

$$graph(f) = \{(x, f(x)) \mid x \in dom(f)\}.$$

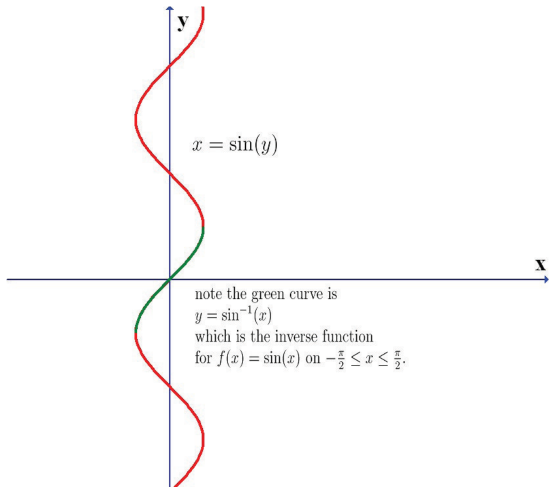
We know this is quite restrictive. We must satisfy the vertical line test if we say our curve is the graph of a function.

Example 12.1.2. *To form a circle centered at the origin of radius R we need to glue together two graphs. In particular we solve the equation $x^2 + y^2 = R^2$ for $y = \sqrt{R^2 - x^2}$ or $y = -\sqrt{R^2 - x^2}$. Let $f(x) = \sqrt{R^2 - x^2}$ and $g(x) = -\sqrt{R^2 - x^2}$ then we find $graph(f) \cup graph(g)$ gives us the whole circle.*

Example 12.1.3. *On the other hand, if we wish to describe the set of all points such that $\sin(y) = x$ we also face a similar difficulty if we insist on functions having independent variable x. Naturally, if we allow for functions with y as the independent variable then $f(y) = \sin(y)$ has graph $graph(f) = \{(f(y), y) \mid y \in$*

[1] all of these constructions find parallel versions when other coordinates such as polar, skew-linear or hyperbolics are used to describe $\mathbb{R}^2$.

$dom(f)\}$. You might wonder, is this correct? I would say a better question is, "is this allowed?". Different books are more or less imaginative about what is permissible as a function. This much we can say, if a shape fails both the vertical and horizontal line tests then it is not the graph of a single function of x or y.



Example 12.1.4. *Let $f(x) = mx + b$ for some constants m, b then $y = f(x)$ is the line with slope m and y-intercept b.*

12.2 level curves

Level curves are amazing. The full calculus of level curves is only partially appreciated even in calculus III, but trust me, this viewpoint has many advantages as you learn more. For now it's simple enough:

Definition 12.2.1. *Level Curve.*

> A level curve is given by a function of two variables $F : dom(F) \subseteq \mathbb{R}^2 \to \mathbb{R}$ and a constant k. In particular, the set of all $(x, y) \in \mathbb{R}^2$ such that $F(x, y) = k$ is called the level-set of F, but more commonly we just say $F(x, y) = k$ is a level curve.

In an algebra class you might have called this the "graph of an equation", but that terminology is dead to us now. For us, it is a level curve. Moreover, for a particular set of points $C \subseteq \mathbb{R}^2$ we can find more than one function F which produces C as a level set. Unlike functions, for a particular curve there is not just one function which returns that curve. This means that it might be important to give both the level-function F and the level k to specify a level curve $F(x, y) = k$.

Example 12.2.2. *A circle of radius R centered at the origin is a level curve $F(x, y) = R^2$ where $F(x, y) = x^2 + y^2$. We call F the level function (of two variables).*

Example 12.2.3. *To describe $\sin(y) = x$ as a level curve we simply write $\sin(y) - x = 0$ and identify the level function is $F(x, y) = \sin(y) - x$ and in this case $k = 0$. Notice, we could just as well say it is the level curve $G(x, y) = 1$ where $G(x, y) = x - \sin(y) + 1$.*

Note once more this type of ambiguity is one distinction of the level curve langauge, in constrast, the graph *$graph(f)$* of a function $y = f(x)$ and the function f are interchangeable. Some mathematicians insist the rule $x \mapsto f(x)$ defines a function whereas others insist that a function is a set of pairs $(x, f(x))$. I prefer the mapping rule because it's how I think about functions in general whereas the idea of a graph is much less useful in general.

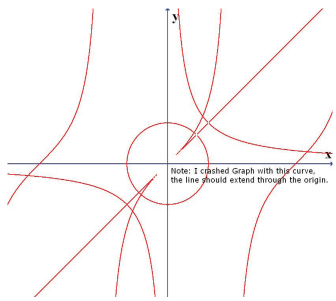
Example 12.2.4. *A line with slope m and y-intercept b can be described by $F(x, y) = mx + b - y = 0$. Alternatively, a line with x-intercept x_o and y-intercept y_o can be described as the level curve $G(x, y) = \frac{x}{x_o} + \frac{y}{y_o} = 1$.*

Example 12.2.5. *Level curves need not be simple things. They can be lots of simple things glued together in one grand equation:*

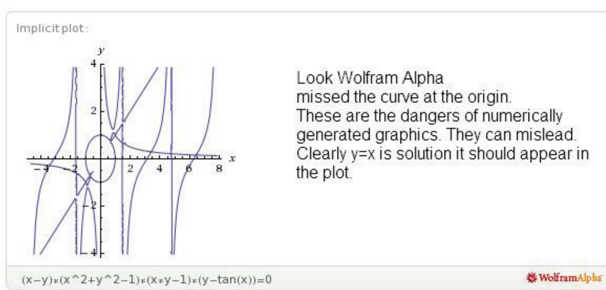
$$F(x, y) = (x - y)(x^2 + y^2 - 1)(xy - 1)(y - \tan(x)) = 0.$$

Solutions to the equation above include the line $y = x$, the unit circle $x^2 + y^2 = 1$, the tilted-hyperbola known more commonly as the reciprocal function $y = \frac{1}{x}$ and finally the graph of the tangent. Some of these intersect, others are disconnected from each other.

It is sometimes helpful to use software to plot equations. However, we must be careful since they are not as reliable as you might suppose. The example above is not too complicated but look what happens with Graph:



Wolfram Alpha shares the same fate:



I hope Mathematica proper fairs better...

Theorem 12.2.6. *any graph of a function can be wriiten as a level curve.*

If $y = f(x)$ is the graph of a function then we can write $F(x, y) = f(x) - y = 0$ hence the graph $y = f(x)$ is also a level curve.

Not much of a theorem. But, it's true. The converse is not true without a lot of qualification. Its a little harder. Basically, the theorem below gives a criteria for when we can undo a level curve and rewrite is as a single function of x.

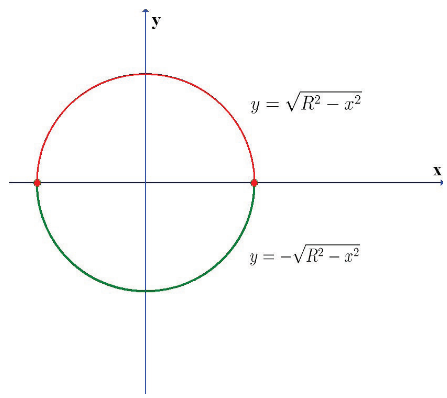
Theorem 12.2.7. *sometimes a level curve can be locally represented as the graph of a function.*

Suppose (x_o, y_o) is a point on the level curve $F(x, y) = k$ hence $F(x_o, y_o) = k$. We say the level curve $F(x, y) = k$ is **locally represented by a function** $y = f(x)$ at (x_o, y_o) iff $F(x, f(x)) = k$ for all $x \in B_\delta(x_o)$ for some $\delta > 0$. Claim: if

$$\frac{\partial F}{\partial y}(x_o, y_o) = \left(\frac{d}{dy} F(x_o, y)\right)\bigg|_{y=y_o} \neq 0$$

and the $\frac{\partial F}{\partial y}$ is continuous near (x_o, y_o) then $F(x, y) = k$ is locally represented by some function near (x_o, y_o).

The theorem above is called the **implicit function theorem** and its proof is nontrivial. Its proper statement is given in Advanced Calculus (Math 332). I'll just illustrate with the circle: $F(x, y) = x^2 + y^2 = R^2$ has $\frac{\partial F}{\partial y} = 2y$ which is continuous everywhere, however at $y = 0$ we have $\frac{\partial F}{\partial y} = 0$ which means the implicit function theorem might fail. On the circle, $y = 0$ when $x = \pm R$ which are precisely the points where we cannot write $y = f(x)$ for just one function. For any other point we may write either $y = \sqrt{R^2 - x^2}$ or $y = -\sqrt{R^2 - x^2}$ as a local solution of the level curve.



Remark 12.2.8. *finding the formula for a local solution generally a difficult problem.*

The implicit function theorem is an existence theorem. It merely says there exists a solution given a certain criteria, howeve it does not tell us how to solve the equation to find the formula for the local function. Sometimes we are just content to have an equation which implicitly defines a function of x. For example, most sane creatures do not try to solve $y^5 + y^4 + 3y^2 - y + 1 = x$ for y. Or $\sin(xy) = 3 + y$. When faced with such curves we prefer the level curve description. We've already done some work on this in Calculus I when we did implicit differentiation. The idea of implicit differentiation is only logical when we can write $y = f(x)$, so we must rely on the implicit function theorem as a backdrop to any implicit differentiation problem (see §4.10 if you forgot about implicit differentiation).

12.3 parametrized curves

The idea of a parametrized curve is probably simpler than the definition below appears. In short, we want to take the real number line, or some subset, and paste it into the plane. Think of taking a string and placing it on a table. You can place the string in a great variety of patterns. Imagine the string has little markers placed along it which say $t = 1$, $t = 2$ etc... the value of this label will tell us at which point we are on the string[2]. That label is called the "parameter".

Definition 12.3.1. *Parametrization of a curve.*

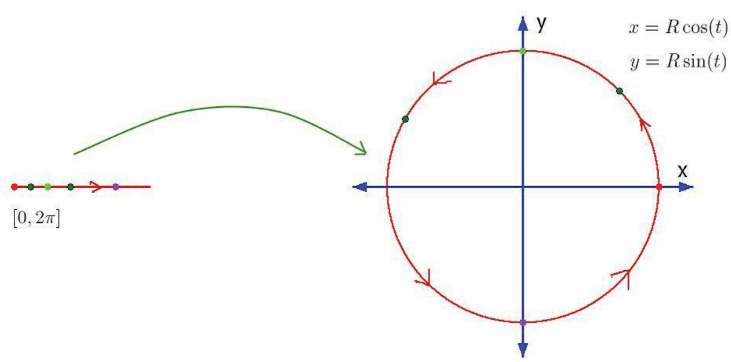
Let C be some curve in the plane. A parametrization of the curve C is a pair of functions $f, g : J \subseteq \mathbb{R} \to \mathbb{R}$ such that $C = \{(f(t), g(t)) \mid t \in J\}$. In other words, a parametric curve is a mapping from $J \subseteq \mathbb{R} \to \mathbb{R}^2$ given by the rule $t \mapsto (f(t), g(t))$ for each $t \in J$. We say that t is the parameter and that the parametric equations for the curve are $x = f(t)$ and $y = g(t)$. In the case $J = [a, b]$ we say that $(f(a), g(a))$ is the initial point and $(f(b), g(b))$ is the terminal point. We often use the notation x for f and y for g when it is convenient. Furthermore, the notation t is just one choice for the label of the parameter, we also may use s or λ or other demarcations.

Finding the parametric equations for a curve does require a certain amount of creativity. However, it's almost always some slight twist on the examples I give in this section.

Example 12.3.2. *Let $x = R\cos(t)$ and $y = R\sin(t)$ for $t \in [0, 2\pi]$. This is a parametrization of the circle of radius R centered at the origin. We can check this by substituting the equations back into our standard Cartesian equation for the circle:*

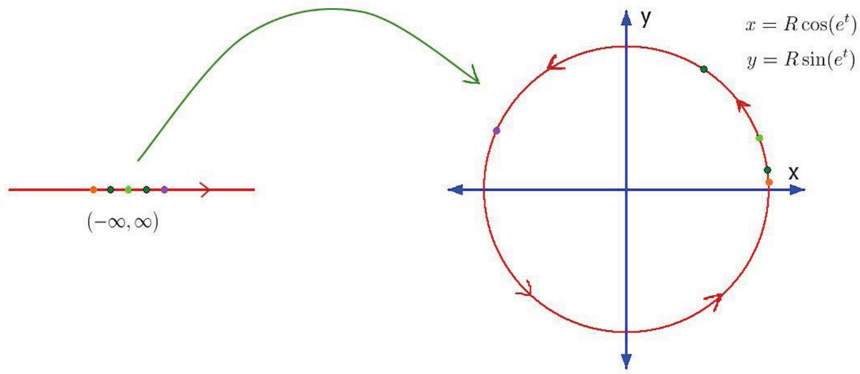
$$x^2 + y^2 = (R\cos(t))^2 + (R\sin(t))^2 = R^2(\cos^2(t) + \sin^2(t))$$

Recall that $\cos^2(t) + \sin^2(t) = 1$ therefore, $x(t)^2 + y(t)^2 = R^2$ for each $t \in [0, 2\pi]$. This shows that the parametric equations do return the set of points which we call a circle of radius R. Moreover, we can identify the parameter in this case as the standard angle from standard polar coordinates.



[2]the parameter can also be thought of as time as the next chapter discusses at length

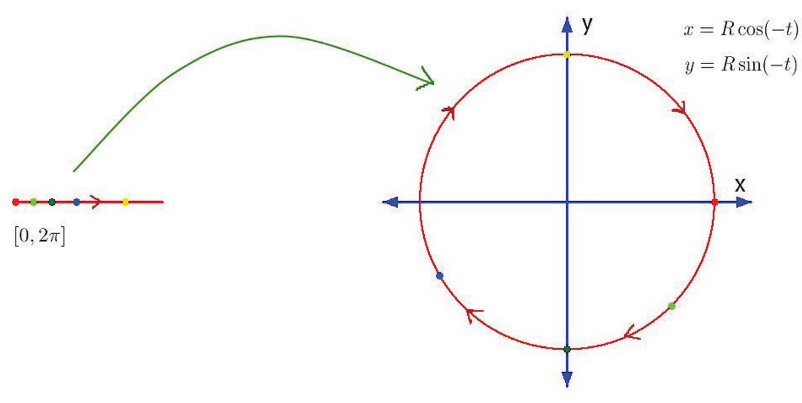
Example 12.3.3. *Let $x = R\cos(e^t)$ and $y = R\sin(e^t)$ for $t \in \mathbb{R}$. We again cover the circle at t varies since it is still true that $(R\cos(e^t))^2 + (R\sin(e^t))^2 = R^2(\cos^2(e^t) + \sin(e^t)) = R^2$. However, since range$(e^t) = [1, \infty)$ it is clear that we will actually wrap around the circle infinitly many times. The parametrizations from this example and the last do cover the same set, but they are radically different parametrizations: the last example winds around the circle just once whereas this example winds around the circle ∞-ly many times.*



Example 12.3.4. *Let $x = R\cos(-t)$ and $y = R\sin(-t)$ for $t \in [0, 2\pi]$. This is a parametrization of the circle of radius R centered at the origin. We can check this by substituting the equations back into our standard Cartesian equation for the circle:*

$$x^2 + y^2 = (R\cos(-t))^2 + (R\sin(-t))^2 = R^2(\cos^2(-t) + \sin^2(-t))$$

Recall that $\cos^2(-t) + \sin^2(-t) = 1$ therefore, $x(t)^2 + y(t)^2 = R^2$ for each $t \in [0, 2\pi]$. This shows that the parametric equations do return the set of points which we call a circle of radius R. Moreover, we can identify the parameter an angle measured CW[3] from the positive x-axis. In contrast, the standard polar coordinate angle is measured CCW from the postive x-axis. Note that in this example we cover the circle just once, but the direction of the curve is opposite that of Example 12.3.2.



The idea of directionality is not at all evident from Cartesian equations for a curve. Given a graph $y = f(x)$ or a level-curve $F(x, y) = k$ there is no intrinsic concept of direction ascribed to the curve. For example, if I

[3] CW is an abbreviation for ClockWise,whereas CCW is an abbreviation for CounterClockWise.

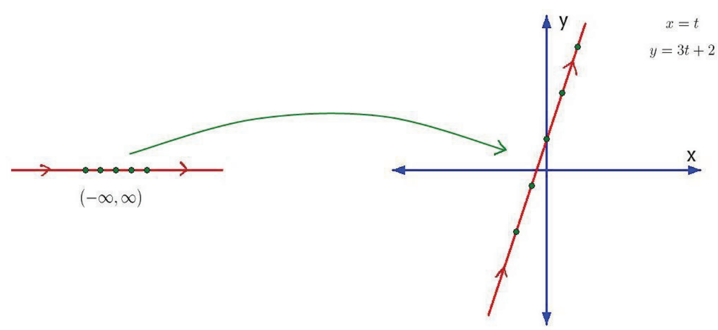
ask you whether $x^2 + y^2 = R^2$ goes CW or CCW then you ought not have an answer. I suppose you could ad-hoc pick a direction, but it wouldn't be natural. This means that if we care about giving a direction to a curve we need the concept of the parametrized curve. We can use the ordering of the real line to induce an ordering on the curve.

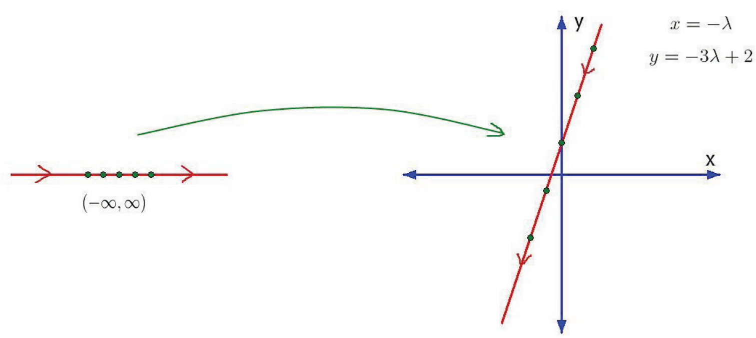
Definition 12.3.5. *oriented curve.*

Suppose $f, g : J \subseteq \mathbb{R} \to \mathbb{R}$ are $1-1$ functions. We say the set $\{(f(t), g(t)) \mid t \in J\}$ is an **oriented curve** and say $t \to (f(t), g(t))$ is a consistently oriented **path** which covers C. If $J = [a, b]$ and $(f(a), g(a)) = p$ and $(f(b), g(b)) = q$ then we can say that C is a curve from to p to q.

I often illustrate the orientation of a curve by drawing little arrows along the curve to indicate the direction. Furthermore, in my previous definition of parametrization I did not insist the parametric functions were $1-1$, this means that those parametrizations could reverse direction and go back and forth along a given curve. What is meant by the terms "path", "curve" and "parametric equations" may differ from text to text so you have to keep a bit of an open mind and try to let context be your guide when ambguity occurs. I will try to be uniform in my langauge within this course.

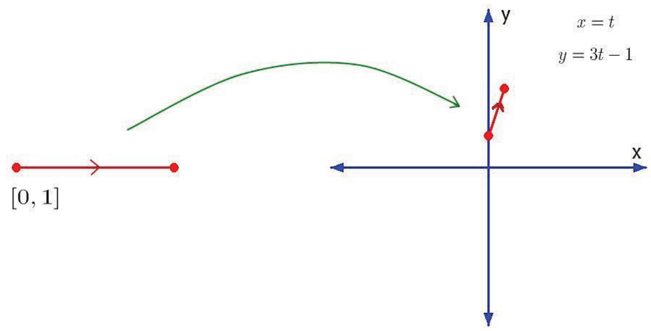
Example 12.3.6. *The line $y = 3x + 2$ can be parametrized by $x = t$ and $y = 3t + 2$ for $t \in \mathbb{R}$. This induces an orientation which goes from left to right for the line. On the other hand, if we use $x = -\lambda$ and $y = -3\lambda + 2$ then as λ increases we travel from right to left on the curve. So the λ-equations give the line the opposite orientation.*

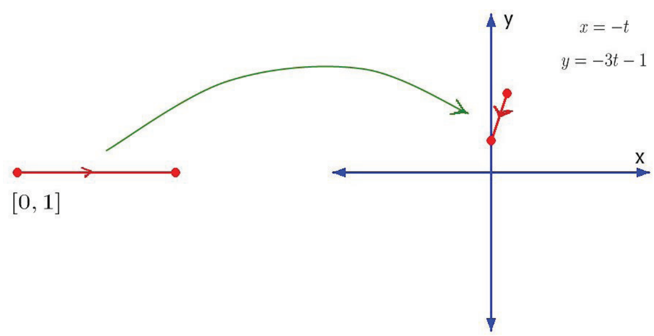




To reverse orientation for $x = f(t), y = g(t)$ for $t \in J = [a, b]$ one may simply replace t by $-t$ in the parametric equations, this gives new equations which will cover the same curve via $x = f(-t), y = g(-t)$ for $t \in [-a, -b]$.

Example 12.3.7. *The line-segment from $(0, -1)$ to $(1, 2)$ can be parametrized by $x = t$ and $y = 3t - 1$ for $0 \leq t \leq 1$. On the other hand, the line-segment from $(1, 2)$ to $(0, -1)$ is parametrized by $x = -t, y = -3t - 1$ for $-1 \leq t \leq 0$.*





The other method to graph parametric curves is simply to start plugging in values for the parameter and assemble a table of values to plot. I have illustrated that in part by plotting the green dots in the domain of the parameter together with their images on the curve. Those dots are the results of plugging in the parameter to find corresponding values for x, y. I don't find that is a very reliable approach in the same way I find plugging in values to $f(x)$ provides a very good plot of $y = f(x)$. That sort of brute-force approach is more appropriate for a CAS system. There are many excellent tools for plotting parametric curves, hopefully I will have some posted on the course website. In addition, the possibility of animation gives us an even more exciting method for visualization of the time-evolution of a parametric curve. In the next chapter we connect the parametric viewpoint with physics and such an animation actually represents the physical motion of some object. My focus in the remainder of this chapter is almost purely algebraic, I could draw pictures to explain, but I wanted the notes to show you that the pictures are not necessary when you understand the algebraic process. That said, the best approach is to do some combination of algebraic manipulation/figuring and graphical reasoning.

Remark 12.3.8. *Ahhhhhh! what is t ???*

If you are at all like me when I first learned about parametric curves you're probably wondering what is t ? You probably, like me, suppose incorrectly that t should be just like x or y. There is a crucial difference between x and y and t. The notations x and y are actually shorthands for the Cartesian coordinate maps $x : \mathbb{R}^2 \to \mathbb{R}$ and $y : \mathbb{R}^2 \to \mathbb{R}$ where $x(a, b) = a$ and $y(a, b) = b$. When I use the notaion $x = 3$ then you know what I mean, you know that I'm focusing on the vertical line with first coordinate 3. On the other hand, if I say $t = 3$ and ask where is it? Then you should say, you question doesn't make sense. The concept of t is tied to the curve for which it is the parameter. There are infinitely many geometric meanings for t. In other words, if you try to find t in the xy-plane without regard to a curve then you'll never find an answer. It's a meaningless question.

On the other hand if we are given a curve and ask what the meaning of t is for that curve then we ask a meaningful question. There are two popular meanings.

1. the parameter s measures the arclength from some base point on the given curve.
2. the parameter t gives the time along the curve.

In case (1.) for an oriented curve this actually is uniquely specified if we have a starting point. Such a parameterization is called the **arclength parametrization** or **unit-speed** parametrization of a curve. These play a fundamental role in the study of the differential geometry of curves. In case (2.) we have in mind that the curve represents the physical trajectory of some object, as t increases, time goes on and the object moves. I tend to use (2.) as my conceptual backdrop. But, keep in mind that these are just applications of parametric curves. In general, the parameter need not be time or arclength. It might just be what is suggested by algebraic convenience: that is my primary motivator in this chapter.

12.4 converting to and from the parametric viewpoint

Let's change gears a bit, we've seen that parametric equations for curves give us a new method to describe particular geometric concepts such as orientability or multiple covering. Without the introduction of the parametric concept these geometric ideas are not so easy to describe. That said, I now turn to the question of how to connect parametric descriptions with Cartesian descriptions of a curve. We'd like to understand how to go both ways if possible:

1. how can we find the Cartesian form for a given parametric curve?
2. how can we find a parametrization of a given Cartesian curve?

In case (2.) we mean to include the ideas of level curves and graphs. It turns out that both questions can be quite challenging for certain examples. However, in other cases, not so much: for example any graph $y = f(x)$ is easily recast as the set of parametric equations $x = t$ and $y = f(t)$ for $t \in dom(f)$. For the standard graph of a function we use x as the parameter.

12.4.1 how can we find the Cartesian form for a given parametric curve?

Example 12.4.1. *What curve has parametric equations $x = t$ for $y = t^2$ for $t \in \mathbb{R}$? To find Cartesian equation we eliminate the parameter (when possible)*

$$t^2 = x^2 = y \quad \Rightarrow \quad y = x^2$$

Thus the Cartesian form of the given parametrized curve is simply $y = x^2$.

Example 12.4.2. *Example 15.2.2: Find parametric equations to describe the graph $y = \sqrt{x+3}$ for $0 \leq x < \infty$. We can use $x = t^2$ and $y = \sqrt{t^2+3}$ for $t \in \mathbb{R}$. Or, we could use $x = \lambda$ and $y = \sqrt{\lambda+3}$ for $\lambda \in [0, \infty)$.*

Example 12.4.3. *What curve has parametric equations $x = t$ for $y = t^2$ for $t \in [0, 1]$? To find Cartesian equation we eliminate the parameter (when possible)*

$$t^2 = x^2 = y \quad \Rightarrow \quad y = x^2$$

Thus the Cartesian form of the given parametrized curve is simply $y = x^2$, however, given that $0 \leq t \leq 1$ and $x = t$ it follows we do not have the whole parabola, instead just $y = x^2$ for $0 \leq x \leq 1$.

Example 12.4.4. *Identify what curve has parametric equations $x = \tan^{-1}(t)$ and $y = \tan^{-1}(t)$ for $t \in \mathbb{R}$. Recall that range$(\tan^{-1}(t)) = (-\pi/2, \pi/2)$. It follows that $-\pi/2 < x < \pi/2$. Naturally we just equate inverse tangent to obtain $\tan^{-1}(t) = y = x$. The curve is the open line-segment with equation $y = x$ for $-\pi/2 < x < \pi/2$. This is an interesting parameterization, notice that as $t \to \infty$ we approach the point $(\pi/2, \pi/2)$, but we never quite get there.*

Example 12.4.5. *Consider $x = \ln(t)$ and $y = e^t - 1$ for $t \geq 1$. We can solve both for t to obtain*

$$t = e^x = \ln(y+1) \quad \Rightarrow \quad y = -1 + exp(exp(x)).$$

The domain for the expression above is revealed by analyzing $x = \ln(t)$ for $t \geq 1$, the image of $[1, \infty)$ under natural log is precisely $[0, \infty)$; $\ln[1, \infty) = [0, \infty)$.

Example 12.4.6. *Suppose $x = \cosh(t) - 1$ and $y = 2\sinh(t) + 3$ for $t \in \mathbb{R}$. To eliminate t it helps to take an indirect approach. We recall the most important identity for the hyperbolic sine and cosine: $\cosh^2(t) - \sinh^2(t) = 1$. Solve for hyperbolic cosine; $\cosh(t) = x+1$. Solve for hyperbolic sine; $\sinh(t) = \frac{y-3}{2}$. Now put these together via the identity:*

$$\cosh^2(t) - \sinh^2(t) = 1 \quad \Rightarrow \quad (x+1)^2 - \frac{(y-3)^2}{4} = 1.$$

Note that $\cosh(t) \geq 1$ hence $x + 1 \geq 1$ thus $x \geq 0$ for the curve described above. On the other hand y is free to range over all of $\mathbb{R}$ since hyperbolic sine has range $\mathbb{R}$. You should[4] recognize the equation as a hyperbola centered at $(-1, 3)$.

[4] many students need to review these at this point, we use circles, ellipses and hyperbolas as examples in this course. I'll give examples of each in this chapter.

12.4.2 how can we find a parametrization of a given Cartesian curve?

I like this topic more, the preceding bunch of examples, while needed, are boring. The art of parameterizing level curves is much more fun.

Example 12.4.7. Find parametric equations for the circle centered at (h, k) with radius R.

To begin recall the equation for such a circle is $(x-h)^2 + (y-k)^2 = R^2$. Our inspiration is the identity $\cos^(t) + \sin^2(t) = 1$. Let $x - h = R\cos(t)$ and $y - k = R\sin(t)$ thus

$$\boxed{x = h + R\cos(t)} \qquad \text{and} \qquad \boxed{y = k + R\sin(t).}$$

I invite the reader to verify these do indeed parametrize the circle by explicitly plugging in the equations into the circle equation. Notice, if we want the whole circle then we simply choose any interval for t of length 2π or longer. On the other hand, if you want to select just a part of the circle you need to think about where sine and cosine are positive and negative. For example, if I want to parametrize just the part of the circle for which $x > h$ then I would choose $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

The reason I choose that intuitively is that the parametrization given for the circle above is basically built from polar coordinates[5] centered at (h, k). That said, to be sure about my choice of parameter domain I like to actually plug in some of my proposed domain and make sure it matches the desired criteria. I think about the graphs of sine and cosine to double check my logic. I know that $\cos(-\frac{\pi}{2}, \frac{\pi}{2}) = (0, 1]$ whereas $\sin(-\frac{\pi}{2}, \frac{\pi}{2}) = (-1, 1)$, I see it in my mind. Then I think about the parametric equations in view of those facts,

$$x = h + R\cos(t) \qquad \text{and} \qquad y = k + R\sin(t).$$

I see that x will range over $(h, h+R]$ and y will range over $(k-R, k+R)$. This is exactly what I should expect geometrically for half of the circle. Visualize that $x = h$ is a vertical line which cuts our circle in half. These are the thoughts I think to make certain my creative leaps are correct. I would encourage you to think about these matters. Don't try to just memorize everything, it will not work for you, there are simply too many cases. It's actually way easier to just understand these as a consequence of trigonometry, algebra and analytic geometry.

Example 12.4.8. Find parametric equations for the level curve $x^2 + 2x + \frac{1}{4}y^2 = 0$ which give the ellipse a CW orientation.

To begin we complete the square to understand the equation:

$$x^2 + 2x + \frac{1}{4}y^2 = 0 \quad \Rightarrow \quad (x+1)^2 + \frac{1}{4}y^2 = 1.$$

We identify this is an ellipse centered at $(-1, 0)$. Again, I use the pythagorean trig. identity as my guide: I want $(x+1)^2 = \cos^2(t)$ and $\frac{1}{4}y^2 = \sin^2(t)$ because that will force the parametric equations to solve the ellipse equation. However, I would like for the equations to describe CW direction so I replace the t with $-t$ and propose:

$$\boxed{x = -1 + \cos(-t)} \qquad \text{and} \qquad \boxed{y = 2\sin(-t)}$$

If we choose $t \in [0, 2\pi)$ then the whole ellipse will be covered. I could simplify $\cos(-t) = \cos(t)$ and $\sin(-t) = -\sin(t)$ but I have left the minus to emphasize the idea about reversing the orientation. In the preceding example we gave the circle a CCW orientation.

---

[5] we will discuss further in a later section, but this should have been covered in at least your precalculus course.

Example 12.4.9. Find parametric equations for the part of the level curve $x^2 - y^2 = 1$ which is found in the first quadrant.

We recognize this is a hyperbola which opens horizontally since $x = 0$ gives us $-y^2 = 1$ which has no real solutions. Hyperbolic trig. functions are built for a problem just such as this: recall $\cosh^2(t) - \sinh^2(t) = 1$ thus we choose $x = \cosh(t)$ and $y = \sinh(t)$. Furthermore, the hyperbolic sine function $\sinh(t) = \frac{1}{2}\left(e^t - e^{-t}\right)$ is everywhere increasing since it has derivative $\cosh(t)$ which is everywhere positive. Moreover, since $\sinh(0) = 0$ we see that $\sinh(t) \geq 0$ for $t \geq 0$. Choose non-negative t for the domain of the parametrization:

$$\boxed{x = \cosh(t), \qquad y = \sinh(t), \qquad t \in [0, \infty).}$$

Example 12.4.10. Find parametric equations for the part of the level curve $x^2 - y^2 = 1$ which is found in the third quadrant.

Based on our thinking from the last example we just need to modify the solution a bit:

$$\boxed{x = -\cosh(t), \qquad y = \sinh(t), \qquad t \in (-\infty, 0].}$$

Note that if $t \in (-\infty, 0]$ then $-\cosh(t) \leq -1$ and $\sinh(t) \leq 0$, this puts us in the third quadrant. It is also clear that these parametric equations to solve the hyperbola equation since

$$(-\cosh(t))^2 - (\sinh(t))^2 = \cosh^2(t) - \sinh^2(t) = 1.$$

The examples thus far are rather specialized, and in general there is no method to find parametric equations. This is why I said it is an art.

Example 12.4.11. Find parametric equations for the level curve $x^2y^2 = x - 2$.

This example is actually pretty easy because we can solve for $y^2 = \frac{x-2}{x^2}$ hence $y = \pm\sqrt{\frac{x-2}{x^2}}$. We can choose x as parameter so the parametric equations are just

$$x = t \quad \textit{and } y = \sqrt{\frac{t-2}{t^2}}$$

for $t \geq 2$. Or, we could give parametric equations

$$x = t \quad \textit{and } y = -\sqrt{\frac{t-2}{t^2}}$$

for $t \geq 2$. These parametrizations simply cover different parts of the same level curve.