Chapter 14

variational calculus

14.1 history

The problem of variational calculus is almost as old as modern calculus. Variational calculus seeks to answer questions such as:

Remark 14.1.1.

1. what is the shortest path between two points on a surface ?
2. what is the path of least time for a mass sliding without friction down some path between two given points ?
3. what is the path which minimizes the energy for some physical system ?
4. given two points on the x-axis and a particular area what curve has the longest perimeter and bounds that area between those points and the x-axis?

You'll notice these all involve a variable which is not a real variable or even a vector-valued-variable. Instead, the answers to the questions posed above will be **paths** or **curves** depending on how you wish to frame the problem. In variational calculus the variable is a function and we wish to find extreme values for a **functional**. In short, a functional is an abstract function of functions. A functional takes as an input a function and gives as an output a number. The space from which these functions are taken varies from problem to problem. Often we put additional **contraints** or **conditions** on the **space of admissable solutions**. To read about the full generality of the problem you should look in a text such as Hans Sagan's. Our treatment is introductory in this chapter, my aim is to show you why it is plausible and then to show you how we use variational calculus.

We will see that the problem of finding an extreme value for a functional is equivalent to solving the Euler-Lagrange equations or Euler equations for the functional. Euler predates Lagrange in his discovery of the equations bearing their names. Eulers's initial attack of the problem was to chop the hypothetical solution curve up into a polygonal path. The unknowns in that approach were the coordinates of the vertices in the polygonal path. Then through some ingenious calculations he arrived at the Euler-Lagrange equations. Apparently there were logical flaws in Euler's original treatment. Lagrange later derived the same equations using the viewpoint that the variable was a function and the **variation** was one of shifting by an arbitrary function. The treatment of variational calculus in Edwards is neither Euler nor Lagrange's approach, it is a refined version which takes in the contributions of generations of mathematicians working on the subject and then merges it with careful functional analysis. I'm no expert of the full history, I just give you a rough sketch of what I've gathered from reading a few variational calculus texts.

Physics played a large role in the development of variational calculus. Lagrange was a physicist as well as a mathematician. At the present time, every physicist takes course(s) in *Lagrangian Mechanics.* Moreover, the use of variational calculus is fundamental since Hamilton's principle says that all physics can be derived from the principle of least action. In short this means that nature is lazy. The solutions realized in the physical world are those which minimize the action. The action

$$S[y] = \int L(y, y', t)\, dt$$

is constructed from the Lagrangian $L = T - U$ where T is the kinetic energy and U is the potential energy. In the case of classical mechanics the Euler Lagrange equations are precisely Newton's equations. The Hamiltonian $H = T + U$ is similar to the Lagrangian except that the fundamental variables are taken to be momentum and position in contrast to velocity and position in Lagrangian mechanics. Hamiltonians and Lagrangians are used to set-up new physical theories. Euler-Lagrange equations are said to give the so-called *classical limit* of modern field theories. The concept of a force is not so useful to quantum theories, instead the concept of energy plays the central role. Moreover, the problem of quantizing and then renormalizing field theory brings in very sophisiticated mathematics. In fact, the math of modern physics is not understood. In this chapter I'll just show you a few famous classical mechanics problems which are beatifully solved by Lagrange's approach. We'll also see how expressing the Lagrangian in non-Cartesian coordinates can give us an easy way to derive forces that arise from geometric contraints. Hopefully we can derive the coriolis force in this manner. I also plan to include a problem or two about Maxwell's equations from the variational viewpoint. There must be at least a dozen different ways to phrase Maxwell's equations, one reason I revisit them is to give you a concrete example as to the fact that physics has many formulations.

I am following the typical physics approach to variational calculus. Edwards' last chapter is more natural mathematically but I think the math is a bit much for your first exposure to the subject. The treatment given here is close to that of Arfken and Weber's Mathematical Physics text, however I suspect you can find these calculations in dozens of classical mechanics texts. More or less our approach is that of Lagrange.

14.2 the variational problem

Our goal in what follows here is to maximize or minimize a particular function of functions. Suppose $\mathcal{F}_o$ is a set of functions with some particular property. For now, we may could assume that all the functions in $\mathcal{F}_o$ have graphs that include (x_1, y_1) and (x_2, y_2). Consider a functional $J : \mathcal{F}_o \to \mathcal{F}_o$ which is defined by an integral of some function f which we call the **Lagrangian**,

$$J[y] = \int_{x_1}^{x_2} f(y, y', x)\, dx.$$

We suppose that f is given but y is a variable. Consider that if we are given a function $y^* \in \mathcal{F}_o$ and another function η such that $\eta(x_1) = \eta(x_2) = 0$ then we can reach a whole family of functions indexed by a real variable α as follows (relabel $y^*(x)$ by $y(x,0)$ so it matches the rest of the family of functions):

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x)$$

Note that $x \mapsto y(x, \alpha)$ gives a function in $\mathcal{F}_o$. We define the **variation** of y to be

$$\boxed{\delta y = \alpha\eta(x)}$$

This means $y(x, \alpha) = y(x, 0) + \delta y$. We may write J as a function of α given the variation we just described:

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x, \alpha), y(x, \alpha)', x)\, dx.$$

It is intuitively obvious that if the function $y^*(x) = y(x, 0)$ is an extremum of the functional then we ought to expect

$$\left[\frac{\partial J(\alpha)}{\partial \alpha}\right]_{\alpha=0} = 0$$

Notice that we can calculate the derivative above using multivariate calculus. Remember that $y(x, \alpha) = y(x, 0) + \alpha\eta(x)$ hence $y(x, \alpha)' = y(x, 0)' + \alpha\eta(x)'$ thus $\frac{\partial y}{\partial \alpha} = \eta$ and $\frac{\partial y'}{\partial \alpha} = \eta' = \frac{d\eta}{dx}$. Consider that:

$$\begin{aligned}
\frac{\partial J(\alpha)}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\left[\int_{x_1}^{x_2} f(y(x, \alpha), y(x, \alpha)', x)\, dx\right] \\
&= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}\right) dx \\
&= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}\right) dx
\end{aligned} \tag{14.1}$$

Observe that

$$\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\eta\right] = \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\eta + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}$$

Hence continuing Equation 14.1 in view of the product rule above,

$$\begin{aligned}\frac{\partial J(\alpha)}{\partial \alpha} &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\eta\right] - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\eta \right) dx \\ &= \frac{\partial f}{\partial y'}\eta\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\eta \right) dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] \right)\eta \, dx\end{aligned} \tag{14.2}$$

Note we used the conditions $\eta(x_1) = \eta(x_2)$ to see that $\frac{\partial f}{\partial y'}\eta\big|_{x_1}^{x_2} = \frac{\partial f}{\partial y'}\eta(x_2) - \frac{\partial f}{\partial y'}\eta(x_1) = 0$. Our goal is to find the extreme values for the functional J. Let me take a few sentences to again restate our set-up. Generally, we take a function y then J maps to a new function $J[y]$. The family of functions indexed by α gives a whole ensemble of functions in $\mathcal{F}_o$ which are near y^* according to the formula,

$$y(x, \alpha) = y^*(x) + \alpha\eta(x)$$

Let's call this set of functions W_η. If we took another function like η, say ζ such that $\zeta(x_1) = \zeta(x_2) = 0$ then we could look at another family of functions:

$$y(x, \alpha) = y^*(x) + \alpha\zeta(x)$$

and we could denote the set of all such functions generated from ζ to be W_ζ. The total variation of y based at y^* should include all possible families of functions in $\mathcal{F}_o$. You could think of W_η and W_ζ be two different subspaces in $\mathcal{F}_o$. If $\eta \neq \zeta$ then these subspaces of $\mathcal{F}_o$ are likely disjoint except for the proposed extremal solution y^*. It is perhaps a bit unsettling to realize there are infinitely many such subspaces because there are infinitely many choices for the function η or ζ. In any event, each possible variation of y^* must satisfy the condition $\left[\frac{\partial J(\alpha)}{\partial \alpha}\right]_{\alpha=0} = 0$ since we **assume** that y^* is an extreme value of the functional J. It follows that the Equation 14.2 holds for all possible η. Therefore, we ought to expect that any extreme value of the functional $J[y] = \int_{x_1}^{x_2} f(y, y', x)\, dx$ must solve the **Euler Lagrange Equations:**

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0 \quad \text{Euler-Lagrange Equations for} \quad J[y] = \int_{x_1}^{x_2} f(y, y', x)\, dx}$$

14.3 variational derivative

The role that η played in the discussion in the preceding section is somewhat similar to the role that the "h" plays in the definition $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. You might hope we could replace arguments in η with a more direct approach. Physicists have a heuristic way of making such arguments in terms of the variation δ. They would cast the arguments in the last page by just "taking the variation of J". Let me give you their formal argument,

$$\begin{aligned} \delta J &= \delta\Big[\int_{x_1}^{x_2} f(y,y',x)\,dx \Big] \\ &= \Big[\int_{x_1}^{x_2} \delta f(y,y',x)\,dx \Big] \\ &= \int_{x_1}^{x_2} \Big(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta\big(\tfrac{dy}{dx}\big) + \frac{\partial f}{\partial x}\delta x \Big)\,dx \\ &= \int_{x_1}^{x_2} \Big(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y) \Big)\,dx \\ &= \frac{\partial f}{\partial y'}\delta y\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \Big(\frac{\partial f}{\partial y} - \frac{d}{dx}\Big[\frac{\partial f}{\partial y'}\Big] \Big)\delta y\,dx \end{aligned} \tag{14.3}$$

Therefore, since $\delta y = 0$ at the endpoints of integration, the Euler-Lagrange equations follow from $\delta J = 0$. Now, if you're like me, the argument above is less than satisfying since we never actually defined what it means to "take δ" of something. Also, why could I commute the variational δ and $\frac{d}{dx}$)? That said, the formal method is not without use since it allows the focus to be on the Euler Lagrange equations rather than the technical details of the variation.

Remark 14.3.1.

The more adept reader at this point should realize the hypocrisy of me calling the above calculation formal since even my presentation here was formal. I also used an analogy, I assumed that the theory of extreme values for multivariate calculus extends to function space. But, $\mathcal{F}_o$ is not $\mathbb{R}^n$, it's much bigger. Edwards builds the correct formalism for a rigourous calculation of the variational derivative. To be careful we'd need to develop the norm on function space and prove a number of results about infinite dimensional linear algebra. Take a look at the last chapter in Edwards' text if you're interested. I don't believe I'll have time to go over that material this semester.

14.4 Euler-Lagrange examples

I present a few standard examples in this section. We make use of the calculation in the last section. Also, we will use a result from your homework which states an equivalent form of the Euler-Lagrange equation is

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\Big[f - y'\frac{\partial f}{\partial y'}\Big] = 0.$$

This form of the Euler Lagrange equation yields better differential equations for certain examples.

14.4.1 shortest distance between two points in plane

If s denotes the arclength in the xy-plane then the pythagorean theorem gives $ds^2 = dx^2 + dy^2$ infinitesimally. Thus, $ds = \sqrt{1 + \frac{dy}{dx}^2}\, dx$ and we may add up all the little distances ds to find the total length between two given points (x_1, y_1) and (x_2, y_2):

$$J[y] = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$$

Identify that we have $f(y, y', x) = \sqrt{1 + (y')^2}$. Calculate then,

$$\frac{\partial f}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}}.$$

Euler Lagrange equations yield,

$$\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = \frac{\partial f}{\partial y} \qquad \Rightarrow \qquad \frac{d}{dx}\left[\frac{y'}{\sqrt{1 + (y')^2}}\right] = 0 \qquad \Rightarrow \qquad \frac{y'}{\sqrt{1 + (y')^2}} = k$$

where $k \in \mathbb{R}$ is constant with respect to x. Moreover, square both sides to reveal

$$\frac{(y')^2}{1 + (y')^2} = k^2 \qquad \Rightarrow \qquad (y')^2 = \frac{k^2}{1 - k^2} \qquad \Rightarrow \qquad \frac{dy}{dx} = \pm\sqrt{\frac{k^2}{1 - k^2}} = m$$

where I have defined m is defined in the obvious way. We find solutions $y = mx + b$. Finally, we can find m, b to fit the given pair of points (x_1, y_1) and (x_2, y_2) as follows:

$$y_1 = mx_1 + b \qquad \text{and} \qquad y_2 = mx_2 + b \qquad \Rightarrow \qquad y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

provided $x_1 \neq x_2$. If $x_1 \neq x_2$ and $y_1 \neq y_2$ then we could perform the same calculation as above with the roles of x and y interchanged,

$$J[x] = \int_{y_1}^{y_2} \sqrt{1 + (x')^2}\, dy$$

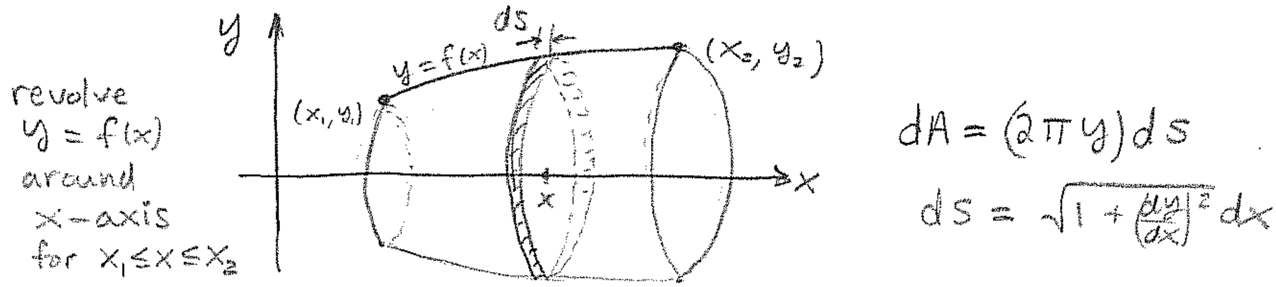
where $x' = dx/dy$ and the Euler Lagrange equations would yield the solution

$$x = x_1 + \frac{x_2 - x_1}{y_2 - y_1}(y - y_1).$$

Finally, if both coordinates are equal then $(x_1, y_1) = (x_2, y_2)$ and the shortest path between these points is the trivial path, the armchair solution. Silly comments aside, we have shown that a straight line provides the curve with the shortest arclength between any two points in the plane.

14.4.2 surface of revolution with minimal area

Suppose we wish to revolve some curve which connects (x_1, y_1) and (x_2, y_2) around the x-axis. A surface constructed in this manner is called a **surface of revolution**. In calculus we learn how to calculate the surface area of such a shape. One can imagine deconstructing the surface into a sequence of ribbons. Each ribbon at position x will have a "radius" of y and a width of dx however, because the shape is tilted the area of the ribbon works out to $dA = 2\pi y ds$ where ds is the arclength.



If we choose x as the parameter this yields $dA = 2\pi y\sqrt{1+(y')^2}\,dx$. To find the surface of minimal surface area we ought to consider the functional:

$$A[y] = \int_{x_1}^{x_2} 2\pi y\sqrt{1+(y')^2}\,dx$$

Identify that $f(y, y', x) = 2\pi y\sqrt{1+(y')^2}$ hence $f_y = 2\pi\sqrt{1+(y')^2}$ and $f_{y'} = 2\pi y y'/\sqrt{1+(y')^2}$. The usual Euler-Lagrange equations are not easy to solve for this problem, it's easier to work with the equations you derived in homework,

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0.$$

Hence,

$$\frac{d}{dx}\left[2\pi y\sqrt{1+(y')^2} - \frac{2\pi y (y')^2}{\sqrt{1+(y')^2}}\right] = 0$$

Dividing by 2π and making a common denominator,

$$\frac{d}{dx}\left[\frac{y}{\sqrt{1+(y')^2}}\right] = 0 \qquad \Rightarrow \qquad \frac{y}{\sqrt{1+(y')^2}} = k$$

where k is a constant with respect to x. Squaring the equation above yields

$$\frac{y^2}{1+(\frac{dy}{dx})^2} = k^2 \qquad \Rightarrow \qquad y^2 - k^2 = k^2(\tfrac{dy}{dx})^2$$

Solve for dx, integrate, assuming the given points are in the first quadrant,

$$x = \int dx = \int \frac{k\,dy}{\sqrt{y^2-k^2}} = k\cosh^{-1}(\tfrac{y}{k}) + c$$

Hence,

$$\boxed{y = k\cosh\left(\frac{x-c}{k}\right)}$$

generates the surface of revolution of least area between two points. These shapes are called **Catenoids** they can be observed in the formation of soap bubble between rings. There is a vast literature on this subject and there are many cases to consider, I simply exhibit a simple solution. For a given pair of points it is not immediately obvious if there exists a solution to the Euler-Lagrange equations which fits the data. (see page 622 of Arfken).

14.4.3 Braichistochrone

Suppose a particle slides freely along some curve from (x_1, y_1) to $(x_2, y_2) = (0, 0)$ under the influence of gravity where we take y to be the vertical direction. **What is the curve of quickest descent?** Notice that if $x_1 = 0$ then the answer is easy to see, however, if $x_1 \neq 0$ then the question is not trivial. To solve this problem we must first offer a functional which accounts for the time of descent. Note that the speed $v = ds/dt$ so we'd clearly like to minimize $J = \int_{(0,0)}^{(x_1,y_1)} \frac{ds}{v}$. Since the object is assumed to fall freely we may assume that energy is conserved in the motion hence

$$\frac{1}{2}mv^2 = mg(y - y_1) \qquad \Rightarrow \qquad v = \sqrt{2g(y_1 - y)}$$

As we've discussed in previous examples, $ds = \sqrt{1 + (y')^2}dt$ so we find

$$J[y] = \int_0^{x_1} \underbrace{\sqrt{\frac{1 + (y')^2}{2g(y_1 - y)}}}_{f(y,y',x)}\, dx$$

Notice that the modified Euler-Lagrange equations $\frac{\partial f}{\partial x} - \frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0$ are convenient since $f_x = 0$. We calculate that

$$\frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{\frac{1+(y')^2}{2g(y_1-y)}}}\frac{2y'}{2g(y_1 - y)} = \frac{y'}{\sqrt{2g(y_1 - y)(1 + (y')^2)}}$$

Hence there should exist some constant $1/(k\sqrt{2g})$ such that

$$\sqrt{\frac{1 + (y')^2}{2g(y_1 - y)}} - \frac{(y')^2}{\sqrt{2g(y_1 - y)(1 + (y')^2)}} = \frac{1}{k\sqrt{2g}}$$

It follows that,

$$\frac{1}{\sqrt{(y_1 - y)(1 + (y')^2)}} = \frac{1}{k} \qquad \Rightarrow \qquad \left(y_1 - y\right)\left(1 + \left(\frac{dy}{dx}\right)^2\right) = k^2$$

We need to solve for dy/dx,

$$(y_1 - y)\left(\frac{dy}{dx}\right)^2 = k^2 - y_1 + y \qquad \Rightarrow \qquad \left(\frac{dy}{dx}\right)^2 = \frac{y + k^2 - y_1}{y_1 - y}$$

Or, relabeling constants $a = y_1$ and $b = k^2 - y_1$ and we must solve

$$\frac{dy}{dx} = \pm\sqrt{\frac{b+y}{a-y}} \qquad \Rightarrow \qquad x = \pm \int \sqrt{\frac{a-y}{b+y}}\, dy$$

The integral is not trivial. It turns out that the solution is a cycloid (Arfken p. 624):

$$\boxed{x = \frac{a+b}{2}\Big(\theta + \sin(\theta)\Big) - d \qquad y = \frac{a+b}{2}\Big(1 - \cos(\theta)\Big) - b}$$

This is the curve that is traced out by a point on a wheel as it travels. If you take this solution and calculate $J[y_{cycloid}]$ you can show the time of descent is simply

$$T = \frac{\pi}{2}\sqrt{\frac{y_1}{2g}}$$

if the mass begins to descend from (x_2, y_2). But, this point has no connection with (x_1, y_1) except that they both reside on the same cycloid. It follows that the period of a pendulum that follows a cycloidal path is indpendent of the starting point on the path. This is not true for a circular pendulum in general, we need the small angle approximation to derive simple harmonic motion. It turns out that it is possible to make a pendulum follow a cycloidal path if you let the string be guided by a frame which is also cycloidal. The neat thing is that even as it loses energy it still follows a cycloidal path and hence has the same period. The "Brachistochrone" problem was posed by Johann Bernoulli in 1696 and it actually predates the variational calculus of Lagrange by some 50 or so years. This problem and ones like it are what eventually prompted Lagrange and Euler to systematically develop the subject. Apparently Galileo also studied this problem however lacked the mathematics to crack it.

$$t = \int \frac{ds}{v} = \int \frac{\sqrt{dx^2+dy^2}}{\sqrt{2g(x-x_1)}}$$ then we note by 6.26 $\begin{cases} x = a(1-\cos\theta) \\ y = a(\theta - \sin\theta) \end{cases}$ $\begin{aligned} dx &= a\sin\theta\, d\theta \\ dy &= a(1-\cos\theta)d\theta \end{aligned}$

$$= \int \left(\frac{1+y'^2}{2g(x-x_1)} \right)^{1/2} dx \qquad y' = \frac{dy}{dx} = \frac{a(1-\cos\theta)d\theta}{a\sin\theta\, d\theta} = \frac{1}{\sin\theta} - \frac{1}{\tan\theta} = y'$$

$$= \int \left(\frac{1 + \left(\frac{1}{\sin\theta} - \frac{1}{\tan\theta}\right)^2}{2g\left(a(1-\cos\theta) - a(1-\cos\theta_1)\right)} \right)^{1/2} a\sin\theta\, d\theta$$

$$= \int \left(\frac{\sin^2\theta + \left(\frac{\sin\theta}{\sin\theta} - \frac{\sin\theta}{\tan\theta}\right)^2}{2g\left(a(1-\cos\theta) - a(1-\cos\theta_1)\right)} \right)^{1/2} a\, d\theta$$

$$= \int \left(\frac{\left(\sin^2\theta + (1-\cos\theta)^2\right)a^2}{2g\left(a(1-\cos\theta) - a(1-\cos\theta_1)\right)} \right)^{1/2} d\theta$$

$$= \int \sqrt{\frac{a}{g}} \left(\frac{\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}{2(\cos\theta_1 - \cos\theta)} \right)^{1/2} d\theta$$

$$= \int \sqrt{\frac{a}{g}} \left(\frac{2 - 2\cos\theta}{2(\cos\theta_1 - \cos\theta)} \right)^{1/2} d\theta$$

$$= \int \sqrt{\frac{a}{g}} \left(\frac{1-\cos\theta}{\cos\theta_1 - \cos\theta} \right)^{1/2} d\theta$$

$$= \int \sqrt{\frac{a}{g}} \left(\frac{2\sin^2(\theta/2)}{\cos\theta_1 - 1 + 2\sin^2(\theta/2)} \right)^{1/2} d\theta$$

$$= \int \sqrt{\frac{a}{g}} \left(\frac{2\sin^2(\theta/2)}{\tilde{\alpha} + 2\sin^2(\theta/2)} \right)^{1/2} d\theta \qquad \text{let} \quad \begin{aligned} \tilde{\alpha} &= \cos\theta_1 - 1 \\ u &= \cos(\theta/2) \\ du &= -\tfrac{1}{2}\sin(\theta/2)\, d\theta \end{aligned}$$

$$= \int \sqrt{\frac{a}{g}} \left(\frac{\sin(\theta/2)\, d\theta}{(\alpha^2 - \cos^2(\theta/2))^{1/2}} \right) \qquad \alpha^2 = \frac{\cos\theta_1 + 1}{2}$$

$$= \int \sqrt{\frac{a}{g}} \left(\frac{-2\, du}{(\alpha^2 - u^2)^{1/2}} \right) = \sqrt{\frac{a}{g}}(-2) \int \frac{du}{(\alpha^2 - u^2)^{1/2}} = \left\{ \sin^{-1}\left(\frac{u}{\alpha}\right) \right\} \sqrt{\frac{a}{g}}(-2)$$

$$= \sqrt{\frac{a}{g}}(-2)\left[\sin^{-1}\left(\frac{\cos\theta_2/2}{\alpha}\right) - \sin^{-1}\left(\frac{\cos\theta_1/2}{\alpha}\right) \right] \qquad \text{evaluating from } \theta_1 \longrightarrow \theta_2, \quad \theta_2 = \pi$$

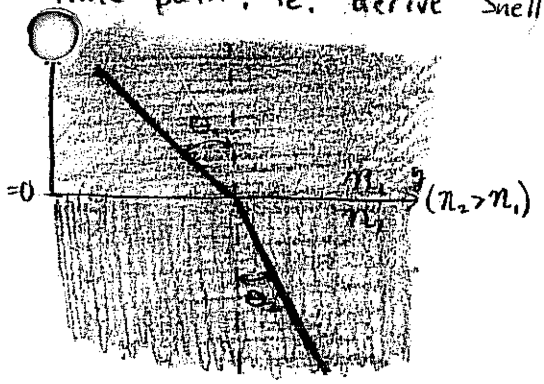
$$= \sqrt{\frac{a}{g}}(-2)\left[\sin^{-1}(0) - \sin^{-1}\left(\frac{\cos\theta_1/2}{\alpha}\right) \right]$$

$$= \left(\sqrt{\frac{a}{g}}\right)(2)\left[\sin^{-1}\left(\frac{\cos\theta_1/2}{\sqrt{\frac{\cos\theta_1+1}{2}}}\right) \right] = \sqrt{\frac{a}{g}}(2)\left\{ \sin^{-1}\left(\frac{\cos\theta_1/2}{\cos\theta_1/2}\right) \right\} \quad \text{as} \quad \cos(\theta_1/2) = \sqrt{\frac{\cos\theta_1+1}{2}}$$

$$= \sqrt{\frac{a}{g}}(2)\left(\sin^{-1}(1)\right) = \sqrt{\frac{a}{g}}(2)\frac{\pi}{2} = \boxed{\pi\sqrt{\frac{a}{g}} = t}$$

SNELL'S LAW DERIVED VIA VARIATIONAL CALCULUS

Consider light passing from medium with index of refraction n_1 into another medium of index of refraction n_2. Use Fermat's principle to find least time path, ie. derive Snell's Law $n_1 \sin\theta_1 = n_2 \sin\theta_2$.



$n = c/v$ thus $v_1 = c/n_1$, $v_2 = c/n_2$

c = speed of light in vaccum

$$t = \int \frac{ds}{v(x)} = \int \frac{dx^2 + dy^2}{v(x)} = \int \frac{\sqrt{dx^2+dy^2}}{v(x)} = \int \frac{\sqrt{1+y'^2}}{v(x)}\,dx \qquad y' = \frac{dy}{dx}$$

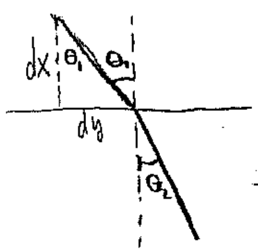
the function $f = \dfrac{\sqrt{1+y'^2}}{v}$ then $\dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial f}{\partial y'} = \dfrac{y'}{v\sqrt{1+y'^2}}$

Euler-Lagrange yields minimum functional with $\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) - \dfrac{\partial f}{\partial y} = 0$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \Rightarrow \frac{\partial f}{\partial y'} = \text{const. with respect to } x = a = \frac{y'}{v\sqrt{1+y'^2}}$$

$$a = \frac{y'}{v\sqrt{1+y'^2}} = \frac{-\tan\theta}{v\sqrt{1+\tan^2\theta}} = \frac{-\tan\theta}{v\sqrt{\sec^2\theta}} = \frac{-\sin\theta}{\cos\theta}\left(\frac{1}{\frac{v}{\cos\theta}}\right) = -\sin\theta$$

$$-a = \frac{\sin\theta}{v} = \frac{n\sin\theta}{c}$$



$\Rightarrow \tan\theta = \dfrac{dx}{-dy}$

$\Rightarrow \tan\theta = \dfrac{-dx}{dy}$

$\Rightarrow$ const $= n\sin\theta$

and n could well have been a function of x. That is $n = n(x)$ which could be used to desribe the gradual bending of starlight by the differing types of air at different altitudes

In general then $\boxed{n(x)\sin\theta = \text{const.}}$

specifically if $n(x) = \begin{cases} n_1 & \text{for } x \geq 0 \\ n_2 & \text{for } x < 0 \end{cases}$ then $\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$

14.5 Euler-Lagrange equations for several dependent variables

We still consider problems with just one independent parameter underlying everything. For problems of classical mechanics this is almost always time t. In anticipation of that application we choose to use the usual physics notation in the section. We suppose that our functional depends on functions $y_1, y_2, \dots, y_n$ of time t along with their time derivatives $\dot{y}_1, \dot{y}_2, \dots, \dot{y}_n$. We again suppose the functional of interest is an integral of a **Lagrangian** function f from time t_1 to time t_2,

$$J[(y_i)] = \int_{t_1}^{t_2} f(y_i, \dot{y}_i, t)\, dt$$

here we use (y_i) as shorthand for $(y_1, y_2, \dots, y_n)$ and $(\dot{y}_i)$ as shorthand for $(\dot{y}_1, \dot{y}_2, \dots, \dot{y}_n)$. We suppose that n-conditions are given for each of the endpoints in this problem; $y_i(t_1) = y_{i1}$ and $y_i(t_2) = y_{i2}$. Moreover, we define $\mathcal{F}_o$ to be the set of paths from $\mathbb{R}$ to $\mathbb{R}^n$ subject to the conditions just stated. We now set out to find necessary conditions on a proposed solution to the extreme value problem for the functional J above. As before let's assume that an extremal solution $y* \in \mathcal{F}_o$ exists. Moreover, imagine varying the solution by some variational function $\eta = (\eta_i)$ which has $\eta(t_1) = (0, 0, \dots, 0)$ and $\eta(t_2) = (0, 0, \dots, 0)$. Consequently the family of paths defined below are all in $\mathcal{F}_o$,

$$y(t, \alpha) = y^*(t) + \alpha\eta(t)$$

Thus $y(t, 0) = y^*$. In terms of component functions we have that

$$y_i(t, \alpha) = y_i^*(t) + \alpha\eta_i(t).$$

You can identify that $\delta y_i = y_i(t, \alpha) - y_i^*(t) = \alpha\eta_i(t)$. Since y^* is an extreme solution we should expect that $\left(\frac{\partial J}{\partial \alpha}\right)_{\alpha=0} = 0$. Differentiate the functional with respect to α and make use of the chain rule for f which is a function of some $2n + 1$ variables,

$$\begin{aligned}
\frac{\partial J(\alpha)}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\left[\int_{t_1}^{t_2} f(y_i(t, \alpha), \dot{y}_i(t, \alpha), t)\, dt\right] \\
&= \int_{t_1}^{t_2} \sum_{j=1}^{n} \left(\frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}_j}\frac{\partial \dot{y}_j}{\partial \alpha} \right) dt \\
&= \int_{t_1}^{t_2} \sum_{j=1}^{n} \left(\frac{\partial f}{\partial y_j}\eta_j + \frac{\partial f}{\partial \dot{y}_j}\frac{d\eta_j}{dt} \right) dt \\
&= \sum_{j=1}^{n} \frac{\partial f}{\partial \dot{y}_j}\eta \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_{j=1}^{n} \left(\frac{\partial f}{\partial y_j} - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}_j} \right)\eta_j\, dt
\end{aligned} \tag{14.4}$$

Since $\eta(t_1) = \eta(t_2) = 0$ the first term vanishes. Moreover, since we may repeat this calculation for all possible variations about the optimal solution y^* it follows that we obtain a set of Euler-Lagrange equations for each component function of the solution:

$$\boxed{\frac{\partial f}{\partial y_j} - \frac{d}{dt}\left[\frac{\partial f}{\partial \dot{y}_j}\right] = 0 \quad j = 1, 2, \dots n \quad \text{Euler-Lagrange Eqns. for } J[(y_i)] = \int_{t_1}^{t_2} f(y_i, \dot{y}_i, t)\, dt}$$

Often we simply use $y_1 = x$, $y_2 = y$ and $y_3 = z$ which denote the position of particle or perhaps just the component functions of a path which gives the geodesic on some surface. In either case we should have 3 sets of Euler-Lagrange equations, one for each coordinate. We will also use non-Cartesian coordinates to describe certain Lagrangians. We develop many useful results for set-up of Lagrangians in non-Cartesian coordinates in the next section.

14.5.1 free particle Lagrangian

For a particle of mass m the kinetic energy K is given in terms of the time derivatives of the coordinate functions x, y, z as follows:

$$K = \tfrac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Construct a functional by integrating the kinetic energy over time t,

$$S = \int_{t_1}^{t_2} \tfrac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\, dt$$

The Euler-Lagrange equations for this functional are

$$\frac{\partial K}{\partial x} = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{x}}\right] \qquad \frac{\partial K}{\partial y} = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{y}}\right] \qquad \frac{\partial K}{\partial z} = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{z}}\right]$$

Since $\frac{\partial K}{\partial \dot{x}} = m\dot{x}$, $\frac{\partial K}{\partial \dot{y}} = m\dot{y}$ and $\frac{\partial K}{\partial \dot{z}} = m\dot{z}$ it follows that

$$\boxed{0 = m\ddot{x} \qquad 0 = m\ddot{y} \qquad 0 = m\ddot{z}.}$$

You should recognize these as Newton's equation for a particle with no force applied. The solution is $(x(t), y(t), z(t)) = (x_o + tv_x, y_o + tv_y, z_o + tv_z)$ which is uniform rectilinear motion at constant velocity (v_x, v_y, v_z). The solution to Newton's equation minimizes the integral of the Kinetic energy. Generally the quantity S is called the **action** and Hamilton's Principle states that the laws of physics all arise from minimizing the action of the physical phenomena. We'll return to this discussion in a later section.

14.5.2 geodesics in $\mathbb{R}^3$

A **geodesic** is the path of minimal length between a pair of points on some manifold. Note we already proved that geodesics in the plane are just lines. In general, for $\mathbb{R}^3$, the square of the infinitesimal arclength element is $ds^2 = dx^2 + dy^2 + dz^2$. The arclength integral from $p = 0$ to $q = (q_x, q_y, q_z)$ in $\mathbb{R}^3$ is most naturally given from the parametric viewpoint:

$$S = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

We assume $(x(0), y(0), z(0)) = (0, 0, 0)$ and $(x(1), y(1), z(1)) = q$ and it should be clear that the integral above calculates the arclength. The Euler-Lagrange equations for x, y, z are

$$\frac{d}{dt}\left[\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right] = 0, \qquad \frac{d}{dt}\left[\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right] = 0, \qquad \frac{d}{dt}\left[\frac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right] = 0.$$

It follows that there exist constants, say a, b and c, such that

$$a = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}, \qquad b = \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}, \qquad c = \frac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}.$$

These equations are said to be **coupled** since each involves derivatives of the others. We usually need a way to uncouple the equations if we are to be successful in solving the system. We can calculate, and equate each with the constant 1:

$$1 = \frac{\dot{x}}{a\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = \frac{\dot{y}}{b\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = \frac{\dot{z}}{c\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}.$$

But, multiplying by the denominator reveals an interesting identity

$$\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} = \frac{\dot{x}}{a} = \frac{\dot{y}}{b} = \frac{\dot{z}}{c}$$

The solution has the form, $x(t) = tq_x$, $y(t) = tq_y$ and $z(t) = tq_z$. Therefore,

$$(x(t), y(t), z(t)) = t(q_x, q_y, q_z) = tq.$$

for $0 \leq t \leq 1$. These are the parametric equations for the line segment from the origin to q.

14.6 the Euclidean metric

The square root in the functional of the last subsection certainly complicated the calculation. It is intuitively clear that if we add up squared line elements ds^2 to give a minimum then that ought to correspond to the minimum for the sum of the positive square roots ds of those elements. Let's check if my conjecture works for $\mathbb{R}^3$:

$$S = \int_0^1 (\ \underbrace{\dot{x}^2+\dot{y}^2+\dot{z}^2}_{f(x,y,z,\dot{x},\dot{y},\dot{z})}\ )\ dt$$

This gives us the Euler Lagrange equations below:

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \qquad \ddot{z} = 0$$

The solution of these equations is clearly a line. In this formalism the equations were uncoupled from the outset.

Definition 14.6.1.

> The Euclidean metric is $ds^2 = dx^2 + dy^2 + dz^2$. Generally, for orthogonal curvelinear coordinates u, v, w we calculate $ds^2 = \frac{1}{||\nabla u||^2}du^2 + \frac{1}{||\nabla v||^2}dv^2 + \frac{1}{||\nabla w||^2}dw^2$. We use this as a guide for constructing functionals which calculate arclength or speed

The beauty of the metric is that it allows us to calculate in other coordinates, consider

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

For which we have implicit inverse coordinate transformations $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$. From these inverse formulas we calculate:

$$\nabla r \ = < x/r, y/r > \qquad \nabla\theta \ = < -y/r^2, x/r^2 >$$

Thus, $||\nabla r|| = 1$ whereas $||\nabla\theta|| = 1/r$. We find that the metric in polar coordinates takes the form:

$$\boxed{ds^2 = dr^2 + r^2 d\theta^2}$$

Physicists and engineers tend to like to think of these as arising from calculating the length of infinitesimal displacements in the r or θ directions. Generically, for u, v, w coordinates

$$dl_u = \frac{1}{||\nabla u||}du \qquad dl_v = \frac{1}{||\nabla v||}dv \qquad dl_w = \frac{1}{||\nabla w||}dw$$

and $ds^2 = dl_u^2 + dl_v^2 + dl_w^2$. So in that notation we just found $dl_r = dr$ and $dl_\theta = rd\theta$. Notice then that cylindircal coordinates have the metric,

$$\boxed{ds^2 = dr^2 + r^2 d\theta^2 + dz^2.}$$

For spherical coordinates $x = r\cos(\phi)\sin(\theta)$, $y = r\sin(\phi)\sin(\theta)$ and $z = r\cos(\theta)$ (here $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$, physics notation). Calculation of the metric follows from the line elements,

$$dl_r = dr \qquad dl_\phi = r\sin(\theta)d\phi \qquad dl_\theta = rd\theta$$

Thus,

$$\boxed{ds^2 = dr^2 + r^2\sin^2(\theta)d\phi^2 + r^2 d\theta^2.}$$

We now have all the tools we need for examples in spherical or cylindrical coordinates. What about other cases? In general, given some p-manifold in $\mathbb{R}^n$ how does one find the metric on that manifold? If we are to follow the approach of this section we'll need to find coordinates on $\mathbb{R}^n$ such that the manifold S is described by setting all but p of the coordinates to a constant. For example, in $\mathbb{R}^4$ we have generalized cylindircal coordinates (r, ϕ, z, t) defined implicitly by the equations below

$$x = r\cos(\phi), \qquad y = r\sin(\phi), \qquad z = z, \qquad t = t$$

On the hyper-cylinder $r = R$ we have the metric $ds^2 = R^2 d\theta^2 + dz^2 + dw^2$. There are mathematicians/physicists whose careers are founded upon the discovery of a metric for some manifold. This is generally a difficult task.

14.7 geodesics

A **geodesic** is a path of smallest distance on some manifold. In general relativity, it turns out that the solutions to Eistein's field equations are geodesics in 4-dimensional curved spacetime. Particles that fall freely are following geodesics, for example projectiles or planets in the absense of other frictional/non-gravitational forces. We don't follow a geodesic in our daily life because the earth pushes back up with a normal force. Also, do be honest, the idea of length in general relativity is a bit more abstract that the geometric length studied in this section. The metric of general relativity is non-Euclidean. General relativity is based on semi-Riemannian geometry whereas this section is all Riemannian geometry. The metric in Riemannian geometry is positive definite. The metric in semi-Riemannian geometry can be written as a quadratic form with both positive and negative eigenvalues. In any event, if you want to know more I know some books you might like.

14.7.1 geodesic on cylinder

The equation of a cylinder of radius R is most easily framed in cylindrical coordinates (r, θ, z); the equation is merely $r = R$ hence the metric reads

$$ds^2 = R^2 d\theta^2 + dz^2$$

Therefore, we ought to minimize the following functional in order to locate the parametric equations of a geodesic on the cylinder: note $ds^2 = \left(R^2 \frac{d\theta^2}{dt^2} + \frac{dz^2}{dt^2}\right)dt^2$ thus:

$$S = \int \left(R^2\dot{\theta}^2 + \dot{z}^2\right) dt$$

Euler-Lagrange equations for the dependent variables θ and z are simply:

$$\ddot{\theta} = 0 \qquad \ddot{z} = 0.$$

We can integrate twice to find solutions

$$\boxed{\theta(t) = \theta_o + At \qquad z(t) = z_o + Bt}$$

Therefore, the geodesic on a cylinder is simply the line connecting two points in the plane which is curved to assemble the cylinder. Simple cases that are easy to understand:

1. Geodesic from $(R\cos(\theta_o), R\sin(\theta_o), z_1)$ to $(R\cos(\theta_o), R\sin(\theta_o), z_2)$ is parametrized by $\theta(t) = \theta_o$ and $z(t) = z_1 + t(z_2 - z_1)$ for $0 \leq t \leq 1$. Technically, there is some ambiguity here since I never declared over what range the t is to range. Could pick other intervals, we could use z at the parameter is we wished then $\theta(z) = \theta_o$ and $z = z$ for $z_1 \leq z \leq z_2$

2. Geodesic from $(R\cos(\theta_1), R\sin(\theta_1), z_o)$ to $(R\cos(\theta_2), R\sin(\theta_2), z_o)$ is parametrized by $\theta(t) = \theta_1 + t(\theta_2 - \theta_1)$ and $z(t) = z_o$ for $0 \leq t \leq 1$.

3. Geodesic from $(R\cos(\theta_1), R\sin(\theta_1), z_1)$ to $(R\cos(\theta_2), R\sin(\theta_2), z_2)$ is parametrized by

$$\theta(t) = \theta_1 + t(\theta_2 - \theta_1) \qquad z(t) = z_1 + t(z_2 - z_1)$$

   You can eliminate t and find the equation $z = \frac{z_2 - z_1}{\theta_2 - \theta_1}(\theta - \theta_1)$ which again just goes to show you this is a line in the curved coordinates.

14.7.2 geodesic on sphere

The equation of a sphere of radius R is most easily framed in spherical coordinates (r, ϕ, θ); the equation is merely $r = R$ hence the metric reads

$$ds^2 = R^2 \sin^2(\theta) d\phi^2 + R^2 d\theta^2.$$

Therefore, we ought to minimize the following functional in order to locate the parametric equations of a geodesic on the sphere: note $ds^2 = \big(R^2 \sin^2(\theta)\frac{d\phi^2}{dt^2} + R^2\frac{d\theta^2}{dt^2}\big)dt^2$ thus:

$$S = \int \Big(\underbrace{R^2 \sin^2(\theta)\dot{\phi}^2 + R^2\dot{\theta}^2}_{f(\theta,\phi,\dot{\theta},\dot{\phi})} \Big)\, dt$$

Euler-Lagrange equations for the dependent variables ϕ and θ are simply: $f_\theta = \frac{d}{dt}(f_{\dot{\theta}})$ and $f_\phi = \frac{d}{dt}(f_{\dot{\phi}})$ which yield:

$$2R^2 \sin(\theta)\cos(\theta)\dot{\phi}^2 = \tfrac{d}{dt}(2R^2\dot{\theta}) \qquad 0 = \frac{d}{dt}\Big(2R^2 \sin^2(\theta)\dot{\phi}\Big).$$

We find a **constant of motion** $L = 2R^2 \sin^2(\theta)\dot{\phi}$ inserting this in the equation for the azmuthial angle θ yields:

$$2R^2 \sin(\theta)\cos(\theta)\dot{\phi}^2 = \tfrac{d}{dt}(2R^2\dot{\theta}) \qquad 0 = \frac{d}{dt}\Big(2R^2 \sin^2(\theta)\dot{\phi}\Big).$$

If you can solve these and demonstrate through some reasonable argument that the solutions are great circles then I will give you points. I have some solutions but nothing looks too pretty.

Remark 14.7.1.

I'd like to add a few more examples here, but time is up. There are a few more examples in homework. In particular, the homework has the geodesic problem set-up in a more tractable manner. It's easier to solve the geodesic problem if we use one of the coordinates on the sphere as the parameter for calculation of arclength. I should have anticipated this in view of the examples I've already given. The parametric equations for a geodesic will be more general, for example in the case of the plane we found horizontal and vertical lines at once whereas one or the other is lost if x or y is taken as the parameter, and hence harder to solve.

Kinetic Energy In Other Coordinates

Basically, can just divide metric by dt^2. Or we can derive these through chain/product rules applied to coordinate transformations.

POLAR COORDINATES:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Supposing x, y & r, θ are all functions of time t with $\dot{x}, \dot{y}, \dot{r}, \dot{\theta}$ denoting their respective t-derivatives we find they must satisfy the following relations,

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \longrightarrow \dot{x}^2 = \dot{r}^2\cos^2\theta - 2r\dot{r}\sin\theta\cos\theta\,\dot{\theta} + r^2\sin^2\theta\,\dot{\theta}^2$$
$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} \longrightarrow \dot{y}^2 = \dot{r}^2\sin^2\theta + 2r\dot{r}\sin\theta\cos\theta\,\dot{\theta} + r^2\cos^2\theta\,\dot{\theta}^2$$

Note the cross-terms cancel once we add, we find the kinetic energy in polar coordinates from $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$,

$$\boxed{T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)} \quad \text{or} \quad \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$$

(notation sometimes altered)

Spherical Coordinates: $r = \sqrt{x^2+y^2+z^2}$, ϕ = polar angle, θ = azimuthal angle

$$\underbrace{\boxed{T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right)}}$$

follows from differentiating $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$ and substituting into $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$.

Likewise, CYLINDRICAL COORDINATES

$$\boxed{T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right)}$$

Supposing that
$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}\right\} \text{ makes } \varphi \text{ a "polar" angle.}$$
$$z = z$$

14.8 Lagrangian mechanics

14.8.1 basic equations of classical mechanics summarized

Classical mechanics is the study of massive particles at relatively low velocities. Let me refresh your memory about the basics equations of Newtonian mechanics. Our goal in this section will be to rephrase Newtonian mechanics in the variational langauge and then to solve problems with the Euler-Lagrange equations. Newton's equations tell us how a particle of mass m evolves through time according to the net-force impressed on m. In particular,

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

If m is not constant then you may recall that it is better to use momentum $\vec{P} = m\vec{v} = m\frac{d\vec{r}}{dt}$ to set-up Newton's 2nd Law:

$$\frac{d\vec{P}}{dt} = \vec{F}$$

In terms of components we have a system of differential equations with indpendent variable time t. If we use position as the dependent variable then Newton's 2nd Law gives three second order ODEs,

$$m\ddot{x} = F_x \qquad m\ddot{y} = F_y \qquad m\ddot{z} = F_z$$

where $\vec{r} = (x, y, z)$ and the dots denote time-derivatives. Moreover, $\vec{F} =< F_x, F_y, F_z >$ is the sum of the forces that act on m. In contrast, if you work with momentum then you would want to solve six first order ODEs,

$$\dot{P}_x = F_x \qquad \dot{P}_y = F_y \qquad \dot{P}_z = F_z$$

and $P_x = m\dot{x}$, $P_y = m\dot{y}$ and $P_z = m\dot{z}$. These equations are easiest to solve when the force is not a function of velocity or time. In particular, if the force $\vec{F}$ is conservative then there exists a potential energy function $U : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = -\nabla U$. We can prove that in the case the force is conservative the total energy is conserved.

14.8.2 kinetic and potential energy, formulating the Lagrangian

Recall the kinetic energy is $T = \frac{1}{2}m||\vec{v}||^2$, in Cartesian coordinates this gives us the formula:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

If $\vec{F}$ is a conservative force then it is independent of path so we may construct the potential energy function as follows:

$$U(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{F} \cdot d\vec{r}$$

Here $\mathcal{O}$ is the origin for the potential and we can prove that the potential energy constructed in this manner has $\vec{F} = -\nabla U$. We can prove that the total (mechanical) energy $E = T + U$ for a conservative system is a constant; $dE/dt = 0$. Hopefully these comments are at least vaguely familiar from some physics course in your distant memory. If not relax, calculationally this chapter is self-contained, read onward.

We already calculated that if we use T as the Lagrangian then the Euler-Lagrange equations produce Newton's equations in the case that the force is zero (see 14.5.1). Suppose that we define the Lagrangian to be $L = T - U$ for a system governed by a conservative force with potential energy function U. We seek to prove the Euler-Lagrange equations are precisely Newton's equations for this conservative system[1] Generically we have a Lagrangian of the form

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z).$$

We wish to find extrema for the functional $S = \int L(t)\, dt$. This yields three sets of Euler-Lagrange equations, one for each dependent variable x, y or z

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] = \frac{\partial L}{\partial x} \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}}\right] = \frac{\partial L}{\partial y} \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}}\right] = \frac{\partial L}{\partial z}.$$

Note that $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$, $\frac{\partial L}{\partial \dot{y}} = m\dot{y}$ and $\frac{\partial L}{\partial \dot{z}} = m\dot{z}$. Also note that $\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F_x$, $\frac{\partial L}{\partial y} = -\frac{\partial U}{\partial y} = F_y$ and $\frac{\partial L}{\partial z} = -\frac{\partial U}{\partial z} = F_z$. It follows that

$$\boxed{m\ddot{x} = F_x \qquad m\ddot{y} = F_y \qquad m\ddot{z} = F_z.}$$

Of course this is precisely $m\vec{a} = \vec{F}$ for a net-force $\vec{F} =< F_x, F_y, F_z >$. We have shown that **Hamilton's principle** reproduces Newton's Second Law for conservative forces. Let me take a moment to state it.

Definition 14.8.1. Hamilton's Principle:

If a physical system has generalized coordinates q_j with velocities $\dot{q}_j$ and Lagrangian $L = T - U$ then the solutions of physics will minimize the action S defined below:

$$\boxed{S = \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\, dt}$$

Mathematically, this means the variation $\delta S = 0$ for physical trajectories.

This is a necessary condition for solutions of the equations of physics. Sufficient conditions are known, you can look in any good variational calculus text. You'll find analogues to the second derivative test for variational differentiation. As far as I can tell physicists don't care about this logical gap, probably because the solutions to the Euler-Lagrange equations are the ones for which they are looking.

[1]don't mistake this example as an admission that Lagrangian mechanics is limited to conservative systems. Quite the contrary, Lagrangian mechanics is actually more general than the orginal framework of Newton!

14.8.3 easy physics examples

Now, you might just see this whole exercise as some needless multiplication of notation and formalism. After all, I just told you we just get Newton's equations back from the Euler-Lagrange equations. To my taste the impressive thing about Lagrangian mechanics is that you get to start the problem with energy. Moreover, the Lagrangian formalism handles non-Cartesian coordinates with ease. If you search your memory from classical mechanics you'll notice that you either do constant acceleration, circular motion or motion along a line. What if you had a particle constrained to move in some frictionless ellipsoidal bowl. Or what if you had a pendulum hanging off another pendulum? How would you even write Newtons' equations for such systems? In contrast, the problem is at least easy to set-up in the Lagrangian approach. Of course, solutions may be less easy to obtain.

Example 14.8.2. Projectile motion: *take z as the vertical direction and suppose a bullet is fired with initial velocity $v_o =< v_{ox}, v_{oy}, v_{oz} >$. The potential energy due to gravity is simply $U = mgz$ and kinetic energy is given by $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Thus,*

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

Euler-Lagrange equations are simply:

$$\frac{d}{dt}\Big[m\dot{x}\Big] = 0 \qquad \frac{d}{dt}\Big[m\dot{y}\Big] = 0 \qquad \frac{d}{dt}\Big[m\dot{z}\Big] = \frac{\partial}{\partial z}(-mgz) = -mg.$$

Integrating twice and applying initial conditions gives us the (possibly familiar) equations

$$x(t) = x_o + v_{ox}t, \qquad y(t) = y_o + v_{oy}t, \qquad z(t) = z_o + v_{oz}t - \tfrac{1}{2}gt^2.$$

Example 14.8.3. Simple Pendulum: *let θ denote angle measured off the vertical for a simple pendulum of mass m and length l. Trigonmetry tells us that*

$$x = l\sin(\theta) \qquad y = l\cos(\theta) \qquad \Rightarrow \qquad \dot{x} = l\cos(\theta)\dot{\theta} \qquad y = -l\sin(\theta)\dot{\theta}$$

Thus $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\theta}^2$. Also, the potential energy due to gravity is $U = -mgl\cos(\theta)$ which gives us

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos(\theta)$$

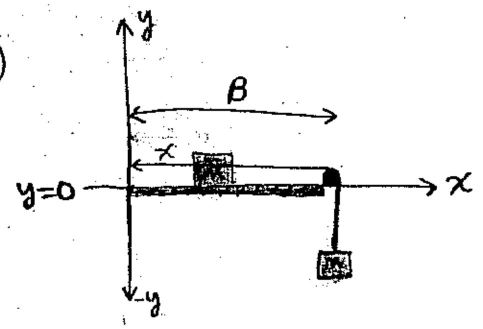
Then, the Euler-Lagrange equation in θ is simply:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] = \frac{\partial L}{\partial \theta} \qquad \Rightarrow \qquad \frac{d}{dt}(ml^2\dot{\theta}) = -mgl\sin(\theta) \qquad \Rightarrow \qquad \ddot{\theta} + \frac{g}{l}\sin(\theta) = 0.$$

In the small angle approximation, $\sin(\theta) = \theta$ then we have the solution $\theta(t) = \theta_o\cos(\omega t + \phi_o)$ for angular frequency $\omega = \sqrt{g/l}$

Ex TWO BLOCKS OF MASS M FALL OR SLIDE AS PICTURED W/O FRICTION

(a)



$$(\beta - x) - y = l \quad \text{where } \beta, l \text{ are just constants}$$

$$-x - y = l - \beta \implies -\dot{x} - \dot{y} = 0 \implies \dot{x} = -\dot{y}$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}M\dot{y}^2 = M\dot{y}^2$$

$$U = Mgy$$

$$L = T - U = M\dot{y}^2 - Mgy = 0$$

$$\frac{\partial L}{\partial \dot{y}} = 2M\dot{y} \qquad \frac{\partial L}{\partial y} = -Mg$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 2M\ddot{y} + Mg = 0 \implies \ddot{y} = \frac{-g}{2}$$

, this says the acceleration in y is constant thus integrating twice we obtain the standard form constant acceleration formulae we know and love from freshman physics,

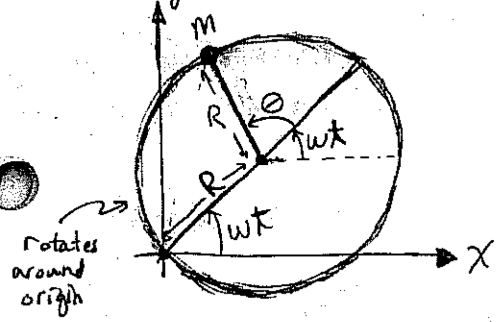
$$\left(\begin{array}{l}\dot{y}_o = \text{intial } V_y \\ y_o = \text{intial position}\end{array}\right)$$

$$\int \ddot{y}\,dt = \int -\frac{g}{2}\,dt \implies \dot{y} = -\frac{g}{2}(t) + \dot{y}_o$$

$$\int\left(\dot{y}_o - \frac{g}{2}t\right)dt = y_o - \frac{gt^2}{4} + \dot{y}_o t \,, \quad \boxed{y(t) = y_o + \dot{y}_o t - \frac{g}{4}t^2}$$

(in general we would have $y(t) = C_1 + C_2 t - \frac{g}{4}t^2$ but I applied the intial conditions)

example: Pendulum mounted on center of wheel rotating around point on rim.



$$x = R\cos(\omega t) + R\cos(\omega t + \theta)$$
$$y = R\sin(\omega t) + R\sin(\omega t + \theta)$$
$$\dot{x} = -R\omega\sin(\omega t) - R(\omega + \dot{\theta})\sin(\omega t + \theta)$$
$$\dot{y} = R\omega\cos(\omega t) + R(\omega + \dot{\theta})\cos(\omega t + \theta)$$

$$\dot{x}^2 = \left(-R\omega\sin(\omega t) - R(\omega+\dot{\theta})\sin(\omega t+\theta)\right)\left(-R\omega\sin(\omega t) - R(\omega+\dot{\theta})\sin(\omega t+\theta)\right), \quad \text{Let } \beta = \omega t.$$
$$= R^2\omega^2\sin^2(\beta) + 2R(\omega+\dot{\theta})\sin(\beta+\theta)R\omega\sin(\beta) + R^2(\omega+\dot{\theta})^2\sin^2(\omega t+\theta)$$

$$\dot{y}^2 = \left(R^2\omega^2\cos^2(\beta) + 2R^2\omega(\omega+\dot{\theta})\cos(\beta)\cos(\beta+\theta) + R^2(\omega+\dot{\theta})^2\cos^2(\omega t+\theta)\right)$$

$$T = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) = \tfrac{1}{2}m\Big\{R^2\omega^2(\sin^2\beta+\cos^2\beta) + 2R^2\omega(\omega+\dot{\theta})\left[\sin(\beta)\sin(\beta+\theta) + \cos(\beta)\cos(\beta+\theta)\right]$$
$$+ R^2(\omega+\dot{\theta})^2\left[\sin^2(\beta+\theta) + \cos^2(\beta+\theta)\right]\Big\}$$

$$\sin(\beta)\sin(\beta+\theta) + \cos(\beta)\cos(\beta+\theta) = \sin(\beta)\left[\sin\beta\cos\theta + \cos\beta\sin\theta\right] + \cos\beta\left[\cos\beta\cos\theta - \sin\theta\sin\beta\right]$$
$$= \cos\theta(\sin^2\beta + \cos^2\beta) + \sin\beta\cos\beta\sin\theta - \cos\beta\sin\theta\sin\beta$$
$$= \cos\theta$$

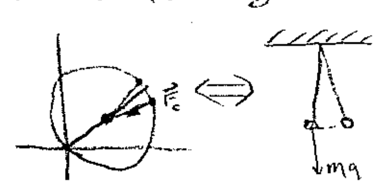
$$T = \tfrac{1}{2}m\left(R^2\omega^2 + 2R^2\omega(\omega+\dot{\theta})\cos\theta + R^2(\omega+\dot{\theta})^2\right)$$

$L = T - U = T.$ as no U was given.

$$\frac{\partial L}{\partial\dot{\theta}} = mR^2\omega\cos\theta + mR^2(\omega+\dot{\theta})$$
$$\frac{\partial L}{\partial\theta} = -mR^2\omega(\omega+\dot{\theta})\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = \frac{d}{dt}\left(mR^2\omega\cos\theta + mR^2(\omega+\dot{\theta})\right) + mR^2\omega(\omega+\dot{\theta})\sin\theta$$
$$= -mR^2\omega\dot{\theta}\sin\theta + mR^2\ddot{\theta} + mR^2\omega\dot{\theta}\sin\theta + mR^2\omega^2\sin\theta$$
$$= mR^2\ddot{\theta} + mR^2\omega^2\sin\theta = 0$$
$$\Rightarrow \boxed{\ddot{\theta} + \omega^2\sin\theta = 0} \quad (*)$$

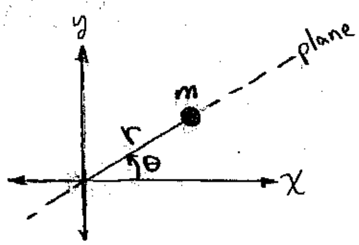
now if θ is small $\sin\theta \approx \theta \Rightarrow (*) \longrightarrow \ddot{\theta} + \omega^2\theta = 0$ which is the equation of a pendulum for small angular dispacements WITH THE NORMAL PENDULUM $\omega = -\frac{g}{\ell}$. Our solution reduces to the classic pendulum equation provided θ is small, that is to say the mass is "about" the end of the diameter



Example: moving plane, mass slides on moving plane.

A PARTICLE OF MASS m RESTS ON A SMOOTH PLANE. THE PLANE IS RAISED TO AN INCLINATION ANGLE θ AT A CONSTANT RATE α.

$$\dot{\theta} = \alpha \Rightarrow \theta = \alpha t + \text{const}, \quad \theta(0) = 0 \Rightarrow \theta(t) = \theta = \alpha t$$



SUPPOSE THE PLANE TILTS ABOUT THE z AXIS. THE TRANSFORMATIONS ARE

$$\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{Bmatrix} \quad \text{and as } \theta = \alpha t \Rightarrow \begin{Bmatrix} x = r\cos\alpha t \\ y = r\sin\alpha t \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{x} = \dot{r}\cos\alpha t - r\alpha\sin\alpha t \\ \dot{y} = \dot{r}\sin\alpha t + r\alpha\cos\alpha t \end{Bmatrix}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}\cos\alpha t - r\alpha\sin\alpha t)^2 + \frac{1}{2}m(\dot{r}\sin\alpha t + r\alpha\cos\alpha t)^2$$

$$= \frac{1}{2}m\left(\dot{r}^2\cos^2\alpha t - 2\, r\alpha\cancel{\sin\alpha t\,\dot{r}\cos\alpha t} + r^2\alpha^2\sin^2\alpha t\right)$$

$$+ \frac{1}{2}m\left(\dot{r}^2\sin^2\alpha t + 2\dot{r}\cancel{\sin\alpha t\, r\alpha\cos\alpha t} + r^2\alpha^2\cos^2\alpha t\right)$$

$$= \frac{1}{2}m\left(\dot{r}^2(\cos^2\alpha t + \sin^2\alpha t) + r^2\alpha^2(\sin^2\alpha t + \cos^2\alpha t)\right) = \boxed{\frac{1}{2}m(\dot{r}^2 + r^2\alpha^2) = T}$$

For simplicity I choose gravity to act in the y direction in the usual way so,

$$U = mgy = \boxed{mgr\sin\alpha t = U}$$

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\alpha^2) - mgr\sin\alpha t \quad \text{(LANGRANGIAN IN } r\text{)}$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial r} = mr\alpha^2 + mg\sin\alpha t$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{d}{dt}(m\dot{r}) - mr\alpha^2 - mg\sin\alpha t = 0$$

ELIMINATING m we have the second order differential equation with constant coefficients and a non-homogeneous term to solve.

Moving Plate Example Continued,

$$\boxed{\ddot{r} - r\alpha^2 = g\sin\alpha t}$$, first find r_H.

$\ddot{r}_H - r_H\alpha^2 = 0 \Rightarrow z^2 - \alpha^2 = 0 \Rightarrow z = \pm\alpha \Rightarrow$ sol^n by the method of constant coefficients, based on writing the characteristic equation (in z here) is just $r_H = c_1 e^{\alpha t} + c_2 e^{-\alpha t} = c_3\cosh(\alpha t) + c_4\sinh(\alpha t)$

Then suppose $r_p = a\cos\alpha t + b\sin\alpha t$ and substitute into DE.

$$\ddot{r}_p - r_p\alpha^2 = -a\alpha^2\cos\alpha t - b\alpha^2\sin\alpha t - \alpha^2 a\cos\alpha t - \alpha^2 b\sin\alpha t = g\sin\alpha t$$

$$\cos\alpha t(-2a\alpha^2) + \sin\alpha t(2b\alpha^2) = \sin\alpha t(g)$$

$$\Rightarrow -2a\alpha^2 = 0 \quad , \quad 2b\alpha^2 = g$$

$$\Rightarrow a = 0 \quad , \quad b = \frac{g}{2\alpha^2}$$

Then the general Solution is

$$r = r_H + r_P = C_3\cosh(\alpha t) + C_4\sinh(\alpha t) + \frac{g}{2\alpha^2}\sin\alpha t$$

Let r_0 be the intial position for the mass then,

$$r_0 = C_3 + C_4\sinh(0) + \frac{g}{2\alpha^2}\sin(0) = C_3$$

Then as the particle was intially at rest we know $\dot{r} = 0$.

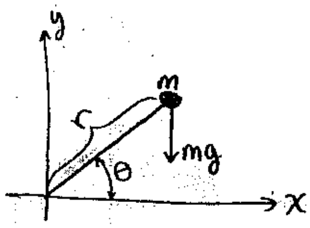
$$\dot{r} = \alpha r_0\sinh(\alpha t) + \alpha C_4\cosh(\alpha t) + \frac{g}{2\alpha}\cos(\alpha t)$$

$$\dot{r}(0) = \alpha C_4 + \frac{g}{2\alpha} = 0 \Rightarrow C_4 = \frac{g}{2\alpha^2}$$

Finally

$$\boxed{r(t) = r_0\cosh(\alpha t) + \frac{g}{2\alpha^2}(\sin\alpha t - \sinh\alpha t)}$$

Gravity & A general radial force analyzed:



$$\vec{F} = -mg\,\hat{j} - Ar^{\alpha-1}\,\hat{r} = \vec{F}_g + \vec{F}_r$$

$$-\nabla U = -\nabla(U_g + U_r) = \vec{F}_g + \vec{F}_r$$

we may superpose the two potential energy functions (radial and gravitational) to find the total potential energy -

$U_r = \dfrac{Ar^\alpha}{\alpha}$ from problem 7.4 as the radial force is identical here.

$U_g = mgy = mgr\sin\theta$, using $y = r\sin\theta$ as before.

I choose radial coordinates r, θ.

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ from (7.4)

$U = \dfrac{Ar^\alpha}{\alpha} + mgr\sin\theta$

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{Ar^\alpha}{\alpha} - mg\,r\sin\theta$$

$\dfrac{\partial L}{\partial \dot{r}} = m\dot{r}$ $\qquad \dfrac{\partial L}{\partial r} = mr\dot{\theta}^2 - Ar^{\alpha-1} - mg\sin\theta$

$\dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ $\qquad \dfrac{\partial L}{\partial \theta} = -mgr\cos\theta$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + Ar^{\alpha-1} + mg\sin\theta = \boxed{m\ddot{r} - mr\dot{\theta}^2 + Ar^{\alpha-1} + mg\sin\theta = 0}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(mr^2\dot{\theta}) + mgr\cos\theta = m\frac{dr^2}{dt}\dot{\theta} + mr^2\ddot{\theta} + mgr\cos\theta = 0$$

$$\Rightarrow 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} + gr\cos\theta = 0$$

$$\Rightarrow \boxed{2\dot{r}\dot{\theta} + r\ddot{\theta} + g\cos\theta = 0}$$

In general we have defined the momenta of a coordinate q as $\dfrac{\partial T}{\partial \dot{q}}$ here we consider θ. $P_\theta = \dfrac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$. This time we may not claim that P_θ is conserved. $\dfrac{d}{dt}(mr^2\dot{\theta}) = -mgr\cos\theta \Rightarrow$ that P_θ is not the same for all time, there is some change with time thus it is not conserved, this is a consequence of gravity, if this were not the case then we would be able to spin things in vertical planes, one could "swing" little children in complete orbits around the swing set.

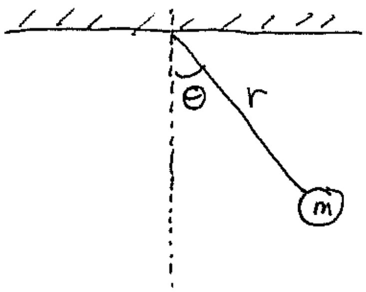
Lagrange Multipliers

The Euler–Lagrange Eq^{n}s are modified to include constraints $f_k(q_1, q_2, .., q_n, t) = 0$ for $k = 1, 2, .., m$ as follows,

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \sum_{k=1}^{m} \lambda_k(t)\frac{\partial f_k}{\partial q_j} = Q_j} \quad (*)$$

for $j = 1, 2, ..., n$ where q_j are generalized coordinates for the system considered.

Example: simple pendulum



$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$U = -mgr\cos\theta$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta$$

The constraint here is $r = l$ which we can write as $f_1(r,\theta) = r - l$. This constaint is time independent. Using $(*)$ as our guide,

① $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda_r\frac{\partial f_1}{\partial r} = \lambda_r = Q_r$$

$$\Rightarrow \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 - mg\cos\theta = \lambda_r \quad (\mathrm{I})$$

② $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda_r\frac{\partial f_1}{\partial \theta} = 0$$

$$\Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) + mgr\sin\theta = 0 \quad (\mathrm{II})$$

③ $r = l$.

Obviously ③ simplifies things considerably. Note $\dot{r} = 0$ thus

$(\mathrm{I}) \longrightarrow \quad -mr\dot{\theta}^2 - mg\cos\theta = \lambda_r \quad \longleftarrow$ centripetal acceleration & more.

$(\mathrm{II}) \longrightarrow \quad \frac{d}{dt}(ml^2\dot{\theta}) = -mgl\sin\theta \longrightarrow \frac{dL}{dt} = -mgl\sin\theta$

think angular mom. & torque!

Ex] $L_{free} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

Constraint: constant velocity $f_1 = x - V_1 t$, $f_2 = y - V_2 t$.

$$\hookrightarrow \quad L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \lambda_1 f_1 - \lambda_2 f_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \longrightarrow m\ddot{x} = \lambda_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \longrightarrow m\ddot{y} = \lambda_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}_1}\right) - \frac{\partial L}{\partial \lambda_1} = f_1 = 0 \rightarrow x = v_1 t \longrightarrow \dot{x} = v_1 \rightarrow \ddot{x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}_2}\right) - \frac{\partial L}{\partial \lambda_2} = f_2 = 0 \rightarrow y = V_2 t \longrightarrow \dot{y} = V_2 \longrightarrow \ddot{y} = 0$$

Hence the generalized forces λ_1 & λ_2 are both zero.

Ex] $L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \lambda f$ where $f = y - g(x)$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial}{\partial x}\lambda f = 0 \rightarrow m\ddot{x} = \lambda \frac{\partial f}{\partial x} = -\lambda \frac{dg}{dx}.$$

$$m\ddot{y} = \lambda \frac{\partial f}{\partial y} = \lambda$$

If we want $y = g(x) = x^2$ then $\frac{dg}{dx} = 2x$

and we need $m\ddot{x} = -2x\lambda$ & $m\ddot{y} = \lambda$ for $y = x^2$.

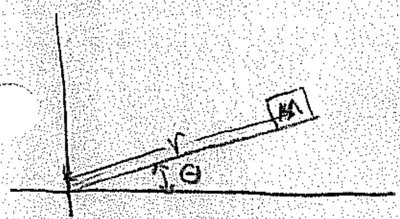
Note, $\dot{y} = 2x\dot{x}$ & $\ddot{y} = 2\dot{x}^2 + 2x\ddot{x}$

$$\Rightarrow m\ddot{y} = m(2\dot{x}^2 + 2x\ddot{x}) = \lambda$$

$$\hookrightarrow m\ddot{x} = -2x\lambda = -2x(m(2\dot{x}^2 + 2\ddot{x}))$$

$$\Rightarrow m\ddot{x} + 4x\ddot{x} = -4mx\dot{x}^2 \ldots$$

Use LAGRANGE Multipliers to find constraint forces, for Moving Plane Example



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$T_{POLAR} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \qquad U = mgr\sin\theta$$

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\sin\theta$$

Constraint : $\dot{\theta} = \alpha \Rightarrow \theta = \alpha t + const.$ and $\theta(t=0) = 0 \Rightarrow const = 0 \Rightarrow \theta = \alpha t$

thus $f_1(r,\theta) = \theta - \alpha t = 0.$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda_1 \frac{\partial f_1}{\partial r} = 0 = \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 - mg\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda_1 \frac{\partial f_1}{\partial \theta} = \lambda_1(1) = \lambda_1 = \frac{d}{dt}\left(mr^2\dot{\theta}\right) + mgr\cos\theta$$

Three Equations, Three unknowns, r, θ, λ_1

$$\left\{\begin{array}{l} \theta - \alpha t = 0 \\ m\ddot{r} - mr\dot{\theta}^2 - mg\sin\theta = 0 \\ \frac{d}{dt}\left(mr^2\dot{\theta}\right) + mgr\cos\theta = \lambda_1 \end{array}\right\}$$

then as $\theta = \alpha t$, $\dot{\theta} = \alpha$

$$\left\{\begin{array}{l} m\ddot{r} - mr\alpha^2 - mg\sin\alpha t = 0 \\ (*)\ \frac{d}{dt}\left(mr^2\alpha\right) + mgr\cos\alpha t = \lambda_1 \end{array}\right\}$$

Now I solve $\ddot{r} - \alpha^2 r - g\sin\alpha t = 0$ to find $r(t)$ as before

I will omit the details.

$$r(t) = r_0\cosh(\alpha t) + \frac{g}{2\alpha^2}\left(\sin\alpha t - \sinh\alpha t\right)$$

Then we substitute $r(t)$ into $(*)$ to get

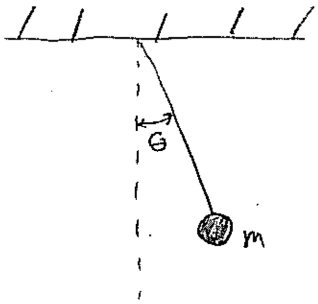
$$\frac{d}{dt}\left(mr^2\alpha\right) + mgr\cos\alpha t = \lambda_1$$

$$m(2r)\dot{r}\alpha + mgr\cos\alpha t = \lambda_1$$

$$\boxed{mr\left(2\alpha\dot{r} + g\cos\alpha t\right) = \lambda_1} = \sum_k \lambda_k \frac{\partial f_k}{\partial \theta} \equiv Q_{\theta} \ : \text{ the generalized force of constraint in } \theta.$$

and $\sum \lambda_k \frac{\partial f_k}{\partial r} \equiv Q_r = 0$: generalized constraint force for r.

Hamiltonian for Simple Pendulum



$$\dot{l} = -\alpha$$
$$l = -\alpha t + l_0$$
$$x = l\sin\theta = (-\alpha t + l_0)\sin\theta$$
$$y = -l\cos\theta = -(-\alpha t + l_0)\cos\theta$$
$$\dot{x} = \dot{l}\sin\theta + l\dot{\theta}\cos\theta = -\alpha\sin\theta + (-\alpha t + l_0)\dot{\theta}\cos\theta$$
$$\dot{y} = -\dot{l}\cos\theta - l\dot{\theta}\sin\theta = +\alpha\cos\theta - (-\alpha t + l_0)\dot{\theta}\sin\theta$$

$$L = \frac{1}{2}m\left(\dot{l}^2 + l^2\dot{\theta}^2\right) + mgl\cos\theta = \boxed{\frac{1}{2}m\left(\alpha^2 + l^2\dot{\theta}^2\right) + mgl\cos\theta = L}$$

$$P_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} \implies \dot{\theta} = P_\theta / ml^2$$

$$H = \dot{\theta}P_\theta - L = \frac{P_\theta^2}{ml^2} - \frac{1}{2}m\left(\alpha^2 + l^2\dot{\theta}^2\right) - mgl\cos\theta$$
$$= \frac{P_\theta^2}{ml^2} - \frac{1}{2}m\left(\alpha^2 + \frac{P_\theta^2}{m^2l^2}\right) - mgl\cos\theta$$

$$\boxed{H(l,\theta,P_\theta) = \frac{P_\theta^2}{2ml^2} - \frac{1}{2}m\alpha^2 - mgl\cos\theta}$$

$$E = T + U = \frac{1}{2}m\left(\alpha^2 + l^2\dot{\theta}^2\right) - mgl\cos\theta$$
$$= \frac{1}{2}m\alpha^2 + \frac{1}{2}\frac{P_\theta^2}{ml^2} - mgl\cos\theta$$

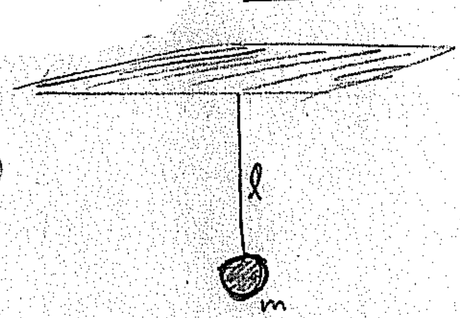
sign of $\frac{1}{2}m\alpha^2$ differs from $H \to E$.

note that the total energy is not the same as the Hamiltonian with this formulation of the problem, because I accounted for the explicit time dependence of the transformation Eq's from the rectangular coord. to the generalized coordinates.

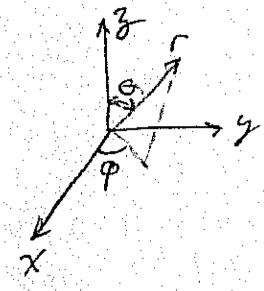
$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\alpha^2 + \frac{1}{2}\frac{P_\theta^2}{ml^2} - mg(\alpha t + l_0)\cos\theta\right) \neq 0$$

thus energy is not conserved.

Hamiltonian for Spherical Pendulum



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$



$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right)$$
$$U = mgr\cos\theta$$

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) - mgr\cos\theta$$

$$\left\{\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \\ p_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} = mr^2\sin^2\theta\,\dot{\varphi} \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \dot{r} &= p_r/m \\ \dot{\theta} &= p_\theta/mr^2 \\ \dot{\varphi} &= p_\varphi/mr^2\sin^2\theta \end{aligned}\right\}$$

$$H = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\phi}p_\phi - L$$
$$= \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\varphi^2}{mr^2\sin^2\theta} - \frac{1}{2}m\left\{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right\} + mgr\cos\theta$$
$$= \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\varphi^2}{mr^2\sin^2\theta} - \frac{1}{2}m\left\{\frac{p_r^2}{m^2} + r^2\left(\frac{p_\theta^2}{m^2r^4}\right) + r^2\sin^2\theta\left(\frac{p_\varphi^2}{m^2r^4\sin^4\theta}\right)\right\} + mgr\cos\theta$$

$$\boxed{H = \cancel{\frac{p_r^2}{2m}}^{\,0} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2\sin^2\theta} + mgr\cos\theta}$$

Combining p_φ and θ terms, $U_{eff}(p_\phi, \theta) = \dfrac{p_\varphi^2}{2mr^2\sin^2\theta} + mgr\cos\theta$

Discuss and Sketch V as a function of θ

(1) $p_\varphi = 0$

(2) several $p_\varphi \neq 0$

(3) Conical pendulum (θ = constant) with reference to $V-\theta$ digram