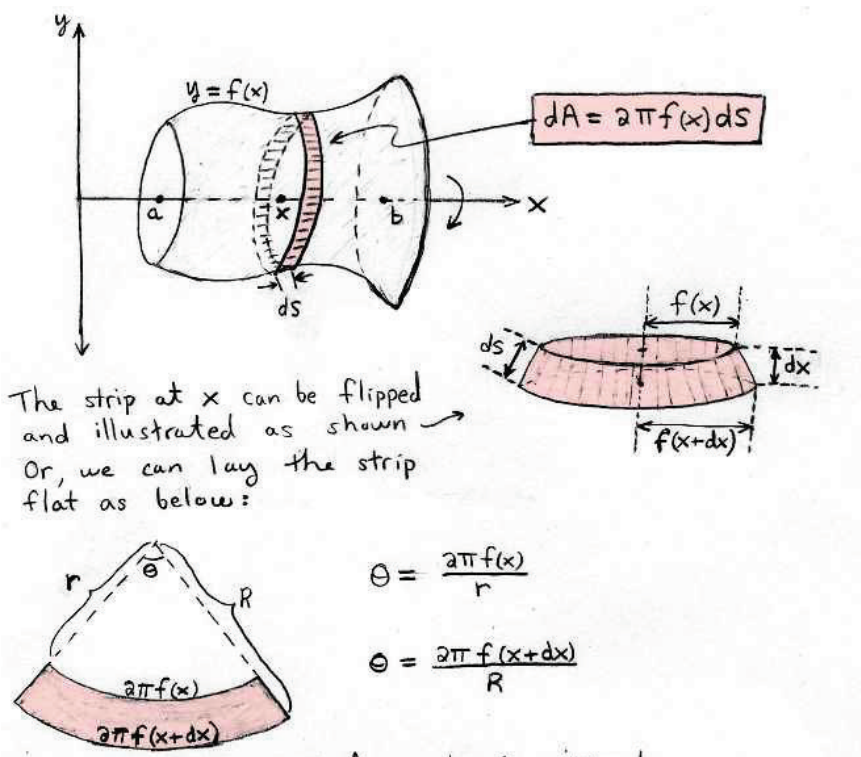
Chapter 14

Further Applications of Integral Calculus

In this short chapter we examine two applications of integral calculus. I will utilize the infinitesimal method to motivate both sections. The surface area problem we consider in this chapter is just the simple case of a surface of revolution. The general problem of surface area is dealt with properly in calculus III. Physics provides a wealth of applied problems, we content ourselves with the problems of work, hydrostatic force on a dam and the center of mass or centroid problem. Our emphasis is once more on mathematical set-up as opposed to physical concept. I will attempt to provide the conceptual framework and then you simply need to work out various geometries in my framework.

14.1 surface area

Imagine we take a graph $y = f(x)$ for $a \leq x \leq b$ and rotate it around the x-axis. We suppose $f(x) \geq 0$ for the purposes of this discussion. This creates a surface of revolution. You may recall from calculus I that we calculated the volume contained inside such surfaces for a variety of cases. For now we will just focus on the case of a surface revolved around the x-axis. Let us focus on just a small portion of the surface. In particular the bit from x to $x + dx$. Suppose ds is the arclength of the graph from $(x, f(x))$ to $(x + dx, f(x + dx))$. The straight-line distance from $(x, f(x))$ to $(x + dx, f(x + dx))$ is identical to ds in this infinitesimal limit. Notice then we can calculate the area of this conical ribbon which has radii $f(x)$ and $f(x + dx)$ at its edges and a length of ds along the edge as described below[1]:



The area of this partial annulus is given by

$$\begin{aligned} dA &= \pi R^2 \left(\frac{\theta}{2\pi}\right) - \pi r^2 \left(\frac{\theta}{2\pi}\right) \\ &= \frac{1}{2}\theta\,(R^2 - r^2) \\ &= \frac{1}{2}\theta\,(R+r)(R-r) \quad : \left[\begin{array}{l}\text{observe } \theta \approx \pi \text{ since} \\ f(x+dx) \approx f(x) \text{ as} \\ dx \approx 0.\end{array}\right] \\ &= \frac{1}{2}(\pi)(2f(x))\,ds \\ &= \underline{2\pi f(x)\,ds}. \end{aligned}$$

Or, in terms of dx, $dA = 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ where $y = f(x)$.

[1] thanks to Ginny for this idea

Furthermore, while we assumed an increasing function for the ease of visualization this formula holds for the case that f is decreasing. Note that ds is positive and we assumed from the outset that $f(x) \geq 0$. Recall $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}dx$ hence the total surface area is thus found from the following integration:

$$\boxed{A = \int_a^b 2\pi f(x)\sqrt{1 + \frac{df}{dx}^2}\, dx}$$

All of this said, we can state a more general formula for parametric curves around an arbitrary axis in the plane. Suppose that $t \mapsto \langle x(t), y(t)\rangle$ is a parametric curve and $\mathcal{L}$ is a line in the plane. Suppose this parametric curve does not cross the axis and any perpendicular bisector of the axis crosses the curve in at most one point. Let $r(t)$ be the distance from the curve to the axis then we can by the same argument as given for $y = f(x)$ derive that the area of the surface or revolution formed by rotation the parameterized curve for $a \leq t \leq b$ is simply:

$$\boxed{A = \int_a^b 2\pi r(t)\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt}$$

In the case of the graph we had $t = x$ and $r(t) = f(x)$. For an abitrary example the real problem is geometrically determining the formula for $r(t)$.

Example 14.1.1. Problem: *Let $R > 0$ and $x = R\cos(t)$ and $y = R\sin(t)$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Find the surface area of the surface formed by revolving the given curve around the y-axis.*

Solution: *the cruve we consider is a half-circle centered at the origin with radius R. Since the axis $\mathcal{L}$ is the y-axis the distance*[2] *to a point $(x(t), y(t))$ on the curve is clearly $x(t)$. Recall $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}dt$ hence we calculate $ds = \sqrt{R^2\sin^2(t) + R^2\cos^2(t)}dt = Rdt$. The area of a typical infinitesimal ribbon is $dA = 2\pi r(t)ds = 2\pi(R\cos(t))(Rdt)$ for each $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Add together all the little $dA's$ by integration to find the total surface area:*

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi R^2\cos(t)dt = 2\pi R^2\sin(t)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \boxed{4\pi R^2.}$$

Therefore, the surface area of a sphere of radius R is $4\pi R^2$. It is interesting to note that the integral of this formula with respect to R yields the volume of a sphere; $V = \frac{4}{3}\pi R^3$. Likewise, the circumference of the disk of radius R is $2\pi R$ which once integrated yields the area πR^2. This pattern does not hold for all solids. For example, if you think about a cube of side length x then the $V = x^3$ whereas the surface area is $A = 6x^2$. The symmetry of the sphere or circle is very special and the pattern continues for higher dimensional spheres[3]

[2] distance from a point to a set is by definition the distance from the point to the closest point in the set

[3] $x^2 + y^2 = 1$ gives unit circle or S_1, $x^2 + y^2 + z^2 = 1$ gives unit spherical shell or S_2, $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$ gives the unit n-sphere S_n. You can find a derivation of the hypervolume of the higher dimensional spheres in Apostol if you're interested.

Example 14.1.2. Problem: *Find the surface area of the surface of revolution formed by rotating $y = \sqrt{x}$ around the x-axis for $0 \leq x \leq 1$.*

Solution: *We have*

$$\begin{aligned} dA &= 2\pi\sqrt{x}\sqrt{1 + \left[\frac{1}{2\sqrt{x}}\right]^2}\ dx \\ &= 2\pi\sqrt{x\left[1 + \frac{1}{4x}\right]}\ dx \\ &= 2\pi\sqrt{x + \frac{1}{4}}\ dx \\ &= \pi\sqrt{4x+1}\ dx \end{aligned}$$

We can integrate the expression above with the help of a $u = 4x + 1$ subsitution. Note

$$\int \sqrt{4x+1}\ dx = \int \sqrt{u}\frac{du}{4} = \frac{2u^{\frac{3}{2}}}{12} + c = \frac{1}{6}(4x+1)^{\frac{3}{2}} + c.$$

Thus,

$$A = \int_0^1 \pi\sqrt{4x+1}\ dx = \frac{\pi}{6}(4x+1)^{\frac{3}{2}}\Big|_0^1 = \boxed{\frac{\pi}{6}(\sqrt{125} - 1).}$$

Example 14.1.3. Problem: *Find the surface area of an open right-circular cone of height h and radius R.*

Solution: *we can view this as a surface of revolution. Take the line $y = Rx/h$ for $0 \leq x \leq h$ and rotate it around the x-axis. Observe that*

$$dA = 2\pi\left(\frac{Rx}{h}\right)\sqrt{1 + \frac{R^2}{h^2}}\ dx = \frac{2\pi Rx}{h^2}\sqrt{h^2 + R^2}\ dx.$$

Now integrate over $0 \leq x \leq h$ to find the total area:

$$A = \int_0^h \frac{2\pi Rx}{h^2}\sqrt{h^2 + R^2}\ dx = \frac{2\pi Rx^2}{2h^2}\sqrt{h^2 + R^2}\Big|_0^h = \boxed{\pi R\sqrt{h^2 + R^2}.}$$

Notice that the formula above checks nicely in the limits $h \to 0$ and $R \to 0$ where we find $A \to \pi R^2$ and $A \to 0$ respective. Can you see why this makes sense?

14.2 physics

In this section we examine a few variable force work problems, variable pressure hydrostatic force problems and finally the center of mass problem for a homogeneous laminate in the plane.

14.2.1 work and force with calculus

The basic physical concepts used here are as follows:

1. work W due to a force F over a displacement Δx is defined to be $W = F\Delta x$ provided the force is exerted in the direction of the displacement and is constant.
2. the force F exerted over an area A by a pressure P is defined to be $F = PA$ provided the pressure is constant over the area A.

In the examples we consider in this section we cannot simply multiply as described above because the requisite idealizations are not met in our examples in the finite case. In other words, the forces are variable and the pressures are not constant. However, if we instead consider an infinitesimal displacement dx or an infinitesimal area dA we can in fact realize the idealized physical laws. It is true that $dW = Fdx$ because the F does not change over the tiny displacement dx. For the problem of the dam, we can say $dF = PdA$ if our dA is a horizontal strip since the pressure is constant over a certain depth. I'll leave the rest of the details for the examples. Mainly we need the following physical equations to complete the examples:

1. $F = mg$, near the surface of the earth this is the force of gravity on a mass m.
2. $P = \rho gd$, is the pressure due to water at a depth d where $\rho \approxeq 1000kg/m^3$ is the density of water.

Example 14.2.1. . *This is a variable work due to variable mass problem.*

Find magnitude of work required to lift a 10m cable with a uniformly distributed mass of 10 kg.

Lets define l to be the length of the cable being lifted, this will vary from $l = 10\,m \longrightarrow l = 0\,m$ as the cable is lifted. We should find the work dW done as we lift length l a distance dl,

$$\begin{aligned} dW &= F\,dl \\ &= -mg\,dl \\ &= -\lambda l g\,dl \end{aligned}$$

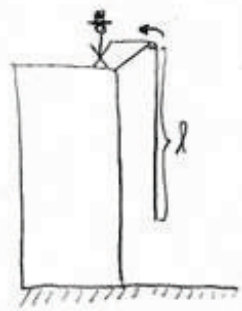
$\lambda = \frac{\text{mass}}{\text{length}} = \frac{10\,kg}{10\,m} = 1\ kg/m$

"λ is linear mass density"

$m = \lambda l$.

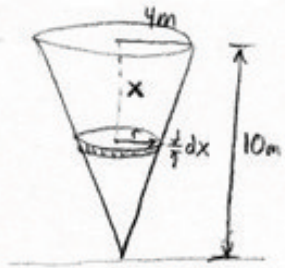
Now add up the work

$$\begin{aligned} W &= \int_{10}^{0} -\lambda g\, l\, dl \\ &= -\lambda g \left(\tfrac{1}{2} l^2\right)\Big|_{10}^{0} \\ &= \lambda g\, \tfrac{1}{2}(10)^2 \\ &= 1 \cdot 9.8 \cdot 50 \\ &= \boxed{490\ J} \end{aligned}$$

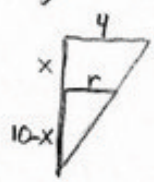


Example 14.2.2. . *This is a variable work due to variable mass problem.*

Find minimum work to pump water out of a full cone.



$$dm = \rho\, dV = \rho \pi r^2 dx$$



$$\frac{4}{10} = \frac{r}{10-x}$$

$$r = \frac{2}{5}(10-x)$$

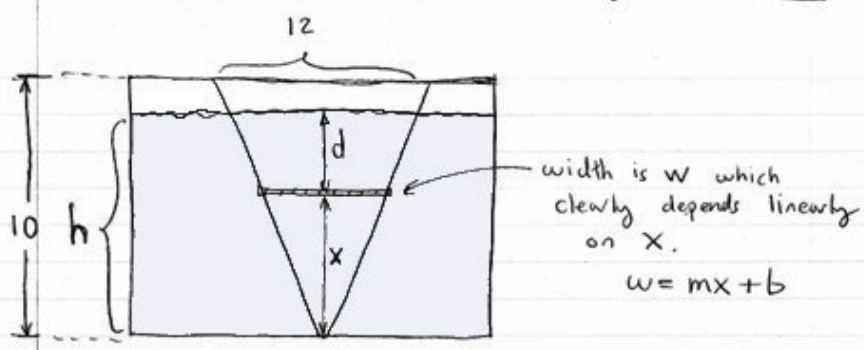
$$\begin{aligned}
dW &= (dm)g \cdot x \\
&= (\rho \pi r^2 dx) g\, x \\
&= \rho g \pi \frac{4(10-x)^2}{25} x\, dx \\
&= \frac{4\pi\rho g}{25}(100x - 20x^2 + x^3)\, dx
\end{aligned}$$

Remark: this is a magic cone, we cannot just tip it over or drill a hole in the bottom. We are forced to pump out the top ☺.

$$\begin{aligned}
W &= \int_0^{10} \frac{4\pi\rho g}{25}(100x - 20x^2 + x^3)\, dx \\
&= \frac{4\pi\rho g}{25}\left(50\times 10^2 - \frac{20}{3}\times 10^3 + \frac{10^4}{4}\right) \\
&= \frac{4\pi(1000)(9.8)}{25}\left(50\times 10^2 - \frac{20}{3}\times 10^3 + \frac{10^4}{4}\right) \\
&= \boxed{4.11\times 10^6\ \mathrm{J}}
\end{aligned}$$

Example 14.2.3. . *The triangular dam problem.*

Find the hydrostatic force on the triangular region pictured below. Assume the water of density ρ is upto level h



$$w(0) = m(0) + b = 0 \quad \therefore \quad \underline{b = 0}$$
$$w(10) = m(10) + 0 = 12 \quad \therefore \quad m = \frac{12}{10} = \frac{6}{5} \quad \therefore \quad \boxed{w = \frac{6}{5}x}$$

This formula for w checks because $w(10) = \frac{6}{5}(10) = 12$ as it should.
Then the area of strip is $dA = w\,dx = \frac{6}{5}x\,dx$.
Now set up the pressure $P = \rho g d$,

$$x + d = h \Rightarrow d = h - x \Rightarrow P = \rho g(h - x)$$

Again the connection between force & pressure is $P = \frac{dF}{dA}$
so $dF = P\,dA$, thus

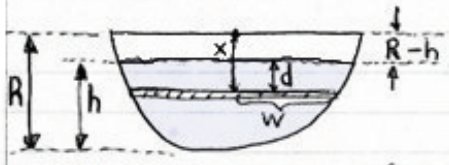
$$dF = \rho g(h - x)\frac{6}{5}x\,dx$$

Now sum the forces for the strips at x in $0 \le x \le h$,

$$F = \int_0^h \frac{6\rho g}{5}(hx - x^2)\,dx$$
$$= \frac{6\rho g}{5}\left(\frac{1}{2}hx^2 - \frac{1}{3}x^3\right)\Big|_0^h$$
$$= \frac{6\rho g}{5}\left(\frac{1}{2}h^3 - \frac{1}{3}h^3\right)$$
$$= \boxed{\frac{1}{5}\rho g h^3}$$

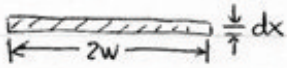
Example 14.2.4. . *The hemispherical dam problem.*

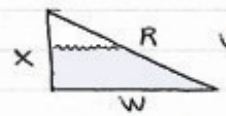
Find hydrostatic force on half-barrel pictured below, well just the end piece. The radius of barrel is R and the water of density ρ is filled to height h. Lets find force dF on a strip of area dA at position x and <u>depth</u> d below the surface.



$$\Rightarrow \quad d = x - (R-h)$$

From picture above we find $P = \rho g d = \rho g (x - R + h)$. (I set up the pressure wrong in notes, it is the <u>depth</u> that should determine the pressure, specifically $P = \rho g d$.) Next;





$$w = \sqrt{R^2 - x^2}$$

$$dA = 2w\,dx = 2\sqrt{R^2 - x^2}\,dx$$

Ok so our choice of x makes dA relatively pretty, (you can try defining x differently but it'll make the square root nasty ...) Ok, we know $P = \frac{dF}{dA}$ so $dF = P\,dA$

$$dF = \rho g\,(x - R + h)\,2\sqrt{R^2 - x^2}\,dx$$

Now we just need to add-up the forces, $R-h \le x \le R$

$$\begin{aligned} F &= \int_{R-h}^{R} 2\rho g\,(x - R + h)\sqrt{R^2 - x^2}\,dx \\ &= \int_{R-h}^{R} 2\rho g\,x\sqrt{R^2 - x^2}\,dx + \int_{R-h}^{R} 2\rho g\,(h - R)\sqrt{R^2 - x^2}\,dx \\ &= 2\rho g \underbrace{\int_{R-h}^{R} x\sqrt{R^2 - x^2}\,dx}_{u\text{-substitute}} + 2\rho g\,(h - R) \underbrace{\int_{R-h}^{R} \sqrt{R^2 - x^2}\,dx}_{\text{trig-substitute.}} \end{aligned}$$

The hemispherical dam problem continued:

$$\int x\sqrt{R^2-x^2}\,dx = \int \sqrt{w}\,\frac{dw}{-2} \qquad \begin{aligned} w &= R^2 - x^2 \\ dw &= -2x\,dx \\ x\,dx &= \frac{-dw}{2} \end{aligned}$$

$$= -\frac{1}{2}\int w^{\frac{1}{2}}\,dw$$

$$= -\frac{1}{2}\,\frac{2}{3}\,w^{\frac{3}{2}} + C$$

$$= -\frac{1}{3}\left(R^2 - x^2\right)^{3/2} + C$$

$$\int \sqrt{R^2-x^2}\,dx = \int (R\cos\theta)(R\cos\theta\,d\theta) \qquad \begin{aligned} x &= R\sin\theta \\ dx &= R\cos\theta\,d\theta \\ R^2-x^2 &= R^2\cos^2\theta \\ \sqrt{R^2-x^2} &= R\cos\theta \end{aligned}$$

$$= \int R^2\cos^2\theta\,d\theta$$

$$= \int \frac{R^2}{2}\left(1+\cos(2\theta)\right)d\theta$$

$$= \frac{R^2}{2}\int d\theta + \frac{R^2}{2}\int \cos(2\theta)\,d\theta$$

$$= \frac{R^2}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$$

$$F = 2\rho g\int_{R-h}^{R} x\sqrt{R^2-x^2}\,dx + 2\rho g(h-R)\int_{R-h}^{R}\sqrt{R^2-x^2}\,dx$$

$$= \frac{-2\rho g}{3}\left(R^2-x^2\right)^{3/2}\Big|_{R-h}^{R} + \rho g(h-R)R^2\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{x=R-h}^{x=R}$$

$$= +\frac{2}{3}\rho g\left(R^2-(R-h)^2\right)^{3/2} + \rho g(h-R)R^2\left(\sin^{-1}\left(\frac{x}{R}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{x}{R}\right)\right)\right)\Big|_{R-h}^{R}$$

$$= \frac{2}{3}\rho g\left(R^2-(R-h)^2\right)^{3/2} + \rho g(h-R)R^2\left(\sin^{-1}\left(\frac{R}{R}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{R}{R}\right)\right)\right)$$

$$- \rho g(h-R)R^2\left(\sin^{-1}\left(\frac{R-h}{R}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{R-h}{R}\right)\right)\right)$$

$$= \boxed{\frac{2}{3}\rho g\left(R^2-(R-h)^2\right)^{3/2} + \rho g(h-R)R^2\left[\frac{\pi}{2} + \frac{1}{2}\sin\left(2\,\frac{\pi}{2}\right) - \sin^{-1}\left(\frac{R-h}{R}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{R-h}{R}\right)\right)\right]}$$

14.2.2 center of mass

The concept of center of mass is a ubiquituous topic in mechanics. In a nutshell it allows us to idealize shapes with finite size as if they were just a point mass. This is a tremendous simplification as it allows us to think of just one particle at a time rather than the infinity of atoms that make up a solid. I'll prove this idealization is reasonable in physics, but for here we just want to see how the center of mass is calculate via calculus.

Moments & Centers of Mass (com)

In the discrete case we define $\vec{r}_{cm} = \dfrac{\sum m_i \vec{r}_i}{\sum m_i}$ the sum is over all the particles.

For 2 dimensional case we have

$$\vec{r}_{cm} = (x_{cm}, y_{cm})$$

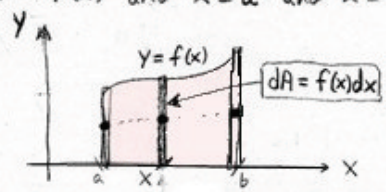
$$x_{cm} = \left(\sum m_i x_i\right) / \left(\sum m_i\right) = M_x / M$$

$$y_{cm} = \left(\sum m_i y_i\right) / \left(\sum m_i\right) = M_y / M$$

M_x = moment of intertia w.r.t x-axis.

M = total mass.

What then is the generalization of this to a continous region. Lets see how to find the c.o.m. of the region bounded by $y = 0$, $y = f(x)$ and $x = a$ and $x = b$. Assume uniform density.



Each strip has its c.o.m. at $(x, \frac{1}{2} f(x))$. We can treat it like a bunch of particles with dm each

$$\rho = \frac{dm}{dA} \rightarrow dm = \rho\, dA$$

So we find the c.o.m. of this system in the natural way,

$$\bar{x} = \frac{\int_a^b \rho x f(x) dx}{\int_a^b \rho f(x) dx} \quad \text{AND} \quad \bar{y} = \frac{\int_a^b \rho \frac{1}{2} f(x) f(x) dx}{\int \rho f(x) dx}$$

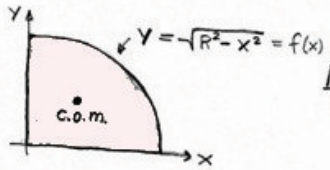
Or in terms of moments & mass we have

$M_y \equiv \rho \int x f(x) dx$	$\bar{x} = \dfrac{M_y}{M}$
$M_x \equiv \rho \int \frac{1}{2} [f(x)]^2 dx$	$\bar{y} = \dfrac{M_x}{M}$
$M = \rho \int f(x) dx = \rho\,(\text{AREA})$	

We assume ρ to be a constant.

Example 14.2.5. . *Center of mass problem.*

Find the moments of intertia and c.o.m for a quarter-circle of uniform density ρ with radius R



$$M = \int_0^R \rho f(x)\,dx$$
$$= \rho \int_0^R f(x)\,dx$$
$$= \boxed{\tfrac{\pi}{4}\rho R^2 = M}$$

mass is simply area times density.

$$M_y = \rho \int_0^R x\sqrt{R^2 - x^2}\,dx$$
$$= \rho \int_{R^2}^{0} -\tfrac{1}{2}\sqrt{u}\,du$$

$u = R^2 - x^2 \qquad u(R) = 0$
$du = -2x\,dx \qquad u(0) = R^2$

$$= \frac{\rho}{2}\,\frac{2}{3}\,u^{3/2}\Big|_0^{R^2}$$
$$= \frac{\rho}{3}\left(R^2\right)^{3/2}$$
$$= \boxed{\tfrac{1}{3}\rho R^3 = M_x}$$

$$M_x = \rho \int_0^R \tfrac{1}{2}(R^2 - x^2)\,dx$$
$$= \frac{\rho}{2}\left(R^2 x - \tfrac{1}{3}x^3\right)\Big|_0^R$$
$$= \boxed{\tfrac{1}{3}\rho R^3 = M_y}$$

Now we can find the center of mass,

$$X_{cm} = \frac{M_y}{M} = \frac{\frac{1}{3}\rho R^3}{\frac{\pi}{4}\rho R^2} = \frac{4}{3\pi}R \cong 0.4244R$$
$$Y_{cm} = \frac{M_x}{M} = \frac{\frac{1}{3}\rho R^3}{\frac{\pi}{4}\rho R^2} = \frac{4}{3\pi}R \cong 0.4244R$$

From the symmetry of the region we could have anticipated $X_{cm} = Y_{cm}$.

Notice we could work problems where the density depended on x without much trouble, however if the density depended on both x and y at once then it would not be easy given our current tools. In calculus III we can treat problems which allow both x and y to vary so we relegate that more interesting class of problems to that course.

Remark 14.2.6. *note format.*

Beyond this point the notes will change format. The notes to follow are from previous years however, I have numbered the equations and sections as to be consistent with the numbering up to this point. The page numbering ceases to be meaningful past this point. You can still refer to the section number without ambiguity provided you clarify if it is Stewart or my notes in question. Sorry for the change in format, I didn't have enough time over break to complete the notes to the level of the notes up to this point. Some of you will rejoice in the sudden reduction in proofy-ness.