Name:

MATH 332-001, MAY. 6, 5PM, 2010,	TEST III (TAKE-HOME)

Do not omit scratch work. I need to see all steps. Skipping details will result in a loss of credit.

Problem 1 [350pts] Use methods of contour integration and/or residue theory to calculate the integral that follows:

$$\int_{-\infty}^{-\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$$

Leave you answer in terms of a, b and π.

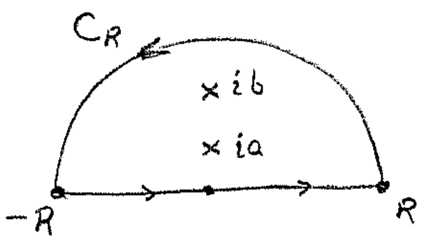
Without loss of generality we may assume $a, b > 0$. Let $f(z) = \frac{1}{(a^2+z^2)(b^2+z^2)}$

Notice $f(z) = \frac{1}{(z+ia)(z-ia)} \frac{1}{(z+ib)(z-ib)}$. Furthermore, note

if $|z| > \max(a,b)$ and $|z| = R$ then

$$|f(z)| \leq \frac{1}{\big||a|^2 - |z^2|\big|\big||b^2| - |z^2|\big|} = \frac{1}{|a^2 - R^2||b^2 - R^2|} = \underbrace{\frac{1}{(R^2-a^2)(R^2-b^2)}}_{M_R}.$$

This bound M_R will be useful in what follows.



[could be $ib \leftrightarrow ia$ never said $a < b$, doesn't modify calculation]

Notice $\underset{z=ia}{\text{Res}}\, f(z) = \left.\frac{1}{(z+ia)(z^2+b^2)}\right|_{z=ia} = \frac{1}{2ia(b^2-a^2)}$.

Likewise, $\underset{z=ib}{\text{Res}}\, f(z) = \left.\frac{1}{(z+ib)(z^2+a^2)}\right|_{z=ib} = \frac{1}{2ib(a^2-b^2)}$.

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$$

Cauchy's Residue Th^{m}

$$\Rightarrow \quad 2\pi i \left(\frac{1}{2ia(b^2-a^2)} + \frac{1}{2ib(a^2-b^2)}\right) = \int_{-R}^{R} \frac{dx}{(a^2+x^2)(b^2+x^2)} + \int_{C_R} f(z)\,dz$$

PROBLEM 1 Continued,

$$0 \leq \left| \int_{C_R} f(z)\,dz \right| \leq M_R\, \ell(C_R) = \frac{2\pi R}{(R^2-a^2)(R^2-b^2)}$$

Note $R \longrightarrow \infty \Rightarrow \int_{C_R} f(z)\,dz = 0$ by squeeze Th^{m}.

Notice as $R \longrightarrow \infty$ we obtain P.V. $\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$ from the $\int_{-R}^{R}$ along the real axis, let me summarize,

$$2\pi i \left(\frac{1}{2ia(b^2-a^2)} + \frac{1}{2ib(a^2-b^2)} \right) = \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{dx}{(a^2+x^2)(b^2+x^2)} + \underbrace{\int_{C_R} f(z)\,dz}_{\to 0} \right)$$

$$\Rightarrow \pi \left[\frac{1}{a(b^2-a^2)} - \frac{1}{b(b^2-a^2)} \right] = \text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$$

$$\Rightarrow \pi \left[\frac{1}{ab} \left[\frac{b-a}{b^2-a^2} \right] \right] = \frac{\pi}{ab(a+b)} = \text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

$$\therefore \boxed{\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{ab(a+b)}}$$

Problem 2 [350pts] Use methods of contour integration and/or residue theory to calculate the integral that follows:

$$\int_0^{2\pi} e^{2\cos(\theta)}\, d\theta.$$

Answer can be left in a closed form infinite series that has a factorial.

Identify $\int_0^{2\pi} e^{2\cos\theta} d\theta = \int_C f(z)dz$ for C being the unit circle: $z = e^{i\theta}$ for $0 \le \theta \le 2\pi$.

then $2\cos\theta = e^{i\theta} + e^{-i\theta} = z + \frac{1}{z}$ on C.

Furthermore, $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$.

Substituting,

$$\int_0^{2\pi} e^{2\cos\theta} d\theta = \int_C \exp\left(z + \frac{1}{z}\right)\frac{dz}{iz}$$

$$= \int_C \underbrace{\frac{-i}{z} e^{z} e^{\frac{1}{z}}}_{f(z)} dz$$

$$= 2\pi i \operatorname*{Res}_{z=0}[f(z)].$$

This is an interesting residue to compute.

$$f(z) = -i\left(1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \cdots\right)\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{2z^3} + \frac{1}{3!z^4} + \cdots\right)$$

$$= -i\left(\frac{1}{z} + \frac{1}{z} + \frac{1}{4z} + \frac{1}{3!3!z} + \cdots\right) \quad : \text{just focusing on } \frac{1}{z} \text{ type terms.}$$

$$= -i\left(\frac{1}{(0!)^2} + \frac{1}{(1!)^2} + \frac{1}{(2!)^2} + \frac{1}{(3!)^2} + \cdots\right)\frac{1}{z} + \cdots$$

$$= \frac{-i}{z}\sum_{n=0}^{\infty}\frac{1}{(n!)^2} + \cdots \Rightarrow \operatorname*{Res}_{z=0} f(z) = \sum_{n=0}^{\infty}\frac{-i}{(n!)^2}$$

$$\therefore \boxed{\int_0^{2\pi} e^{2\cos\theta} d\theta = 2\pi\sum_{n=0}^{\infty}\frac{1}{(n!)^2}}$$

Problem 3 [400pts] Use methods of contour integration and/or residue theory to calculate the integral that follows:

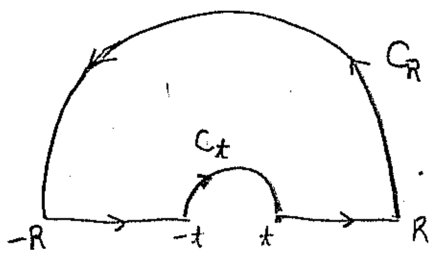
$$\int_0^\infty \frac{\sin^3(x)}{x^3}\,dx$$

Leave you answer in terms of $3, \pi$ and 8.

Observe that $\sin^3(x) = \operatorname{Im}\left(\frac{3}{4}e^{ix} - \frac{1}{4}e^{3ix} - \frac{1}{2}\right)$.

Thus $f(z) = \frac{1}{z^3}\left(\frac{3}{4}e^{iz} - \frac{1}{4}e^{3iz} - \frac{1}{2}\right)$ will reproduce the given integrand $\frac{\sin^3(x)}{x^3}$ along the real axis, but we'll need to take the imaginary component to select it.

Call the complete contour "C"



we have in mind $R \to \infty$ and $t \to 0^+$. Inside this half-annular region it is clear $f(z)$ is analytic. However, as $t \to 0^+$ the pole at $z=0$ will contribute a term.

$$0 = \int_C f(z)\,dz = \underbrace{\int_{-R}^{-t} f(x)\,dx}_{\text{I}} + \underbrace{\int_{C_t} f(z)\,dz}_{\text{III}} + \underbrace{\int_t^R f(x)\,dx}_{\text{II}} + \underbrace{\int_{C_R} f(z)\,dz}_{\text{IV}}$$

Cauchy Residue Thm.

Notice, I & II should combine to yield P.V. $\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,dx$ as we take limits $t \to 0^+$ and $R \to \infty$. In contrast, I expect IV vanishes, let's see why, for $|z| = R$,

$$|f(z)| \leq \frac{1}{|z|^3}\left(\left|\frac{3}{4}e^{iz}\right| + \left|\frac{1}{4}e^{3iz}\right| + \left|\frac{1}{2}\right|\right) = \frac{3}{2R^3} = M_R$$

Hence $\left|\int_{C_R} f(z)\,dz\right| \leq M_R \pi R = \frac{3\pi}{2R^2} \Rightarrow \lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$.

The calculation for C_t requires more thought.

PROBLEM 3 Continued

We wish to calculate $\int_{C_t} f(z)dz$ as $t \to 0^+$.

Notice, $C_t : z = te^{i\theta}$ for $0 \leq \theta \leq \pi$. Moreover,

$$f(z) = \frac{1}{z^3}\left(\frac{3}{4}e^{iz} - \frac{1}{4}e^{3iz} - \frac{1}{2}\right)$$

$$= \frac{1}{z^3}\left\{\frac{3}{4}\left[1 + iz - \frac{1}{2}z^2 - \frac{i}{3!}z^3 + \cdots\right] - \frac{1}{4}\left[1 + 3iz - \frac{9}{2}z^2 - \frac{27i}{3!}z^3 + \cdots\right] - \frac{1}{2}\right\}$$

$$= \frac{1}{z^3}\left\{-\frac{3}{8}z^2 + \frac{9}{8}z^2 + \cdots\right\}$$

$$= \frac{3}{4}\frac{1}{z} + \text{analytic at zero terms.}$$

We find $f(z)$ has simple pole at $z = 0$ hence by the Lemma I proved in lecture,

(because C_t is clockwise)

$$\lim_{t \to 0^+} \int_{C_t} f(z)dz = -\pi i\left(\frac{3}{4}\right) = \qquad \left(\text{half circle gives } \theta = \pi \text{ for "}T_r\text{" lemma}\right)$$

Therefore in the limit $R \to \infty$, $t \to 0^+$ we find,

$$0 = \text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx - \frac{3\pi i}{4}$$

Taking imaginary component of both sides and using the $\text{P.V.} \int_{-\infty}^{\infty} f_{even}(x)dx = \int_0^{\infty} f_{even}(x)dx$ Lemma,

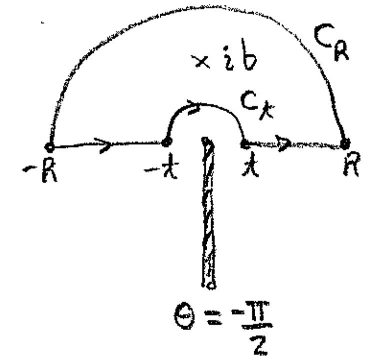
$$\boxed{\int_0^{\infty} \frac{\sin^3(x)}{x^3}\,dx = \frac{3\pi}{8}}$$

Problem 4 [400pts] Use methods of contour integration and/or residue theory to calculate the integral that follows (assume $b > 0$):

$$\int_0^\infty \frac{\ln(x)}{x^2 + b^2}\, dx$$

Leave you answer in terms of $2, \pi$ and b and $\ln(b)$.

Let $\log_\alpha(z) = \log(z)$ such that we have branch-cut at $\theta = \alpha$ and $\log_\alpha(z) = \ln|z| + i\arg_\alpha(z) = \ln(r) + i\theta$ where $\alpha < \theta < 2\pi + \alpha$. Let $\alpha = -\pi/2$ and consider the contour C pictured below,



$$\underbrace{f(z) = \frac{\log_\alpha(z)}{z^2 + b^2}}_{\text{restricts to } \frac{\ln(x)}{x^2+b^2} \text{ for positive real axis.}}, \quad \alpha = \frac{-\pi}{2}$$

There is a pole at $z = ib$ since $f(z) = \dfrac{\log_\alpha(z)}{(z+ib)(z-ib)}$.

We aim to calculate P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\ln|x|}{x^2+b^2}\,dx$.

Cauchy's Residue Th$^{\underline{m}}$ yields that,

$$\underbrace{2\pi i \operatorname*{Res}_{z=ib} f(z)}_{\text{V}} = \underbrace{\int_{-R}^{-t} \frac{\ln|x| + i\pi}{x^2+b^2}\,dx}_{\text{I}} + \underbrace{\int_{C_t} f(z)dz}_{\text{III}} + \underbrace{\int_{t}^{R} \frac{\ln|x|\,dx}{x^2+b^2}}_{\text{II}} + \underbrace{\int_{C_R} f(z)dz}_{\text{IV}}$$

I begin with V, note $z = ib$ is simple pole thus,

$$\operatorname*{Res}_{z=ib}\left(\frac{1}{z-ib}\,\frac{\log_\alpha(z)}{z+ib}\right) = \frac{\log_\alpha(ib)}{2ib}, \quad \underline{b > 0 \Rightarrow b \in \mathbb{R}}\,!$$

$$= \frac{1}{2ib}\log_\alpha\left(b e^{i\frac{\pi}{2}}\right)$$

$$= \underline{\frac{1}{2ib}\left(\ln b + i\frac{\pi}{2}\right)}.$$

note $\frac{-\pi}{2} < \frac{\pi}{2} < \frac{3\pi}{2}$

correct element of arg(ib) to select by def$^{\underline{n}}$ of $\log_\alpha(z)$.

PROBLEM 4 Continued

I suspect IV vanishes as $R \to \infty$. Consider for $z \in C_R$ we have $|z| = R$ thus assuming $R > b$ reasonable to assume as $R \to \infty$

$$|f(z)| = \frac{|\log_\alpha(z)|}{|z^2+b^2|} \leq \frac{|\ln|z| + i\arg_\alpha(z)|}{||z^2| - |b^2||}$$

$$\leq \frac{\ln R + |\arg_\alpha(z)|}{R^2 - b^2} \qquad -\frac{\pi}{2} < \arg_\alpha(z) < \frac{3\pi}{2}$$

$$\leq \frac{\ln R + 3\pi/2}{R^2 - b^2} = M_R$$

Note $0 \leq \left|\int_{C_R} f(z)\,dz\right| \leq M_R\, l(C_R) = M_R \cdot \pi R$. Consider then

$$\lim_{R\to\infty}\left[\frac{(\ln R + 3\pi/2)\pi R}{R^2 - b^2}\right] \underset{(\frac{\infty}{\infty})}{\overset{L}{=}} \lim_{R\to\infty}\left[\frac{\pi \ln(R) + 3\pi/2 + \pi}{2R}\right] \underset{(\frac{\infty}{\infty})}{\overset{L}{=}} \lim_{R\to\infty}\left(\frac{\pi}{R}\right) = 0.$$

Therefore, by squeeze Thm $\lim_{R\to\infty}\left|\int_{C_R} f(z)\,dz\right| = 0$.

hence, $\int_{C_R} f(z)\,dz = 0$. //

Next consider III we have in mind $t \to 0^+$. Again we have that for $z \in C_t$ that $|z| = t$ and since we have $t \to 0^+$ in mind, we assume $t < b$

$$|f(z)| \leq \frac{|\ln|z| + i\arg_\alpha(z)|}{||z^2| - |b^2||} \leq \frac{\ln t + 3\pi/2}{|t^2 - b^2|} \leq \frac{\ln(t) + 2\pi}{b^2 - t^2} = M_t$$

Note, $0 \leq \left|\int_{C_t} f(z)\,dz\right| \leq M_t\, \pi t$. Should calculate $\lim_{t\to 0^+}(\pi t M_t)$,

PROBLEM 4 Continued

$$\lim_{t \to 0^+} \left(\frac{\pi t \ln(t) + 2\pi^2 t}{b^2 - t^2} \right) = \lim_{t \to 0^+} \left[\frac{\pi \ln(t) + 2\pi^2}{\frac{b^2}{t} - t} \right]$$

$$\underset{\left(\frac{\infty}{\infty}\right)}{\overset{\text{L}}{=}} \lim_{t \to 0^+} \left[\frac{\pi / t}{-b^2/t^2 - 1} \right]$$

$$\underset{\left(\frac{\infty}{\infty}\right)}{\overset{\text{L}}{=}} \lim_{t \to 0^+} \left[\frac{-\pi/t^2}{2b^2/t^3} \right]$$

$$= \lim_{t \to 0^+} \left[\frac{-\pi t}{2b^2} \right] = 0.$$

Thus, by squeeze $Th^{\underline{m}}$, note $0 \leq \left| \int_{C_t} f(z)dz \right| \leq \pi t M_t$

$$\Rightarrow \lim_{t \to 0^+} \left| \int_{C_t} f(z)dz \right| = 0 \quad \therefore \quad \underline{\int_{C_t} f(z)dz = 0}.$$

Thus, in the limit $R \to \infty$ and $t \to 0^+$ we obtain,

$$\underbrace{\frac{2\pi i}{2ib}\left(\ln(b) + \frac{i\pi}{2}\right)}_{\pi b \ln(b) + i\frac{\pi^2}{2b}} = P.V.\left(\int_{-\infty}^{\infty} \frac{\ln|x|\, dx}{x^2 + b^2} + \underbrace{i \int_{-\infty}^{\infty} \frac{\pi}{x^2+b^2} dx}_{\text{came from the fact } \theta = \pi \text{ on negative real axis.}} \right)$$

We find two interesting integrals,

$$Re: \quad \frac{\pi \ln(b)}{b} = P.V. \int_{-\infty}^{\infty} \frac{\ln|x|}{x^2+b^2} dx$$

$$Im: \quad \frac{\pi^2}{2b} = P.V. \int_{-\infty}^{\infty} \frac{\pi\, dx}{x^2+b^2}$$

(this not surprising given that $\int \frac{dx}{x^2+b^2} = \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) + C$)

We obtain that

$$\boxed{\int_0^{\infty} \frac{\ln|x|\, dx}{x^2+b^2} = \frac{\pi \ln(b)}{2b}}$$

(and $\ln|x| = \ln(x)$ for $x > 0$)

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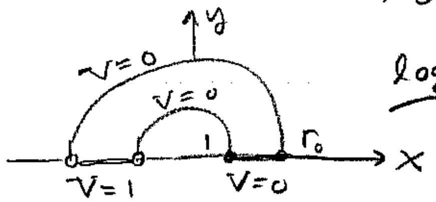
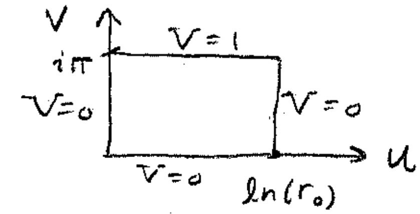


log



$$w = \log_\alpha(z)$$
$$\alpha = \frac{-\pi}{2}$$

$$\log(re^{i\theta}) = \ln(r) + i\theta = u + iv$$

$$\begin{cases} 0 < u < a = \ln r_0 \\ 0 < v < b = \pi \end{cases}$$

$$V(u,v) = \frac{4}{\pi}\sum_{n=1}^{\infty} \underbrace{\frac{\sinh\left(\frac{m\pi v}{\ln(r_0)}\right)}{\sinh\left(\frac{m\pi \pi}{\ln(r_0)}\right)} \frac{\sin\left(\frac{m\pi u}{\ln(r_0)}\right)}{m}}$$

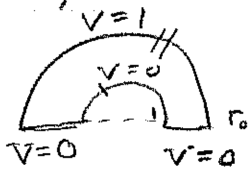
$m = 2n-1$ using Churhill's notation from problem statement.

Let $\alpha_n = \left(\frac{2n-1}{\ln r_0}\right)\pi$ and we obtain, using $\underbrace{u = \ln(r) \;\&\; v = \theta}_{\text{see } w = \log_\alpha(z).}$

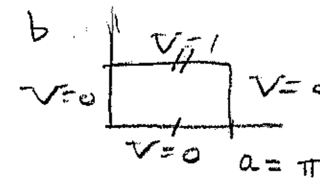
$$\boxed{V(r,\theta) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sinh(\alpha_n\theta)}{\sinh(\alpha_n\pi)} \frac{\sin(\alpha_n \ln(r))}{2n-1}}$$

PROBLEM 11 of pg. 315

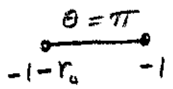
Use Exercise 10 idea to solve similar problem. Solve Dirichlet problem for $1 < r < r_0$, $0 < \theta < \pi$



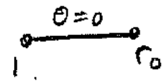
Want a transformation onto UV-plane which gives back the BC with



How do we map the upper half-circle to horizontal line? Also want to map



and

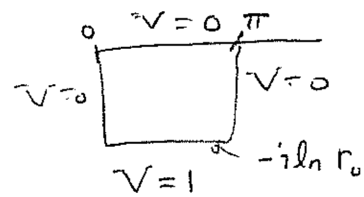


to verticle line segments in u-v plane.

$$z = re^{i\theta} \longmapsto W = i\ln\left(\tfrac{1}{r}\right) + \theta \Leftarrow \underline{u = \theta,\ v = -\ln(r)}.$$
$$= -i\ln(r) + \theta$$
$$= -i(\ln(r) + i\theta)$$
$$= -i\log(z) \Leftarrow \text{analytic} \Rightarrow \text{conformal trns.}$$

Note, $r = 1$, $0<\theta<\pi$ maps to $w = i\ln(1) + \theta = \theta$

Also, $r = r_0$, $0<\theta<\pi$ maps to $w = i\ln(1/r_0) + \theta$ $\Leftarrow$ (not quite right)



Well,

$u = \theta$, $v = -\ln(r)$, $a = \pi$

$$\Rightarrow V(r,\theta) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sinh(m(-\ln(r)))}{\sinh(m(-\ln(r_0)))}\,\frac{\sin(\theta)}{m}, \quad m = 2n-1$$
$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{r^m - r^{-m}}{r_0^{\,m} - r_0^{\,-m}}\right]\frac{\sin\theta}{m}.$$

Since $\sinh(m\ln(r^{-1})) = \frac{1}{2}\left(e^{m\ln(r^{-1})} - e^{-m\ln(r^{-1})}\right) = \frac{1}{2}(r^{-m} - r^m)$ and $\sinh(m(\ln r_0^{-1})) = \frac{1}{2}(r_0^{\,-m} - r_0^{\,m})$, cancelling minus signs yields desired result.

<u>Remark:</u> my "sol$^{\underline{n}}$" not correct, should find $W = f(z)$ mapping to rectangle above u-axis (not below). Anyway, it's close.